Total Vertex Enumeration in Rooted Planar Maps

Leonid M. Koganov
Center of Nonlinear Mechanics and Technology
Russian Academy of Sciences
117334, Moscow, RUSSIA

Valery A. Liskovets*
Institute of Mathematics, National Academy of Sciences
220072, Minsk, BELARUS
liskov@im.bas-net.by

Timothy R. S. Walsh†
Département D'Informatique
Université du Québec à Montréal
Montréal (Québec), CANADA, H3C 3P8
walsh.timothy@uqam.ca

Abstract

Two combinatorial identities are proved.
(1) $H_n(\mathcal{E}) = \frac{n+2}{3} M_n(\mathcal{E})$, where $H_n(\mathcal{E})$ denotes the total number of vertices in all the n-edged rooted planar Eulerian maps and $M_n(\mathcal{E})$ denotes the number of such maps.
(2) $H_n(\mathcal{L}) = \frac{5n^2+13n+2}{2(4n+1)} M_n(\mathcal{L})$, where $H_n(\mathcal{L})$ and $M_n(\mathcal{L})$ are defined similarly for the class $\mathcal{L}$ of loopless maps. Simple closed formulae for $M_n(\mathcal{E})$ and $M_n(\mathcal{L})$ are well known, and they correspond to equivalent binomial sum identities.

Mathematics Subject Classification (1991): 05C30, 05A19

---

*Supported by INTAS under grant INTAS-BELARUS 97-0093.
†Supported by National Science and Engineering Research Council (NSERC) under grant RGPIN-44703.

1 Map Enumeration and Identities

1.1. Motivation. Let $\mathcal{X}$ stand for a class of rooted planar maps and $\mathcal{X}_n$ denote the set of maps in $\mathcal{X}$ having n edges. $M_n = M_n(\mathcal{X}) := |\mathcal{X}_n|$ denotes the number of maps in $\mathcal{X}_n$. Finding M_n is a classical enumerative problem initiated by W. T. Tutte, and it has been solved effectively for many interesting types of maps.

Recently, one of the authors had encountered the problem of enumerating the total number $H_n = H_n(\mathcal{X})$ of vertices in $\mathcal{X}_n$ ([11]; cf. also [10]). This is often a trivial or easy question. Such is the case for cubic or, more generally, s-valent maps and (by Euler's formula) for triangulations or, more generally, maps with all faces of valency $s \geq 3$. In all these cases,

$$H_n = \alpha_n M_n \tag{1.1}$$

for a simple multiplicative constant $\alpha_n = \alpha_n(\mathcal{X})$.

For instance, any n-edged triangulation contains $2 + n/3$ vertices, so that for triangulations, $H_n = \frac{n+6}{3} M_n$.

Moreover, if $\mathcal{X}$ is a *self-dual* class of maps (i.e. $\mathcal{X}^* = \mathcal{X}$ where $\mathcal{X}^*$ consists of the maps topologically dual to the maps in $\mathcal{X}$), then, as can easily be seen, $H_n = \frac{n+2}{2} M_n$. All (planar) maps, non-separable maps, polyhedral maps and maps without either loops or isthmuses provide examples of self-dual classes.

For other types of maps, we do not know any direct inter-connection between the two quantities that are valid a fortiori, although sometimes the counting of H_n is, in a sense, only a technical problem. Namely, $H_n = \sum_k h_{k,n}$ where $h_{k,n} = h_{k,n}(\mathcal{X})$ stands for the total number of vertices of valency k in $\mathcal{X}_n$. As shown in [11], $h_{k,n} = 2n \cdot r_{k,n}/k$, where $r_{k,n} = r_{k,n}(\mathcal{X})$ denotes the number of maps with the root vertex of valency k. Thus,

$$H_n = 2n \sum_k \frac{r_{k,n}}{k} \quad \text{and} \quad M_n = \sum_k r_{k,n}, \tag{1.2}$$

and one needs only to determine $r_{k,n}$. If, instead, the numbers $q_{k,n} = q_{k,n}(\mathcal{X})$ of maps having k vertices and n edges are known, then we have

$$H_n = \sum_k k \cdot q_{k,n} \quad \text{and} \quad M_n = \sum_k q_{k,n}. \tag{1.3}$$

1.2. Enumerative identities. Calculations have suggested, however, that in at least two non-trivial cases, we can attain much more [10]: there should exist closed sum-free expressions for H_n of type (1.1). Namely, for planar Eulerian maps ($\mathcal{E}$),

$$H_n(\mathcal{E}) = \frac{n+2}{3} M_n(\mathcal{E}), \qquad n \geq 1; \tag{1.4}$$

likewise, for planar loopless maps ($\mathcal{L}$),

$$H_n(\mathcal{L}) = \frac{5n^2+13n+2}{2(4n+1)} M_n(\mathcal{L}), \quad n \geq 1 \tag{1.5}$$

This is the sequence 1,2,8,43,268,1824,13156,... The last formula can be expressed even more elegantly in dual form (or, equivalently, for the total number of faces): for isthmusless maps ($\mathcal{L}^*$),

$$H_n(\mathcal{L}^*) = \frac{(3n+2)(n+1)}{2(4n+1)} M_n(\mathcal{L}^*), \quad n \geq 1, \tag{1.5*}$$

(of course, $M_n(\mathcal{X}) = M_n(\mathcal{X}^*)$). This is equivalent to (1.5) by the relation $\alpha_n(\mathcal{X}) + \alpha_n(\mathcal{X}^*) = n+2$, which follows directly from Euler's formula. Note, for completeness, that $\mathcal{E}^*$ represents bipartite planar maps and

$$H_n(\mathcal{E}^*) = \frac{2(n+2)}{3} M_n(\mathcal{E}^*), \quad n \geq 1, \tag{1.4*}$$

if identity (1.4) holds.

It is also clear that $\frac{2nM_n}{H_n} = \mu_n(\mathcal{X})$ is the *mean* vertex valency in $\mathcal{X}_n$. Thus, in (1.1), $\alpha_n = 2n/\mu_n$, and the expressions above can be reformulated in terms of μ_n.

In turn, there are closed sum-free formulae for $M_n(\mathcal{E})$ and $M_n(\mathcal{L})$. Namely, by [15]:

$$M_n(\mathcal{E}) = \frac{3 \cdot 2^{n-1}(2n)!}{n!(n+2)!} \tag{1.6}$$

and by [16]:

$$M_n(\mathcal{L}) = \frac{6(4n+1)!}{n!(3n+3)!}. \tag{1.7}$$

Therefore, from (1.4) we obtain

$$H_n(\mathcal{E}) = \frac{2^{n-1}(2n)!}{n!(n+1)!} = 2^{n-1}C_n, \tag{1.8}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. Numerically, $H_n(\mathcal{E})$ is the sequence $1, 4, 20, 112, 672, 4224, \ldots$ Note also that $H_n(\mathcal{E}^*) = 2^n C_n$. Moreover, for the class of all maps $\mathcal{A}$, $M_n(\mathcal{A}) = \frac{2 \cdot 3^n (2n)!}{n!(n+2)!}$, whence, $H_n(\mathcal{A}) = 3^n C_n$. This is the sequence $1, 3, 18, 135, 1134, 10206, \ldots$

Similarly, from identities (1.5*) and (1.7),

$$H_n(\mathcal{L}^*) = \frac{(4n)!}{n!(3n+1)!} = \frac{1}{3n+1}\binom{4n}{n}. \tag{1.9}$$

Curiously enough, the latter expression has also quite different interpretations. For instance, this is the number of pentagonal n-cell dissections of a (rooted) convex $(3n+2)$-gon (and the corresponding generating function

$f_5(x)$ is defined by the equation $f_5(x) = x(1 + f_5(x))^4)$, cf. [6].

We note that the three integer sequences given above are lacking in [14]; they are presented in [13] with ID numbers A027836, A003645 and A005159, resp.

1.3. Binomial coefficient identities. According to [15], the number of maps in $\mathcal{E}_n$ with k vertices is

$$q_{k,n}(\mathcal{E}) = \frac{2(n-1)!}{(k-1)!(n-k+2)!} \sum_{i=0}^{n-k} \binom{2n}{n-k-i} \binom{k+i}{i}. \tag{1.10}$$

Summing (1.10) over k we obtain $M_n(\mathcal{E})$. So that formula (1.6) gives rise to the next identity:

$$\sum_{k=1}^{n} \frac{2(n-1)!}{(k-1)!(n-k+2)!} \sum_{i=0}^{n-k} \binom{2n}{n-k-i} \binom{k+i}{i} = \frac{3 \cdot 2^{n-1}(2n)!}{n!(n+2)!}. \tag{1.11}$$

Likewise, by (1.3), formula (1.8) (and, thus, (1.4)) is equivalent to the identity

$$\sum_{k=0}^{n} \frac{2k(n-1)!}{(k-1)!(n-k+2)!} \sum_{i=0}^{n-k} \binom{2n}{n-k-i} \binom{k+i}{i} = \frac{2^{n-1}(2n)!}{n!(n+1)!}. \tag{1.12}$$

Now, there is a simple exact formula for the number of loopless maps with n edges and root vertex valency k obtained by E. A. Bender and N. C. Wormald [1]: $r_{k,n}(\mathcal{L}) = L_{k,n}$, where

$$L_{k,n} = \frac{k}{2(2n-k)(3n-k+1)} \binom{4n-2k}{n-k} \binom{2k+2}{k+1}. \tag{1.13}$$

Formulae (1.7) and (1.13) imply

$$\sum_{k=0}^{n} \frac{k}{(2n-k)(3n-k+1)} \binom{4n-2k}{n-k} \binom{2k+2}{k+1} = \frac{12(4n+1)!}{n!(3n+3)!}. \tag{1.14}$$

Moreover, according to [1], the number of rooted 3-connected planar triangulations with $3n+3$ edges and root vertex valency $k+2$ is equal to the number $L_{k,n}$. Now, the number of rooted 3-connected planar triangulations with $3n+3$ edges (all with $n+3$ vertices) is equal to $M_n(\mathcal{L})$. Thus, by (1.2),

$$\sum_{k=0}^{n} \frac{k}{k+2} \frac{n+1}{(2n-k)(3n-k+1)} \binom{4n-2k}{n-k} \binom{2k+2}{k+1} = \frac{2(n+3)(4n+1)!}{n!(3n+3)!} \tag{1.15}$$

(in the last three identities we extended formally the range of summation to $k = 0$). We, however, need to prove the identity

$$\sum_{k=1}^{n} \frac{n}{(2n-k)(3n-k+1)} \binom{4n-2k}{n-k} \binom{2k+2}{k+1} = \frac{3(5n^2+13n+2)(4n)!}{n!(3n+3)!}, \tag{1.16}$$

which is equivalent to formula (1.5) by (1.2).

Presumably these binomial coefficient identities are new (at least, we failed to find them in the available literature, including [4], [5] and [12]).

The main aim of the present note is to prove the identities given above:

1.4. Theorem. *Combinatorial identities* (1.4) *and* (1.5) *are valid. Equivalently: expressions* (1.8) *and* (1.9) *and binomial coefficient identities* (1.12) *and* (1.16) *are valid.*

For both classes of maps we provide here two different proofs, one analytical (by Lagrange inversion) and the other combinatorial (binomial). Both proofs can turn out to be useful for possible generalizations, if any, to other classes of maps or to weight enumerators different from the number of vertices. Of course, it would be very interesting to deduce a direct ("explaining") bijective proof of identities (1.4) and (1.5) (or their duals (1.4*) and (1.5*)).

2 Analytical Proofs

2.1. Eulerian maps. According to [15],

$$q_{k,n}(\mathcal{E}) = \frac{2 \cdot n!}{k!(n-k+2)!} c_{k,n} \tag{2.1}$$

where $c_{k,n}$ is the coefficient of x^n in

$$\left(\frac{(1-4x)^{-1/2}-1}{2} \right)^k. \tag{2.2}$$

From expression (1.3),

$$H_n(\mathcal{E}) = \sum_{k=0}^{n+1} \frac{2k \cdot n!}{k!(n-k+2)!} c_{k,n}, \tag{2.3}$$

which simplifies to the coefficient of x^n in

$$\frac{2}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} \left(\frac{(1-4x)^{-1/2}-1}{2} \right)^{k+1}$$

(the upper limit can be extended from n to $n+1$ because no Eulerian map has more vertices than edges)

$$= \frac{2}{n+1}\left(\frac{(1-4x)^{-1/2}-1}{2}\right)\left(\frac{(1-4x)^{-1/2}-1}{2}+1\right)^{n+1}$$

$$= \frac{2}{n+1}\left[\left(\frac{(1-4x)^{-1/2}-1}{2}+1\right)^{n+2} - \left(\frac{(1-4x)^{-1/2}-1}{2}+1\right)^{n+1}\right]. \tag{2.4}$$

As in [15], put $y := [1-(1-4x)^{1/2}]/(2x)$, so that

$$y = 1 + xy^2 \tag{2.5}$$

and

$$\frac{(1-4x)^{-1/2}-1}{2} + 1 = \frac{1}{2-y}. \tag{2.6}$$

Lagrange's inversion formula (see, e.g., [8]) states that if

$$y = a + x\varphi(y) \tag{2.7a}$$

then

$$f(y) = f(a) + \sum_{n=1}^{\infty} \frac{x^n}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left[f'(x)\{\varphi(x)\}^n\right]_{x=a}, \tag{2.7b}$$

where $\varphi(y)$ and $f(y)$ are arbitrary formal generating functions. In order to apply this formula to series (2.4), taking into account expressions (2.5) and (2.6) we set $a := 1$, $\varphi(y) := y^2$ and

$$f(y) := \frac{2}{n+1}\left[(2-y)^{-(n+2)} - (2-y)^{-(n+1)}\right]. \tag{2.8}$$

Now, applying formula (2.7b) to (2.8), the coefficient of x^n in $f(y)$ is

$$\frac{2}{(n+1)n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left[(n+2)(2-x)^{-(n+3)}x^{2n} - (n+1)(2-x)^{-(n+2)}x^{2n}\right]_{x=1}. \tag{2.9}$$

Setting $z := x - 1$, we find that (2.9) is the coefficient of z^{n-1} in

$$\frac{2}{(n+1)n}\left[(n+2)(1-z)^{-(n+3)}(1+z)^{2n} - (n+1)(1-z)^{-(n+2)}(1+z)^{2n}\right]. \tag{2.10}$$

Now the coefficient of z^{n-1} in $(1-z)^{-(n+2)}(1+z)^{2n}$ is

$$\sum_{k=0}^{n-1}\binom{n+k+1}{k}\binom{2n}{n-k-1} = \frac{(2n)!}{(n+1)!(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{k} = 2^{n-1}\binom{2n}{n-1}.$$

And the coefficient of z^{n-1} in $(1-z)^{-(n+3)}(1+z)^{2n}$ is

$$\sum_{k=0}^{n-1}\binom{n+k+2}{k}\binom{2n}{n-k-1}$$

$$=\sum_{k=0}^{n-1}\binom{n+k+1}{k}\binom{2n}{n-k-1}+\sum_{k=1}^{n-1}\binom{n+k+1}{k-1}\binom{2n}{n-k-1}$$

$$=2^{n-1}\binom{2n}{n-1}+\sum_{k=0}^{n-2}\binom{n+k+2}{k}\binom{2n}{n-k-2}$$

$$=2^{n-1}\binom{2n}{n-1}+\frac{(2n)!}{(n+2)!(n-2)!}\sum_{k=0}^{n-2}\binom{n-2}{k}$$

$$=2^{n-1}\binom{2n}{n-1}+2^{n-2}\binom{2n}{n-2}.$$

Substituting these values into (2.10) and simplifying we obtain (1.8). □

A similar calculation can be used to re-derive formula (1.6).

2.2. Loopless maps. Let

$$a(x,y):=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}L_{k,n}x^n y^k, \tag{2.11}$$

where $L_{k,n}$ is defined by formula (1.13). Then by [1],

$$a(x,y)=\frac{(1+v)^2}{2vy^2}\left(y+3vy-(1+v)^2+(1+v)(1+v-y)\sqrt{1-\frac{4vy}{(1+v)^2}}\right) \tag{2.12}$$

where

$$v=x(1+v)^4. \tag{2.13}$$

From (1.2) we obtain

$$H_n(\mathcal{L})=2n\sum_{k=1}^{n}\frac{L_{k,n}}{k} \tag{2.14}$$

$$=2n\sum_{k=0}^{n}\frac{L_{k,n}}{k}-2n\frac{L_{k,n}}{k}\Big|_{k=0}. \tag{2.15}$$

The first term in formula (2.15) is equal to the coefficient of x^n in

$$2n \int_0^1 \frac{a(x,y)}{y} \mathrm{d}y. \tag{2.16}$$

After the change of variables $w := \sqrt{1 - \frac{4vy}{(1+v)^2}}$ and some tedious calculations, expression (2.16) evaluates to

$$n(2\ln|v| + v^3 - 2v^2 - 5v + 2). \tag{2.17}$$

If functions (2.17) and (2.13) are used for $f(v)$ and $\varphi(v)$ respectively, then from Lagrange's formulae (2.7) with the variable v instead of y, the coefficient of x^n in $f(v)$ is the coefficient of v^{n-1} in

$$\frac{1}{n} f'(v)(1+v)^{4n}. \tag{2.18}$$

(Note that $f'(v)$ contains the term v^{-1}, which has a singularity at $v = 0$; however, Lagrangian inversion is independent of any analytical considerations due to its purely combinatorial proof [8]).

Differentiating function (2.17) and then simplifying, we obtain the coefficient of x^n in power series (2.18):

$$(n+2)\frac{6(4n+1)!}{n!(3n+3)!}, \tag{2.19}$$

which, by (1.7), is just $(n+2)$ times the number of rooted loopless maps with n edges.

The second term in (2.15) is equal to $\frac{1}{3n+1}\binom{4n}{n}$ according to (1.13). Together with (2.19) this gives easily

$$H_n(\mathcal{L}) = 3(5n^2 + 13n + 2)\frac{(4n)!}{n!(3n+3)!}, \tag{2.20}$$

which, due to identity (1.6), is equivalent to (1.5) . □

3 Binomial Coefficient Proofs

3.1. Eulerian maps. From (1.3) and (1.10),

$$H_n(\mathcal{E}) = \sum_{k=1}^{n} \frac{2k(n-1)!}{(k-1)!(n-k+2)!} \sum_{i=0}^{n-k} \binom{2n}{n-k-i}\binom{k+i}{i}. \tag{3.1}$$

Set $j := i + k$. Then the right-hand side of (3.1) becomes

$$\sum_{j=1}^{n} \binom{2n}{n-j} \sum_{k=1}^{j} \frac{2k\,(n-1)!}{(k-1)!(n-k+2)!}\binom{j}{k}$$

$$= \frac{2}{n(n+1)} \sum_{j=1}^{n} j \binom{2n}{n-j} \sum_{k=1}^{j} \binom{j-1}{k-1} \binom{n+1}{k-1}. \tag{3.2}$$

Now, $\binom{j-1}{k-1}$ is the coefficient of x^{k-1} in $(1+x)^{j-1}$, and $\binom{n+1}{k-1}$ is the coefficient of x^{n-k+2} in $(1+x)^{n+1}$. So that in (3.2), the internal sum over k is the coefficient of x^{n+1} in $(1+x)^{n+j}$, which is $\binom{n+j}{n+1}$. Therefore expression (3.2) is equal to

$$\frac{2}{n(n+1)} \sum_{j=1}^{n} j \binom{2n}{n-j} \binom{n+j}{n+1} = \frac{2(2n)!}{n(n+1)(n+1)!} \sum_{j=1}^{n} \frac{[(j-1)+1]}{(n-j)!(j-1)!}. \tag{3.3}$$

The last sum is equal to

$$\frac{1}{(n-2)!} \sum_{j=2}^{n} \binom{n-2}{j-2} + \frac{1}{(n-1)!} \sum_{j=1}^{n} \binom{n-1}{j-1} = \frac{1}{(n-1)!} \left((n-1)2^{n-2} + 2^{n-1} \right).$$

So, (3.3) simplifies to (1.8). □

A similar calculation leads to yet another derivation of formula (1.6), and one can also deduce (1.12) from (1.11).

3.2. Loopless maps. From (1.13), the first term in (2.15) is

$$\sum_{k=0}^{n} \frac{n}{(2n-k)(3n-k+1)} \binom{4n-2k}{n-k} \binom{2k+2}{k+1}. \tag{3.4}$$

By the binomial theorem, $\binom{2k+2}{k+1}$ is the coefficient of x^{k+1} in $(1-4x)^{-1/2}$, which is the coefficient of x^k in

$$\frac{(1-4x)^{-1/2} - 1}{x} = 2(1-4x)^{-1/2} \left[\frac{1-(1-4x)^{1/2}}{2x} \right]. \tag{3.5}$$

To evaluate sum (3.4) we look for a generating function in which the coefficient of x^{n-k} is

$$\frac{1}{(2n-k)(3n-k+1)} \binom{4n-2k}{n-k}. \tag{3.6}$$

According to [7] (formulae (16.8) and (16.11)), the following two expansions are valid:

$$\sum_{n=0}^{\infty} \binom{2n+k}{n} x^n = (1-4x)^{-1/2} \left[\frac{1-(1-4x)^{1/2}}{2x} \right]^k \tag{3.7}$$

and

$$\sum_{n=0}^{\infty} \frac{k}{2n+k} \binom{2n+k}{n} x^n = \left[\frac{1-(1-4x)^{1/2}}{2x} \right]^k. \tag{3.8}$$

These are also formulae (5.72) and (5.70) in [5], and in both books these equalities were deduced in elementary analytical ways. Other analytical proofs are also known in the literature. At the end we provide another, purely combinatorial proof of (3.8).

Now, expression (3.6) is $2n$ times the sum of the following 3 terms:

$$\frac{1}{2n+2(n-k)}\binom{2n+2(n-k)}{n-k}, \tag{3.9a}$$

$$\frac{1}{2n+2+2(n-k-1)}\binom{2n+2(n-k-1)}{n-k-1} \tag{3.9b}$$

and

$$-\frac{1}{2n+1+2(n-k)}\binom{2n+1+2(n-k)}{n-k}. \tag{3.9c}$$

Dividing (3.8) by k, setting n to $n-k$, $n-k-1$ and $n-k$, and setting k to $2n$, $2n+2$ and $2n+1$, we find that (3.9a), (3.9b) and (3.9c) is the coefficient of x^{n-k}, x^{n-k-1} and x^{n-k}, respectively, in

$$\frac{1}{2n}\left(\frac{1-(1-4x)^{1/2}}{2x}\right)^{2n}, \tag{3.10a}$$

$$\frac{1}{2n+2}\left(\frac{1-(1-4x)^{1/2}}{2x}\right)^{2n+2} \tag{3.10b}$$

and

$$-\frac{1}{2n+1}\left(\frac{1-(1-4x)^{1/2}}{2x}\right)^{2n+1}. \tag{3.10c}$$

Multiplying each of these generating functions by function (3.5) we find that expression (3.4) is $4n$ times the sum of the coefficients of x^n, x^{n-1} and x^n, respectively, in

$$\frac{(1-4x)^{-1/2}}{2n}\left(\frac{1-(1-4x)^{1/2}}{2x}\right)^{2n+1}, \tag{3.11b}$$

$$\frac{(1-4x)^{-1/2}}{2n+2}\left(\frac{1-(1-4x)^{1/2}}{2x}\right)^{2n+3} \tag{3.11b}$$

and

$$-\frac{(1-4x)^{-1/2}}{2n+1}\left(\frac{1-(1-4x)^{1/2}}{2x}\right)^{2n+2}. \tag{3.11b}$$

Substituting $k := 2n+1$, $2n+3$ and $2n+2$ into power series (3.7), summing the appropriate coefficients and multiplying by $4n$ we obtain, after some simplification, expression (2.20). The proof of (1.5) is then completed as in 2.2. □

Proofs of expansions (3.8) and (3.7). It is well known that the coefficient of x^n in $[1-(1-4x)^{1/2}]/(2x)$ is the nth Catalan number, which is, in turn, the number of balanced parenthesis systems, or Dyck words, with n left and n right parentheses (see, e.g., [2]). Thus, the right hand series of (3.8) counts ordered k-tuples of Dyck words which have a total of n left and n right parentheses. Now, following [9], we establish a bijection between these k-tuples and the set of all Dyck words beginning with at least $k-1$ left parentheses and containing $n-k+1$ parentheses of each sort. Given such a k-tuple, concatenate the k Dyck words in left-to-right order. Insert a right parenthesis between every adjacent pair of Dyck words, and then add $k-1$ left parentheses at the very beginning. This yields the required Dyck word. This construction is reversible. Indeed, given a Dyck word beginning with at least $k-1$ left parentheses and containing $n-k+1$ parentheses of each sort, consider the mate of each of the first $k-1$ left parentheses as a 'separator'. It is clear that a certain Dyck word lies between the $(k-1)$st left parenthesis and its mate (the first separator), a Dyck word lies between each pair of adjacent separators and a Dyck word lies to the right of the last separator, forming jointly the desired k-tuple.

According to [3, p. 70] the number of Dyck words beginning with at least $k-1$ left parentheses and containing the total of $n-k+1$ parentheses of each sort is just the coefficient of x^n in the left side of (3.8).

To derive (3.7), we note that

$$\sum_{n=0}^{\infty}\binom{2n+k}{n}x^n = 2\sum_{n=0}^{\infty}\frac{n}{2n+k}\binom{2n+k}{n}x^n + \sum_{n=0}^{\infty}\frac{k}{2n+k}\binom{2n+k}{n}x^n$$

$$= \frac{2x}{k}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty}\frac{k}{2n+k}\binom{2n+k}{n}x^n + \sum_{n=0}^{\infty}\frac{n}{2n+k}\binom{2n+k}{n}x^n. \qquad (3.12)$$

Applying (3.8) to both terms in the right-hand side of (3.12) we obtain (3.7). □

References

[1] E. A. Bender and N. C. Wormald, The number of loopless planar maps, *Discr. Math.* **54** (1985), No 2, 235–237.

[2] M. Delest and X. G. Viennot, Algebraic languages and polyominoes enumeration, *Theor. Comput. Sci.* **34** (1984), 169–206.

[3] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol.1, 2nd ed., J. Wiley & Sons, NY, 1957.

[4] H. W. Gould, *Combinatorial Identities*, Morgantown, W.Va., 1972.

[5] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics: a Foundation for Computer Science*, 2nd ed., Addison-Wesley, Amsterdam, 1994.

[6] F. Harary, E. M. Palmer and R. C. Read, On the cell-growth problem for arbitrary polygons, *Discr. Math.* **11** (1975), No 3, 371–389.

[7] L. M. Koganov, *Pseudogenerated Two-Index Sequences,* Nedra, Moscow, 1989 (in Russian).

[8] G. Labelle, Une nouvelle démonstration combinatoire des formules d'inversion de Lagrange, *Advances in Math.* **42** (1981), 217–247.

[9] A. B. Lehman, unpublished manuscript.

[10] V. A. Liskovets, A pattern of asymptotical vertex valency distributions in planar maps, *Preprint* MATH-AL-7-1997, TU-Dresden.

[11] V. A. Liskovets, A pattern of asymptotic vertex valency distributions in planar maps, *J. Combin. Th.* **B75** (1999), No 1, 116–133.

[12] J. Riordan, *Combinatorial Identities*, J. Wiley & Sons, NY, 1968.

[13] N. J. A. Sloane, *On-Line Encyclopedia of Integer Sequences,* AT&T Research, `http://www.research.att.com/~njas/sequences`

[14] N. J. A. Sloane and S. Plouffe, *The Encyclopedia of Integer Sequences,* Academic Press, San Diego (1995).

[15] T. R. S. Walsh, Hypermaps versus bipartite maps, *J. Combin. Th.* **B18** (1975), No 2, 155–163.

[16] T. R. Walsh and A. Lehman, Counting rooted maps by genus, III: Nonseparable maps, *J. Combin. Th.* **B18** (1975), No 3, 222–259.