Bounds on the Total Redundance and Efficiency of a Graph

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Abstract

For a graph G with vertex set V, the total redundance, $TR(G)$, and efficiency, $F(G)$, are defined by the two expressions:
$TR(G) = \min\{\sum_{v\in S}(1+\deg v) : S \subseteq V \text{ and } |N[x]\cap S| \geq 1\ \forall x \in V\}$,
$F(G) = \max\{\sum_{v\in S}(1+\deg v) : S \subseteq V \text{ and } |N[x]\cap S| \leq 1\ \forall x \in V\}$.
That is, TR measures the minimum possible amount of domination if every vertex is dominated at least once, and F measures the maximum number of vertices that can be dominated if no vertex is dominated more than once.

We establish sharp upper and lower bounds on $TR(G)$ and $F(G)$ for general graphs G and, in particular, for trees, and briefly consider related Nordhaus-Gaddum-type results.

1 Introduction

Let G be a graph with vertex set V and edge set E; let $p = |V|$ and $q = |E|$. The minimum and maximum degrees of vertices in G are denoted by $\delta(G)$ (or δ) and $\Delta(G)$ (or Δ), respectively.

A *dominating set* D of G is a subset of V such that every vertex in $V - D$ is adjacent to at least one vertex in D. A dominating set of G of minimum cardinality is a *minimum dominating set* of G, and its cardinality is the *domination number* of G, denoted by $\gamma(G)$. A vertex v of G is said to *dominate* a vertex w if $w = v$ or $wv \in E$, i.e., if w is contained in the closed neighbourhood $N[v]$ of v.

If D is any dominating set of G, the *total redundance of* D, $TR(D)$, is defined by $TR(D) = \sum_{v\in D}(1+\deg v)$. The *total redundance of* G, $TR(G)$, is defined to be the minimum value of $TR(D)$, where the minimum is taken over all dominating sets D of G. These concepts were introduced by Grinstead and Slater in [5] and [6], where $TR(G)$ was called the redundance number of G, and the present terminology was introduced in [7] where further associated concepts were also considered. A dominating set D of G for which $TR(D) = p$ is known as an *efficient dominating set* of G (see [1], [2] and [3]). Related concepts are discussed in [4].

A *packing* of G is a subset P of V such that, for every vertex $v \in V$, $|N[v] \cap P| \leq 1$; i.e., for every pair u, v of distinct vertices in P, the distance between u and v is at least 3. (Also see [8] in which k-packings are introduced for $k \in \mathbf{Z}^+$: 1-packings corresponding to independent sets of vertices and 2-packings to our packings, and, in general, a k-packing being a set S of vertices such that $d(u,v) > k$ for all distinct $u, v \in S$.) A packing of G of maximum cardinality is called a *maximum packing* of G, and its cardinality is called the packing number of G, denoted by $\rho(G)$. For a packing P of G the *efficiency* of P, denoted by $F(P)$, is defined by $F(P) = \sum_{v \in P}(1 + \deg v)$; i.e., $F(P) = |\bigcup_{v \in P} N[v]|$. The *efficient domination number* of G, denoted by $F(G)$, is the maximum value of $F(P)$, taken over all packings P of G. Related concepts are discussed in [6].

Thus, briefly,

$$TR(G) = \min\left\{ \sum_{v \in S}(1 + \deg v) : S \subseteq V \text{ and } |N[x] \cap S| \geq 1 \text{ for all } x \in V \right\}$$

and

$$F(G) = \max\left\{ \sum_{v \in S}(1 + \deg v) : S \subseteq V \text{ and } |N[x] \cap S| \leq 1 \text{ for all } x \in V \right\}.$$

A few observations can be made immediately:

1. For a graph G of order p, $F(G) \leq p \leq TR(G)$; furthermore $F(G) = p$ or $TR(G) = p$ if and only if G possesses a dominating set which is also a packing (i.e., an efficient dominating set). Hence either $TR(G) = F(G)$ or $TR(G) - F(G) \geq 2$.

2. For every integer $m \geq 3$ there exists a graph G such that $TR(G) - F(G) = m$; for instance if $G \cong K_{m-1,m-1}$, then $TR(G) = 2m$ and $F(G) = m$.

3. For any positive integer m there exists a graph G for which $TR(G) = mF(G)$; for instance, $G \cong K_m \times K_m$ is such that $TR(G) = m(2m-1)$ and $F(G) = 2m - 1$.

4. For every pair of integers p, m with $1 \leq m \leq p$ there exists a graph G of order p which possesses an efficient dominating set of cardinality m; for instance $G \cong K_{1,p-2m+1} \cup (m-1)K_2$ if $m \leq p/2$.

5. It has been established (cf. [3]) that the infinite grid graph $P_\infty \times P_\infty$ has an efficient dominating set. However, for $G \cong P_3 \times P_3$, for instance, $F(G) = 7$, $p(G) = 9$ and $TR(G) = 10$.

2 Bounds on $F(G)$ and $TR(G)$

If G is an r-regular graph, then $TR(G) = \gamma(G)(r+1)$ and $F(G) = \rho(G)(r+1)$. But for nonregular graphs the situation is far more complex. The problems of deciding whether, for some input integer k, $TR(G) \leq k$ or $F(G) \geq k$ are NP-complete. In fact, deciding if $F(G) = p = TR(G)$ is NP-complete. (See [2].)

We shall therefore find bounds on $TR(G)$ and $F(G)$. In particular, since $F(G) \leq p \leq TR(G)$, we shall seek a lower bound for $F(G)$ and an upper bound for $TR(G)$, both for general graphs G and for trees.

2.1 Upper bounds on $TR(G)$

If G is a graph of order p, then $TR(G) \geq p$ and this bound is attained if G has an efficient dominating set, for instance, if G is a star. The first theorem gives an upper bound on the total redundance of a tree.

Theorem 1 *If T is a nontrivial tree of order p, then*

$$TR(T) \leq 3p/2 - 1.$$

Proof: Let T be a nontrivial tree with partite sets V_1 and V_2, where $|V_1| \leq |V_2|$. Then V_1 is a dominating set of T and

$$TR(T) \leq TR(V_1) = \sum_{v \in V_1} (1 + \deg v) = |V_1| + q(T) \leq p/2 + p - 1 = 3p/2 - 1,$$

as required. □

For any integer $k \geq 1$ the above bound is attained by the tree of order $4k+2$ obtained from $kP_3 \cup (k+1)K_1$ by the introduction of a new vertex which is adjacent to the central vertex of each component of $kP_3 \cup (k+1)K_1$.

We now turn out attention to general graphs.

Theorem 2 *Let G be a graph of order p. Then the total redundance of G is at most $cp^{3/2} + o(p^{3/2})$ where $c = 4/\sqrt{27e}$.*

Proof: The proof is by probabilistic methods. Let u be a vertex of the maximum degree Δ. Let X denote the set of vertices not in $N[u]$ that have degree at most $p^{3/4}$, and let $|X| = x$.

Assume first that $0.0001p \leq x \leq 0.99p$. Set $\pi = \sqrt{x/e}/(p-x)$. (Note that in particular $1/(100\sqrt{ep}) < \pi < 100/\sqrt{ep}$.) Form the set A by taking each vertex of G independently with probability π. Let B denote the set of those vertices that are dominated by neither A nor u. Then $D = A \cup B \cup \{u\}$ is a dominating set.

For a vertex v, let $y_v = 1 + \deg v$ denote the cardinality of v's closed neighbourhood $N[v]$. Then the expected total redundance of A is given by

$$\mathbf{E}(TR(A)) = \sum_{v \in V} \pi y_v = 2\pi q + \pi p.$$

Since $x \leq p - \Delta$, and $2q \leq xp^{3/4} + (p-x)\Delta$, it follows that

$$\mathbf{E}(TR(A)) \leq (p-x)^2 \pi + O(p^{5/4}).$$

Furthermore, since a vertex v of $V - N[u]$ is contained in B if and only if no vertex of $N[v]$ is contained in A, the probability that v is contained in B is $(1-\pi)^{y_v}$. Since $1 - \pi \leq e^{-\pi}$ (this holds whenever $\pi \geq 0$), it follows that the expected total redundance of B is bounded by

$$\mathbf{E}(TR(B)) \leq \sum_{v \notin N[u]} y_v \, e^{-\pi y_v}.$$

A vertex $v \notin X$ has degree more than $p^{3/4}$ and for it $y_v e^{-\pi y_v}$ is $o(1/p)$ (since $\pi \geq 1/(100\sqrt{ep})$). Also, the maximum of the real function $f(y) = ye^{-\pi y}$ is $1/(\pi e)$ (attained at $y = 1/\pi$). So

$$\mathbf{E}(TR(B)) \leq x/(\pi e) + o(1).$$

Hence the expected total redundance of D is bounded by

$$\mathbf{E}(TR(D)) \leq (p-x)^2 \pi + x/(\pi e) + O(p^{5/4}).$$

Now, substitute the value of π into this expression. Then by calculus it follows that the maximum of the resultant expression $2(p-x)\sqrt{x/e}$ is attained at $x = p/3$. There its value is $cp^{3/2}$ for the above value of c.

If $x < 0.0001p$, then set $\pi = 0.01/\sqrt{p}$ and construct D as before. The expected total redundance of D is at most $p^2\pi + x/(\pi e) + O(p^{5/4})$, which is less than $0.02p^{3/2} + O(p^{5/4})$. If $x > 0.99p$, then set $\pi = 100/\sqrt{p}$ and construct D as before. The expected total redundance of D is at most $(0.01p)^2\pi + p/(\pi e) + O(p^{5/4})$, which is less than $0.02p^{3/2} + O(p^{5/4})$.

Since the expected total redundance of D is at most $cp^{3/2} + o(p^{3/2})$, there must exist a dominating set D with total redundance at most this value. Consequently,

$$TR(G) \leq cp^{3/2} + o(p^{3/2}),$$

as required. □

In general we have:

Theorem 3 *If a graph G has p vertices and $q \geq p$ edges, then the total redundance of G is at most $O((pq)^{1/2})$.*

Proof: The proof is similar to the above proof but simpler. Set $\pi = \sqrt{p/q}$. Then construct A by taking each vertex independently with probability π, and set $D = A \cup (V - N[A])$. Then by a discussion similar to the above it follows that

$$\mathbf{E}(TR(D)) \leq 2q\pi + p/(\pi e) + O(p),$$

whence the bound. □

We can also use probabilistic methods to show that the bound in Theorem 2 is sharp. Define the graph $G(j,k,d)$ as follows: Take disjoint sets J and K where J is an independent set of j vertices and K is a clique of k vertices, and join each vertex of J to d vertices in K independently at random.

Lemma 1 *For $j = p/3$, $k = 2p/3$, and $d = \lfloor\sqrt{4ep/3}\rfloor$, with probability $1 - o(1)$ the total redundance of $G = G(j,k,d)$ is at least $cp^{3/2} - o(p^{3/2})$ where $c = 4/\sqrt{27e}$.*

Proof: We need to show that with probability $1 - o(1)$ there does not exist a dominating set of G with total redundance less than $cp^{3/2} - o(p^{3/2})$.

Any minimal dominating set D of G is of the form $D = A \cup X_A$ where $A \subseteq K$ and $X_A = J - N(A)$. If $|A| > p^{1/2}$, then $TR(D) \geq k|A| > cp^{3/2}$. So we may restrict our attention to those D such that $|A| \leq p^{1/2}$.

Now let A be any subset of K of cardinality $a \leq p^{1/2}$. Any particular vertex x of J is in X_A if and only if $N(x) \subseteq K - A$. So the probability ξ_a that $x \in X_A$ is given by

$$\xi_a = \frac{\binom{k-a}{d}}{\binom{k}{d}} \approx \left(\frac{k-a}{k}\right)^d \approx e^{-ad/k},$$

and $|X_A|$ has the binomial distribution $B(j, \xi_a)$. In particular, $\mathbf{E}(|X_A|) = j\xi_a$.

We can use large-deviation bounds to show that it is extremely unlikely that $|X_A|$ is much smaller than its expectation. From Lecture 4 in [9] it follows that

$$\Pr\left(|X_A| < j\xi_a - p^{7/8}\right) < e^{-p^{7/4}/(2j\xi_a)} < e^{-p^{3/4}}.$$

There are at most $p^{p^{1/2}}$ possibilities for A. Since $p^{p^{1/2}}e^{-p^{3/4}}$ is $o(1)$, with probability $1 - o(1)$ it holds that: For every a up to $p^{1/2}$, and for every $A \subseteq K$ of cardinality a, the set $X_A = J - N(A)$ has cardinality at least $j\xi_a - p^{7/8}$.

The total redundance of $A \cup X_A$ is at least $ak + |X_A|d$. Hence, with probability $1 - o(1)$, the total redundance of the graph G is at least

$$TR(G) \geq \min_a ak + jde^{-ad/k}$$

less some small-order terms. This expression is minimized at a equal to $k \ln(jd^2/h^2)/d$. Hence, for the above values of j, k and d, with probability $1 - o(1)$ the graph $G(j,k,d)$ has total redundance at least $cp^{3/2} - o(p^{3/2})$. □

2.2 Lower bounds on $F(G)$

If G is a graph of order p, then $F(G) \leq p$ and this bound is attained, for instance, if G is a star.

Lemma 2 *For any graph G of order p and maximum degree Δ,*

(a) $F(G) \geq \Delta + 1$,

(b) $F(G) \geq \dfrac{p(\Delta+1)}{\Delta^2+1}$.

Proof:
(a) If v is a vertex of degree Δ, then $F(G) \geq |N[v]| = 1 + \Delta$.
(b) Let S be a packing of G such that $F(S) = F(G)$, and let $T = N[S]$. Then S is a maximal packing and so $N[T] = V(G)$. Every vertex in $N[T] - N[S]$ is adjacent to a vertex in $N[S] - S$, and every vertex in $N[S] - S$ has at least one neighbour in S. Hence $|N[T]| \leq |N[S]| + (|N[S]| - |S|)(\Delta - 1)$. Since $F(G) = |N[S]|$, it follows that

$$p \leq \Delta\, F(G) - |S|(\Delta - 1).$$

Furthermore, $F(G) = |N[S]| \leq |S|(\Delta+1)$. Hence $p \leq \Delta\, F(G) - F(G)(\Delta - 1)/(\Delta + 1)$ (provided $\Delta \geq 1$). Rearranged this gives the desired bound. □

That the bound in (a) is best possible follows from the observation that, for $m \in \mathbf{N}$ and $G \cong K_m \times K_m$, $p(G) = m^2$, $\Delta(G) = 2m - 2$, diam $G \leq 2$ and so $F(G) = 1 + \Delta(G) = 2m - 1$. The bound in (b) is attained by $G = C_5$, for example, but we believe that, in general, this bound is very poor.

As a corollary of the above two bounds we obtain:

Theorem 4 *If G is a graph of order p, then $F(G) = p$ if $p \leq 3$ and $F(G) \geq 1 + \sqrt{p-1}$ if $p \geq 4$.*

An infinite class of graphs of diameter 2 for which $F(G)$ closely approximates $1+\sqrt{p-1}$ may be constructed as follows: Let n be an integral power of a prime, F a field of order n and π the projective plane coordinatised by F. Then the points and lines of π correspond to the one- and two-dimensional subspaces of the three-dimensional vector space $V_3(F)$ over F. A point u and line ℓ of π are said to have homogeneous coordinates (u_1, u_2, u_3) and $[\ell_1, \ell_2, \ell_3]$, respectively, if the corresponding one- and two-dimensional subspaces of $V_3(F)$, are, respectively, the space spanned by (u_1, u_2, u_3) and the solution space of $\ell_1 x_1 + \ell_2 x_2 + \ell_3 x_3 = 0$. Let G be the graph which has the points of π as its vertices and in which distinct vertices u and v, with homogeneous coordinates (u_1, u_2, u_3) and (v_1, v_2, v_3), are adjacent if and only if $u_1v_1 + u_2v_2 + u_3v_3 = 0$. Hence v is adjacent in G to the $n+1$ vertices corresponding to points on the line $[v_1, v_2, v_3]$ of π if $v_1^2+v_2^2+v_3^2 \neq 0$ or to n such vertices if $v_1^2+v_2^2+v_3^2 = 0$; so $p(G) = n^2+n+1$ and so $\Delta(G) = n+1$. Since, furthermore, both u and v are adjacent to the point of π in which the lines with homogeneous coordinates $[u_1, u_2, u_3]$ and $[v_1, v_2, v_3]$ intersect, it follows that $d_G(u, v) \leq 2$ and so diam $G = 2$. Hence $F(G) = 1 + \Delta(G) = n + 2$, whereas $1 + \sqrt{p-1} = 1 + \sqrt{n^2+n}$; so $F(G) = \lceil 1 + \sqrt{p-1} \rceil$.

For trees we can improve the lower bound slightly. For a forest T, a *leaf* is a vertex of degree 1, and a vertex v of T is called a *penult* if v is adjacent to at least one leaf and to at most one non-leaf. For instance, the penultimate vertices of a longest path in T are penults of T.

Theorem 5 *For a tree T of order p, $F(T) \geq \sqrt{8(p+2)} - 4$.*

Proof: We prove that $F(T) \geq \sqrt{8(p+2)} - 4$ for any forest T on p vertices. The proof is by induction on p. The statement is certainly true for $p = 0$ and for an empty forest.

Let w be a penult of T with degree d. Let $T^* = T - N[w]$. Then by the induction hypothesis $F(T^*) \geq \sqrt{8(p-d+1)} - 4$. Let P be a packing of T^* with $F(P) = F(T^*)$, and x a leaf-neighbour of w in T. Since P contains no vertex from $N[w]$, $P \cup \{x\}$ is a packing of T; hence $F(T) \geq F(P) + 2 \geq \sqrt{8(p-d+1)} - 2$. We are done if $\sqrt{8(p-d+1)} \geq \sqrt{8(p+2)} - 2$. Consequently, we may assume that

$$d > \sqrt{2(p+2)} - 3/2$$

for all penults of T.

Suppose there exist penults w and w' of T at distance 3 or more. Then $Q = \{w, w'\}$ is a packing of T. So $F(T) \geq F(Q) > 2(\sqrt{2(p+2)} - 3/2)$, whence we are done. So we may assume that any two penults of T are at distance at most 2, and thus T is a tree of diameter at most 4.

Let v be a central vertex of T. Let it have a penults and b leaves as neighbours, and let d denote the maximum degree of a neighbour w of v. Then

$$F(T) \geq 1 + \deg v = a + b + 1.$$

Further, consider a packing R consisting of w and, for every other penult x adjacent to v, one leaf-neighbour of x. Then

$$F(T) \geq F(R) = d + 1 + 2(a - 1).$$

Note that $p \leq ad + b + 1$. We need to determine where the maximum of the above two bounds is minimized. It can easily be shown that where this occurs the two bounds are equal and $p = ad + b + 1$. It then follows that the minimum occurs when $a = \sqrt{(p+2)/2} - 1$, $d = \sqrt{2(p+2)} - 1$, and $b = \sqrt{9(p+2)/2} - 4$. For these values both lower bounds are $\sqrt{8(p+2)} - 4$. □

The bound in the above theorem is attained, for each value of $a \geq 2$, by the tree T_a, obtained from the star $K_{1,a}$ by attaching $3a - 1$ leaves to the centre of $K_{1,a}$ and $2a$ leaves to each leaf of $K_{1,a}$. Then $p(T_a) = 2a^2 + 4a$ and $F(T_a) = 4a = \sqrt{8(p(T_a) + 2)} - 4$.

On the other hand, caterpillars have a much higher efficiency. Recall that a *caterpillar* T is a tree such that the removal of all leaves from T yields a path, called the *spine* of T.

Theorem 6 *For a nontrivial caterpillar T of order p with n vertices on the spine, $F(T) \geq (p + 2n + 2)/3$.*

Proof: Let the spine of T be given by $v_1 v_2 \ldots v_n$, and let v_0 and v_{n+1} be leaves adjacent to v_1 and v_n respectively. Let P_0, P_1 and P_2 be the packings of T defined, for $i \in \{0, 1, 2\}$, by $P_i = \{ v_j : j \equiv i \pmod 3), 0 \leq j \leq n + 1 \}$. Then every leaf in T is dominated in exactly one of the three packings, except for v_0 and v_{n+1} which are dominated in two of them, and every vertex on the spine of T is dominated in all three packings. So $\sum_{i=0}^{2} F(P_i) \geq (p + 2n + 2)$. It follows that $\max\{F(P_1), F(P_2), F(P_3)\} \geq (p + 2n + 2)/3$. □

This bound is tight for a caterpillar in which the spine has order n a multiple of 3, and every interior vertex on the spine is adjacent to a leaves while the two ends of the spine are adjacent to $a-1$ leaves each (for $a \geq 2$).

3 Nordhaus–Gaddum-Type Bounds

It is possible to establish Nordhaus–Gaddum-type bounds for the packing numbers or total redundances of a graph G and its complement $\bar{G}$.

Theorem 7 *For a graph G of order p,*

(a) $p+1 \leq F(G)+F(\bar{G}) \leq 2p$,

(b) $(p-1)^{3/2}+1 \leq F(G)F(\bar{G}) \leq p^2$.

Proof: The upper bounds follow immediately from $F(G), F(\bar{G}) \leq p$. To establish the lower bounds, let $F = F(G)$ and $\bar{F} = F(\bar{G})$. Then $F + \bar{F} \geq \Delta(G)+\Delta(\bar{G})+2 \geq p+1$. Furthermore, by Theorem 4, $F, \bar{F} \geq \sqrt{p-1}+1$. From calculus it follows that $F\bar{F} \geq f(\sqrt{p-1}+1)$, where $f(x) = x(p+1-x)$. And so $F\bar{F} \geq (p-1)^{3/2}+1$. □

The upper bounds in the above theorem are attained if, for instance, G is complete. To show that the lower bounds are sharp, consider $G \cong C_5[K_n]$, the lexicographic product of C_5 with K_n, for which $p = 5n$, $F(G) = 3n$, $F(\bar{G}) = 2n+1$ and so $F(G)+F(\bar{G}) = 5n+1 = p+1$. The lower bound in (b) is attained if $G \cong C_5$.

The use of similar techniques yields the existence of constants c_1, c_2 and c_3 for which $p^2 \leq R(G)R(\bar{G}) \leq c_1 p^{5/2}$, and, indeed, $c_2 p^{3/2} \leq R(G)F(G) \leq c_3 p^{5/2}$. The determination of "best possible" values of such constants is beyond the scope of this paper, but may merit investigation.

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