A Note on Joint Distributions of Some Random Vectors Defined on Permutations

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Abstract. We derive the exact joint distribution and prove the asymptotic joint normality of the numbers of peaks, double rises, troughs and double falls in a random permutation. A Chi-square randomness test, as a by-product, is also proposed.

1.Introduction

Suppose that $\Pi_n = (\pi_1, ..., \pi_n)$ is a random permutation of $I_n = \{1, ..., n\}$ that is uniformly distributed on the set S_n of all permutations of I_n. The element π_i of a permutation Π_n is called a *peak* if $\pi_{i-1} < \pi_i > \pi_{i+1}$, a *double rise* if $\pi_{i-1} < \pi_i < \pi_{i+1}$, a *trough* if $\pi_{i-1} > \pi_i < \pi_{i+1}$ and a *double fall* if $\pi_{i-1} > \pi_i > \pi_{i+1}$, respectively. Thus, with the convention $\pi_{n+1} = \pi_1$ and $\pi_0 = \pi_n$, each element of a permutation can be classified in one of the above four categories. Denote the numbers of peaks, double rises, troughs and double falls, respectively, in a permutation Π_n by

$$A_n = \sum_{i=1}^n I\{\pi_{i-1} < \pi_i > \pi_{i+1}\},$$
$$B_n = \sum_{i=1}^n I\{\pi_{i-1} < \pi_i < \pi_{i+1}\},$$
$$C_n = \sum_{i=1}^n I\{\pi_{i-1} > \pi_i < \pi_{i+1}\} \qquad \text{and}$$
$$D_n = \sum_{i=1}^n I\{\pi_{i-1} > \pi_i > \pi_{i+1}\},$$

where $I\{\cdot\}$ is the indicator function. Also, with the two ends π_1 and π_n not being counted, *i.e.* replacing $\sum_{i=1}^n$ by $\sum_{i=2}^{n-1}$, denote the corresponding four numbers in a permutation Π_n by A_n^*, B_n^*, C_n^* and D_n^*, respectively.

In this note we are interested in the joint distributions of the above two sets of random variables. In the circular case (A_n, B_n, C_n, D_n), since peaks and troughs always appear alternately in each permutation, it is clear that $A_n = C_n$. Also, since $A_n + B_n + C_n + D_n = n$, $n \geq 2$, we need only take into account A_n and B_n instead of all four variables. In the linear case

$(A_n^*, B_n^*, C_n^*, D_n^*)$, since $A_n^* + B_n^* + C_n^* + D_n^* = n-2$, $n \geq 3$, we can delete D_n^* and only deal with the remaining three variables.

In the literature the study of *turning points, i.e.* peaks and troughs, in a random permutation can be traced back to Bienaymé [1,2] in 1874-1875. He proposed a randomness test based on the number of turning points , which, as pointed out by Heyde and Seneta [8,p.124], is one of the earliest nontrivial nonparametric tests. Similar nonparametric tests have been employed on other problems, *e.g.* the number of peaks is used by Stigler [10] to test and to estimate the serial correlation of the AR(1) model in time series. For other statistical applications, the reader is referred to Stigler [10], Warren and Seneta [11] and references therein. In the past, studies on the distributional aspect of the above random variables are concentrated on peaks(troughs). The generating function of A_n or A_n^* (or a modified version of it) is derived by Carlitz and Scoville [3], David and Barton [5,pp.162-164] and Warren and Seneta [11]. The asymptotic normality of A_n is proved by Chao [4], David and Barton [5,pp.158-162] and Wolfowitz [12]. The most important results about peaks are due to Entringer [6]. He derived the generating function and hence the exact distribution of A_n. Relying on Entringer's results, we derive the generating function and the exact joint distribution of (A_n, B_n) in Section 2. In Section 3 recursive formulae for calculating the joint distribution of (A_n^*, B_n^*, C_n^*) are derived. In Section 4 we prove the asymptotic joint normality for (A_n^*, B_n^*) and (A_n, B_n) using a central limit theorem for m-dependent stationary random variables. As a by-product, a new Chi-square randomness test based on the numbers of peaks and double rises is also proposed.

2.Joint distribution and generating function of (A_n, B_n)

For $n \geq 1$, $a \geq 0$ and $b \geq 0$, let $\psi(n; a, b)$ denote the number of permutations in S_n with $A_n = a$ and $B_n = b$, and let $\psi(n; a, \cdot)$ denote the number of those in S_n with $A_n = a$. We note the following boundary conditions and properties of $\psi(n; a, b)$ and $\psi(n; a, \cdot)$, all of which are obvious :

(i) $\psi(1; 0, \cdot) = \psi(1; 0, 0) = 1$.

(ii) $\psi(n; a, \cdot) \neq 0$ only if $1 \leq a \leq \frac{n}{2}$ or (n, a)=(1,0).

(iii) $\psi(n; a, \cdot) = \sum_{b=0}^{n-2a} \psi(n; a, b)$, $n \geq 2$.

Clearly, due to the uniform distribution on S_n,

$$Pr\{A_n = a, B_n = b\} = \frac{\psi(n;a,b)}{n!} \quad \text{and} \quad Pr\{A_n = a\} = \frac{\psi(n;a,\cdot)}{n!}.$$

Entringer [6] first derives a recursive formula for $\psi(n; a, \cdot)$. This then leads to a partial differential equation whose solution provides a generating function for A_n. Furthermore, by expanding that generating function, he obtains an explicit formula for $\psi(n; a, \cdot)$ in terms of Stirling numbers of the second kind. His results are stated in the following theorem.

Theorem 2.1(Entringer)

(i) $\psi(n+1; a, \cdot) = \frac{n+1}{n}[2a\psi(n; a, \cdot) + (n-2a+2)\psi(n; a-1, \cdot)],$

for $n \geq 1,\ a \geq 1.$

(ii) $\sum_{n=1}^{\infty}\sum_{a=0}^{\infty} \psi(n; a, \cdot)\frac{1}{n!}x^n y^a = x\frac{1-\sqrt{1-y}Tanh(x\sqrt{1-y})}{1-\frac{Tanh(x\sqrt{1-y})}{\sqrt{1-y}}},$

for $-\frac{1}{2} \leq x \leq \frac{1}{2},\ -\frac{1}{2} \leq y \leq \frac{1}{2}.$

(iii) $\psi(n; a, \cdot)$

$$= (-1)^{n+a} n 2^{n-1} \sum_{r=a-1}^{[\frac{n}{2}]-1} \sum_{t=n-2r-1}^{n-1} \frac{(-1)^t t!}{2^t}\binom{r}{a-1}\binom{t-1}{n-2r-2} S(n-1, t),$$

for $n \geq 2,\ a \geq 0$, *where*

$$S(n-1, t) = \frac{(-1)^t}{t!}\sum_{s=1}^{t}(-1)^s\binom{t}{s}s^{n-1}$$

is a Stirling number of the second kind.

We now take B_n into account in addition to A_n and derive an explicit formula for $\psi(n; a, b)$ based on Entringer's result. We state our result in Theorem 2.2 and prove it by a simple geometric approach.

Theorem 2.2 *For* $n \geq 2$,

$$\psi(n; a, b) = \binom{n-2a}{b}\left(\frac{1}{2}\right)^{n-2a}\psi(n; a, \cdot),$$

$a = 1, \cdots, [\frac{n}{2}],\ b = 0, \cdots, n-2a.$

Proof. Since

$$\psi(n; a, b) = Pr\{B_n = b | A_n = a\}\psi(n; a, \cdot),$$

it suffices to show that the conditional distribution of B_n given $A_n = a$ is a binomial distribution with parameters $n - 2a$ and $\frac{1}{2}$.

Consider a permutation in S_n with a peaks, and hence with a troughs. Due to the circular property and the alternate appearances of peaks and troughs, these turning points can be uniquely represented by an ordered *trough-peak chain* of length $2a$, $\langle t_1, p_1, t_2, p_2, \cdots, t_a, p_a \rangle$ with $t_1 = 1$. Geometrically, we can draw consecutive line segments, according to this ordered trough-peak chain, with alternating vertical heights t_i for the troughs p_i for the peaks. Then we have an uphill slope between each pair of (t_i, p_i) and a downhill slope between (p_i, t_{i+1}), where $t_{a+1} = t_1$. Let $E = \{e_1, \cdots, e_{n-2a}\} = I_n \setminus \{t_1, p_1, \cdots, t_a, p_a\}$. Then, each $e_i \in E$ is either a double rise if located on an uphill slope, or a double fall if located on a downhill slope. Next, for each e_i, draw a horizontal line with height e_i. It is clear that each horizontal line intersects an equal number of uphill and downhill slopes, and those intersecting points are possible positions for the e_i. (See Fig. 2.1 for an example). Therefore, each e_i has the same chance of being a double rise or a double fall. Furthermore, the location of each e_i is independent of the location of other elements in E. Hence, for any given ordered trough-peak chain of length $2a$, the proportion of permutations with b double rises is $\binom{n-2a}{b}(\frac{1}{2})^{n-2a}$. Summing over all ordered trough-peak chains of length $2a$ completes the proof. □

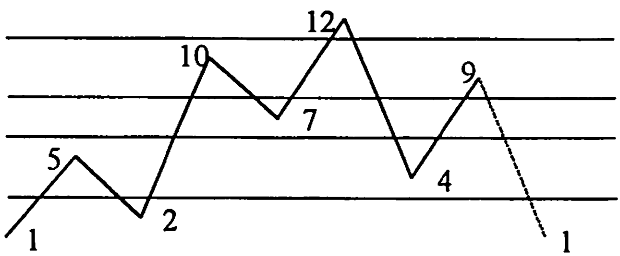

Figure 2.1. An example of Π_{12} with trough-peak chain $\langle 1, 5, 2, 10, 7, 12, 4, 9 \rangle$ and $E = \{3, 6, 8, 11\}$.

It is clear from Theorem 2.2 that $\psi(n; a, \cdot)$ is pivotal for all $\psi(n; a, b)$ and hence the latter can easily be calculated via the recursive formula in Theorem 2.1.

Next, we derive the generating function of (A_n, B_n) based on Entringer's result and Theorem 2.2.

Theorem 2.3

$$\sum_{n=1}^{\infty} \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \frac{\psi(n;a,b)}{n!} x^n y^a z^b$$

$$= \frac{(1-z)x}{2} + \frac{(1+z)x}{2} \frac{1-\sqrt{1-\frac{4y}{(1+z)^2}}Tanh(\frac{(1+z)x}{2}\sqrt{1-\frac{4y}{(1+z)^2}})}{1-\frac{Tanh(\frac{(1+z)x}{2}\sqrt{1-\frac{4y}{(1+z)^2}})}{\sqrt{1-\frac{4y}{(1+z)^2}}}},$$

where $-\frac{1}{2} \le x \le \frac{1}{2}$, $-\frac{1}{32} \le y \le \frac{1}{32}$ *and* $-\frac{1}{2} \le z \le \frac{1}{2}$.

Proof.

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \frac{\psi(n;a,b)}{n!} x^n y^a z^b \\
&= \psi(1;0,0)x + \sum_{n=2}^{\infty}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \frac{\binom{n-2a}{b}(\frac{1}{2})^{n-2a}\psi(n;a,\cdot)}{n!} x^n y^a z^b \\
&= x + \sum_{n=2}^{\infty}\sum_{a=0}^{\infty} (1+z)^{n-2a}(\tfrac{1}{2})^{n-2a}\frac{\psi(n;a,\cdot)}{n!} x^n y^a \\
&= x + \sum_{n=2}^{\infty}\sum_{a=0}^{\infty} \frac{\psi(n;a,\cdot)}{n!}\left[\frac{(1+z)x}{2}\right]^n \left[\frac{4y}{(1+z)^2}\right]^a \\
&= x + g(\tfrac{(1+z)x}{2}, \tfrac{4y}{(1+z)^2}) - \psi(1;0,\cdot)\frac{(1+z)x}{2} \\
&= \frac{(1-z)x}{2} + g(\tfrac{(1+z)x}{2}, \tfrac{4y}{(1+z)^2}),
\end{aligned}$$

where

$$g(s,t) = s\frac{1-\sqrt{1-t}Tanh(s\sqrt{1-t})}{1-\frac{Tanh(s\sqrt{1-t})}{\sqrt{1-t}}},$$

for $-\frac{1}{2} \le s \le \frac{1}{2}$, $-\frac{1}{2} \le t \le \frac{1}{2}$.

It is easy to see that the range conditions for x, y and z stated in the theorem satisfy those for s and t. □

3. Recursive formulae for joint distribution of (A_n^*, B_n^*, C_n^*)

In this section we study the joint distribution of (A_n^*, B_n^*, C_n^*), $n \ge 3$. Owing to its complexity, we can only derive recursive formulae for it. Since the two ends π_1 and π_n of a permutation Π_n are not classified and not counted, this causes difficulties in the derivations. Those difficulties can be resolved by partitioning the set S_n into the following four subsets according to pairs (π_1, π_2) and (π_{n-1}, π_n). Let

$$M_n = \{\Pi_n : \pi_1 < \pi_2 \text{ and } \pi_{n-1} > \pi_n\},$$

$$W_n = \{\Pi_n : \pi_1 > \pi_2 \text{ and } \pi_{n-1} < \pi_n\},$$
$$N_n = \{\Pi_n : \pi_1 < \pi_2 \text{ and } \pi_{n-1} < \pi_n\},$$
$$U_n = \{\Pi_n : \pi_1 > \pi_2 \text{ and } \pi_{n-1} > \pi_n\}.$$

Here, the technique we use is similar to that of adding an auxillary variable in deriving the distributions of various runs used by Fu and Koutras [7].

Note that $C_n^* = A_n^* - 1$ for permutations in M_n, $C_n^* = A_n^* + 1$ for those in W_n and $C_n^* = A_n^*$ for those in $N_n \bigcup U_n$. Now let $\psi(M_n; a, b)$ denote the number of permutations in M_n with $A_n^* = a$ and $B_n^* = b$, and denote $\psi(W_n; a, b)$, $\psi(N_n; a, b)$ and $\psi(U_n; a, b)$ similarly. Then, for $n \geq 3$, we have

$$n!Pr\{A_n^* = a, B_n^* = b, C_n^* = c\} = \begin{cases} \psi(M_n; a, b), & \text{if } c = a - 1, \\ \psi(W_n; a, b), & \text{if } c = a + 1, \\ \psi(N_n; a, b) + \psi(U_n; a, b), & \text{if } c = a, \\ 0, & \text{otherwise.} \end{cases}$$

The counting problem of the above four numbers can be simplified by the symmetry property between M_n and W_n and that between N_n and U_n resulting from the following two bijections. Define the bijection on S_n

$$\phi_1 : \Pi_n = (\pi_1 \cdots \pi_n) \longrightarrow \Pi_n' = (\pi_1' \cdots \pi_n'), \quad \pi_i' = (n+1) - \pi_{n+1-i} \;;$$

see Figure 3.1 for an example. It is easy to show that

$$\psi(W_n; a, b) = \psi(M_n; a+1, b). \tag{3.1}$$

Also, define the bijection on S_n

$$\phi_2 : \Pi_n = (\pi_1 \cdots \pi_n) \longrightarrow \Pi_n' = (\pi_1' \cdots \pi_n'), \quad \pi_i' = (n+1) - \pi_i \;;$$

see Figure 3.2 for an example. Then

$$\psi(U_n; a, b) = \psi(N_n; a, n-2a-b-2). \tag{3.2}$$

Therefore, we need only derive recursive formulae for $\psi(M_n; a, b)$ and $\psi(N_n; a, b)$.

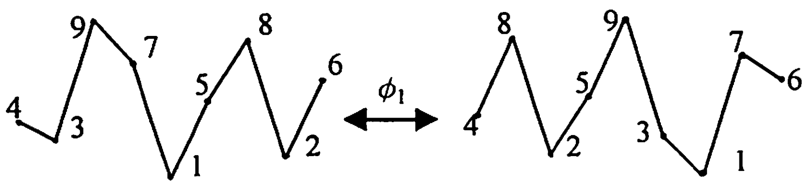

Figure 3.1. An example of ϕ_1.

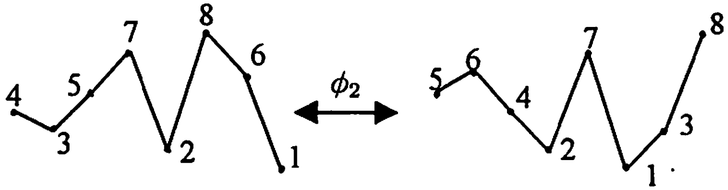

Figure 3.2. An example ofϕ_2.

Theorem 3.1 *For* $n \geq 3$,

$$\psi(M_{n+1};a,b) = \begin{cases} (b+1)\psi(M_n;a-1,b+1) + a\psi(M_n;a,b) \\ +a\psi(M_n;a,b-1) + (n-2a-b+1)\psi(M_n;a-1,b) \\ +\psi(N_n;a-1,b) + \psi(N_n;a-1,n-2a-b), \\ \qquad for\ a=1,\cdots,[\frac{n}{2}], b=0,\cdots,n-2a, \\ 0, \qquad otherwise; \end{cases} \tag{3.3}$$

$$\psi(N_{n+1};a,b) = \begin{cases} (b+1)\psi(N_n;a-1,b+1) + a\psi(N_n;a,b) \\ +(a+1)\psi(N_n;a,b-1) + (n-2a-b)\psi(N_n;a-1,b) \\ +2\psi(M_n;a,b), \\ \qquad for\ a=1,\cdots,[\frac{n-1}{2}], b=0,\cdots,n-2a-1; \\ \qquad or\ (a,b)=(0,n-1), \\ 0, \qquad otherwise, \end{cases} \tag{3.4}$$

with initial conditions

$$\psi(M_3;a,b) = \begin{cases} 2, & for\ (a,b)=(1,0), \\ 0, & otherwise; \end{cases}$$

$$\psi(N_3;a,b) = \begin{cases} 1, & for\ (a,b)=(0,1), \\ 0, & otherwise. \end{cases}$$

Proof. A permutation $\Pi_{n+1} \in M_{n+1}$ can be generated by inserting $n+1$ into any position (I) between π_1 and π_n of a permutation $\Pi_n \in M_n$, (II) between π_{n-1} and π_n of a permutation $\Pi_n \in N_n$, or (III) between π_1 and π_2 of a permutation $\Pi_n \in U_n$.

We first consider how the numbers of peaks and double rises change in case (I). Those changes depend solely on the classification of the two consecutive elements of $\Pi_n \in M_n$ where $n+1$ is to be inserted in-between.

Note that eight among all sixteen pairs of such classifications are impossible. For example, a peak cannot be followed by a peak or a double rise. The remaining eight possible pairs are (1)double rise – double rise, (2)double rise – peak, (3)peak – trough, (4)peak – double fall, (5)double fall – double fall, (6)double fall – trough, (7)trough – double rise and (8)trough – peak. If $n+1$ is inserted into the pair (1)double rise – double rise, then a new peak, $n+1$, is generated while the first double rise stays the same and the second becomes a trough. Hence, $\Delta A^*_{n+1} = A^*_{n+1} - A^*_n = 1$ and $\Delta B^*_{n+1} = B^*_{n+1} - B^*_n = -1$. Inserting $n+1$ into (7)trough – double rise yields the same changes. Thus, inserting $n+1$ (i) in front of a double rise leads to the first term $(b+1)\psi(M_n; a-1, b+1)$ of (3.3). Similar arguments lead to the following results : Inserting $n+1$ (ii) in front of a peak ((2) and (8)) yields $\Delta A^*_{n+1} = \Delta B^*_{n+1} = 0$ and leads to the second term of (3.3), (iii) behind a peak ((3) and (4)) yields $\Delta A^*_{n+1} = 0$ and $\Delta B^*_{n+1} = 1$ and leads to the third term, and (iv) behind a double fall ((5) and (6)) yields $\Delta A^*_{n+1} = 1$ and $\Delta B^*_{n+1} = 0$ and leads to the fourth term.

The fifth term of (3.3) is derived from case (II) and the last term from case (III) by applying (3.2). Thus, the proof of (3.3) is complete.

In order to prove (3.4), we begin with the fact that a permutation $\Pi_{n+1} \in N_{n+1}$ can be generated by inserting $n+1$ (I′) into any position between π_1 and π_{n-1} or behind π_n of a permutation $\Pi_n \in N_n$, (II′) behind π_n of a permutation $\Pi_n \in M_n$, or (III′) between π_1 and π_2 of a permutation $\Pi_n \in W_n$. We omit similar derivations and only mention that (3.1) is applied for combining the two terms corresponding to cases (II′) and (III′) into the last term of (3.4). □

4. Asymptotic joint normality of (A_n, B_n) and (A^*_n, B^*_n) and a randomness test

We begin by stating the means, variances and covariances of (A_n, B_n) and (A^*_n, B^*_n). By using the following simple combinatorial results :

$$Pr\{\pi_1 < \pi_2 > \pi_3 < \pi_4 > \pi_5\} = \tfrac{16}{5!},$$
$$Pr\{\pi_1 < \pi_2 < \pi_3 < \pi_4 > \pi_5\} = \tfrac{4}{5!},$$
$$Pr\{\pi_1 < \pi_2 > \pi_3 < \pi_4 < \pi_5\} = \tfrac{9}{5!},$$
$$Pr\{\pi_1 < \pi_2 < \pi_3 > \pi_4\} = \tfrac{3}{4!},$$

we can obtain by straightforward calculations,

$$E(A_n) = \tfrac{n}{3}, \quad E(B_n) = \tfrac{n}{6},$$
$$E(A_n^2) = \tfrac{n^2}{9} + \tfrac{2n}{45}, \quad Var(A_n) = \tfrac{2n}{45},$$

$$E(B_n^2) = \frac{n^2}{36} + \frac{23n}{180}, \quad Var(B_n) = \frac{23n}{180},$$

$$E(A_nB_n) = \frac{n^2}{18} - \frac{2n}{45} \text{ and } Cov(A_n, B_n) = -\frac{2n}{45}.$$

For (A_n^*, B_n^*), we have

$$E(A_n^*) = \frac{n-2}{3}, \quad E(B_n^*) = \frac{n-2}{6},$$

$$Var(A_n^*) = \frac{2n+2}{45}, \quad Var(B_n^*) = \frac{23n-37}{180},$$

$$Cov(A_n^*, B_n^*) = -\frac{2n}{45} + \frac{29}{360}.$$

A simple way to prove the asymptotic joint normality for (A_n^*, B_n^*) is to use the well-known central limit theorem for a sum of m-dependent stationary random variables. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of *i.i.d* random variables with continuous distribution. Let π_i denote the rank of X_i among $X_1, \cdots, X_n$. Then, $\Pi_n = (\pi_1, \cdots, \pi_n)$ is a uniformly distributed random permutation on S_n. It follows that

$$(A_n^*, B_n^*) \stackrel{d}{=} \Big(\sum_{i=2}^{n-1} 1\{X_{i-1} < X_i > X_{i+1}\}, \sum_{i=2}^{n-1} 1\{X_{i-1} < X_i < X_{i+1}\}\Big),$$

where $\stackrel{d}{=}$ indicates that two random vectors are identical in distribution. The indicator random vectors $(1\{X_{i-1} < X_i > X_{i+1}\}, 1\{X_{i-1} < X_i < X_{i+1}\})$ are 2-dependent and stationary. Therefore, by Theorem 3 of Hoeffding and Robbins [9], we obtain

Theorem 4.1 *As $n \to \infty$, $((A_n^* - E(A_n^*))/\sqrt{n}, (B_n^* - E(B_n^*))/\sqrt{n})$ converges in distribution to a bivariate normal vector with zero-mean and covariance matrix*

$$\Sigma = \begin{pmatrix} \frac{2}{45} & \frac{-2}{45} \\ \frac{-2}{45} & \frac{23}{180} \end{pmatrix}.$$

The above theorem is also valid if we replace A_n^* by A_n and B_n^* by B_n. It then implies that

$$T_n = Y_n \Sigma^{-1} Y_n', \quad \text{where } Y_n = \Big(\frac{A_n - \frac{n}{3}}{\sqrt{n}}, \frac{B_n - \frac{n}{6}}{\sqrt{n}}\Big),$$

has the Chi-square distribution with 2 degrees of freedom as $n \to \infty$. As mentioned in Section 1, Bienaymé [1,2] proposed a nonparametric test for the randomness of observations $X_1, \cdots, X_n$ based on the number of turning points. We now propose the statistic T_n for testing randomness, since intuitively T_n contains more information than Bienaymé's $2A_n$.

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