Bicyclic Antiautomorphisms of Mendelsohn Triple Systems with 0 or 1 Fixed Points

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Abstract

A cyclic triple, (a, b, c), is defined to be the set $\{(a, b), (b, c), (c, a)\}$ of ordered pairs. A Mendelsohn triple system of order v, MTS(v), is a pair (M, β), where M is a set of v points and β is a collection of cyclic triples of pairwise distinct points of M such that any ordered pair of distinct points of M is contained in precisely one cyclic triple of β. An antiautomorphism of a Mendelsohn triple system, (M, β), is a permutation of M which maps β to β^{-1}, where $\beta^{-1} = \{(c, b, a) | (a, b, c) \in \beta\}$. In this paper we give necessary and sufficient conditions for the existence of a Mendelsohn triple system of order v admitting an antiautomorphism consisting of two cycles of equal length and having 0 or 1 fixed points.

1 PRELIMINARIES

A *Steiner triple system of order v*, STS(v), is a pair (S,β), where S is a set of v points and β is a collection of 3-element subsets of S, called *blocks*, such that any pair of distinct points of S is contained in precisely one block of β. Kirkman [2] has shown that there is an STS(v) if and only if $v \equiv 1$ or 3 (mod 6) or $v = 0$.

A *cyclic triple*, (a,b,c), is defined to be the set $\{(a,b),(b,c),(c,a)\}$ of ordered pairs. A *Mendelsohn triple system of order v*, MTS(v), is a pair (M,β), where M is a set of v points and β is a collection of cyclic triples of pairwise distinct points of M, called *triples*, such that any ordered pair of distinct points of M is contained in precisely one element of β. Mendelsohn [3] has shown that an MTS(v) exists if and only if $v \equiv 0$ or 1 (mod 3) with $v \neq 6$.

For an MTS(v), (M,β), we define β^{-1} by $\beta^{-1} = \{(c,b,a)|(a,b,c) \in \beta\}$. Then (M,β^{-1}) is an MTS(v) and is called the *converse* of (M,β). An MTS(v) which is isomorphic to its converse is said to be *self-converse*. An *automorphism* of (M,β) is a permutation of M which maps β to itself. An *antiautomorphism* of (M,β) is a permutation of M which maps β to β^{-1}. Clearly, an MTS(v) is self-converse if and only if it admits an antiautomorphism.

Let (S,β') be an STS(v). Let $\beta = \{(a,b,c),(c,b,a)|\{a,b,c\} \in \beta'\}$. Then (S,β) is called the *corresponding* MTS(v), and the identity map on the point set is an antiautomorphism. Therefore for $v \equiv 1$ or 3 (mod 6) there exists a self-converse MTS(v).

An antiautomorphism, α, on an MTS(v), (M,β), is called *cyclic* if the permutation defined by α consists of a single cycle of length d and $v-d$ fixed points. Carnes [1] has shown that for $v \equiv 0$ or 4 (mod 12) there is an MTS(v) admitting a cyclic antiautomorphism; that is, for $v \equiv 0$ or 4 (mod 12) there is a self-converse MTS(v).

We call an antiautomorphism α on an MTS(v), (M,β), *bicyclic* if the permutation defined by α consists of two cycles each of length $N = (v-f)/2$ and f fixed points. In this paper we consider bicyclic antiautomorphisms on Mendelsohn triple systems with 0 or 1 fixed points.

If N is the length of a cycle, we let the cycles be $(0_i, 1_i, 2_i, \ldots, (N-1)_i)$, $i \in \{0,1\}$, and let ∞ be the fixed point for the 1 fixed point case. Let $D = \{0,1,2,\ldots,(N-1)\}$. We shall use all additions modulo N in the triples. For $a_i, b_j, c_k \in M - \{\infty\}$, $i,j,k \in \{0,1\}$, $(a_i,b_j,c_k) \in \beta$, let *orbit* $(a_i,b_j,c_k) = \{((a+t)_i,(b+t)_j,(c+t)_k)|t \in D, t \text{ even}\} \cup \{((c+t)_k,(b+t)_j,(a+t)_i)|t \in D, t \text{ odd}\}$. If ∞ is a fixed point and $a_i, b_j \in M - \{\infty\}$, $i,j \in \{0,1\}$, $(\infty,a_i,b_j) \in \beta$, let *orbit* $(\infty,a_i,b_j) = \{(\infty,(a+t)_i,(b+t)_j)|t \in D, t \text{ even}\} \cup \{(\infty,(b+t)_j,(a+t)_i)|t \in D, t \text{ odd}\}$. Clearly the orbits partition β.

We say that a collection of triples, $\bar{\beta}$, is a collection of *base triples* of an MTS(v) under α if the orbits of the triples of $\bar{\beta}$ produce β and exactly one triple of $\bar{\beta}$ occurs in each orbit. Also, we say that the *reverse* of the cyclic triple (a,b,c) is the cyclic triple (c,b,a).

2 BICYCLIC ANTIAUTOMORPHISMS WITH 0 FIXED POINTS

LEMMA 1: *Let (M,β) be an* MTS(v) *admitting a bicyclic antiautomorphism with* 0 *fixed points, where $v = 2N$, N being the length of each of the cycles. Then $v \equiv 0$ or* 16 (mod 24).

Proof: Suppose N is odd. Let $a,b,c \in M$. Then $\alpha^N(a,b,c) = (c,b,a)$ and we have an STS(v) so that $v \equiv 1$ or 3 (mod 6), which implies that v is odd. Hence N is even.

So $v \equiv 0 \pmod 4$. Let $(0_i, 1_i, 2_i, \ldots, (N-1)_i)$, $i \in \{0,1\}$, be one of the cycles. $(0_i, (N/2)_i)$ occurs in a triple, say $(0_i, (N/2)_i, a_j)$, $j \in \{0,1\}$. If $N/2$ is odd then $\alpha^{N/2}(0_i,(N/2)_i,a_j) = ((a+N/2)_j, 0_i, (N/2)_i)$ which leads to the contradiction that the ordered pair $(0_i,(N/2)_i)$ is contained in two distinct cyclic triples. Therefore $N/2$ is even so that $N \equiv 0 \pmod 4$, which implies that $v \equiv 0 \pmod 8$. The facts that $v \equiv 0$ or 1 (mod 3) and that $v \equiv 0 \pmod 8$ together imply that $v \equiv 0$ or 16 (mod 24). □

LEMMA 2: *If $v \equiv 0 \pmod{24}$, there exists an* MTS(v) *which admits a bicyclic antiautomorphism with* 0 *fixed points.*

Proof: Let $v = 24k$, $N = 12k$.

For $k = 1$, the base triples are $(0_0, 3_0, 6_0)$ and $(0_1, 3_1, 6_1)$, along with the following and their reverses:
$(0_0,0_1,5_1), (0_0,1_1,2_1), (0_0,4_1,6_1), (0_1,1_0,2_0), (0_1,3_0,5_0), (0_1,4_0,9_0)$,
$(0_0,4_0,8_0), (0_1,4_1,8_1)$.

For $k = 2$, the base triples are $(0_0,0_1,12_1)$ and $(0_1,0_0,12_0)$, along with the following and their reverses:
$(0_0,1_1,11_1), (0_0,2_1,9_1), (0_0,3_1,8_1), (0_0,4_1,7_1), (0_0,5_1,6_1), (0_0,10_1,21_1)$,
$(0_0,13_1,22_1), (0_0,14_1,20_1), (0_0,15_1,19_1), (0_0,16_1,18_1), (0_1,1_0,7_0)$,
$(0_0,4_0,5_0), (0_0,7_0,10_0), (0_0,8_0,16_0), (0_0,9_0,11_0), (0_1,8_1,16_1)$.

For $k \geq 3$ the base triples are $(0_0,0_1,(6k)_1)$ and $(0_1,0_0,(6k)_0)$, along with the following and their reverses:
$(0_0,1_1,(6k-1)_1), (0_0,2_1,(6k-2)_1), \ldots, (0_0,(k-1)_1,(5k+1)_1)$,
$(0_0,k_1,(5k-1)_1), (0_0,(k+1)_1,(5k-2)_1), \ldots, (0_0,(3k-1)_1,(3k)_1)$,
$(0_0,(6k+1)_1,(12k-2)_1), (0_0,(6k+2)_1,(12k-3)_1), \ldots, (0_0,(7k-1)_1,(11k)_1)$,
$(0_0,(7k)_1,(11k-2)_1), (0_0,(7k+1)_1,(11k-3)_1), \ldots, (0_0,(9k-2)_1,(9k)_1)$,
$(0_0,(5k)_1,(11k-1)_1), (0_1,1_0,(3k+1)_0)$,

$(0_0,(2k+2)_0,(4k-1)_0),(0_0,(2k+3)_0,(4k-2)_0),\ldots,(0_0,(3k-1)_0,(3k+2)_0),$
$(0_0,(4k+1)_0,(6k-1)_0),(0_0,(4k+2)_0,(6k-2)_0),\ldots,(0_0,(5k-1)_0,(5k+1)_0),$
$(0_0,(2k)_0,(2k+1)_0),(0_0,(3k+1)_0,(5k)_0),$
$(0_0,(4k)_0,(8k)_0),(0_1,(4k)_1,(8k)_1).$ □

LEMMA 3: *If* $v \equiv 16 \pmod{24}$, *there exists an* MTS(v) *which admits a bicyclic antiautomorphism with* 0 *fixed points.*

Proof: Let $v = 24k+16$, $N = 12k+8$.

For $k=0$ the base triples are $(0_0,0_1,4_1)$ and $(0_1,0_0,4_0)$, along with the following and their reverses:
$(0_0,1_1,2_1),(0_0,3_1,6_1),(0_0,5_1,7_1),(0_0,1_0,3_0).$

For $k=1$ the base triples are $(0_0,0_1,10_1)$ and $(0_1,0_0,10_0)$, along with the following and their reverses:
$(0_0,2_1,8_1),(0_0,3_1,7_1),(0_0,4_1,6_1),(0_0,12_1,19_1),(0_0,13_1,18_1),(0_0,14_1,17_1),$
$(0_1,4_0,5_0),(0_1,9_0,15_0),(0_1,11_0,19_0),(0_0,2_0,5_0),(0_0,4_0,11_0),(0_1,1_1,9_1).$

For $k=2$ the base triples are $(0_0,0_1,16_1)$ and $(0_1,0_0,16_0)$, along with the following and their reverses:
$(0_0,2_1,14_1),(0_0,3_1,13_1),(0_0,4_1,12_1),(0_0,5_1,11_1),(0_0,6_1,10_1),(0_0,7_1,9_1),$
$(0_0,18_1,31_1),(0_0,19_1,30_1),(0_0,20_1,29_1),(0_0,21_1,28_1),(0_0,22_1,27_1),$
$(0_0,23_1,26_1),(0_1,7_0,8_0),(0_1,15_0,24_0),(0_1,17_0,31_0),$
$(0_0,2_0,15_0),(0_0,3_0,11_0),(0_0,4_0,10_0),(0_0,5_0,12_0),(0_1,1_1,15_1).$

For $k \geq 3$ the base triples are $(0_0,0_1,(6k+4)_1)$ and $(0_1,0_0,(6k+4)_0)$, along with the following and their reverses:
$(0_0,2_1,(6k+2)_1),(0_0,3_1,(6k+1)_1),\ldots,(0_0,(3k+1)_1,(3k+3)_1),$
$(0_0,(6k+6)_1,(12k+7)_1),(0_0,(6k+7)_1,(12k+6)_1),\ldots,$
$(0_0,(9k+5)_1,(9k+8)_1),$
$(0_1,(3k+1)_0,(3k+2)_0),(0_1,(6k+3)_0,(9k+6)_0),(0_1,(6k+5)_0,(12k+7)_0),$
$(0_0,(2k+3)_0,(4k+1)_0),(0_0,(2k+4)_0,(4k)_0),\ldots,(0_0,(3k)_0,(3k+4)_0),$
$(0_0,(4k+4)_0,(6k-1)_0),(0_0,(4k+5)_0,(6k-2)_0),\ldots,(0_0,(5k)_0,(5k+3)_0),$
$(0_0,2_0,(6k+3)_0),(0_0,(2k-3)_0,(6k)_0),(0_0,(2k-1)_0,(5k+1)_0),$
$(0_0,(2k)_0,(4k+2)_0),(0_0,(2k+1)_0,(5k+2)_0),(0_1,1_1,(6k+3)_1).$ □

By the previous three lemmas we have the following theorem.

THEOREM 1: *There exists an* MTS(v) *admitting a bicyclic antiautomorphism with* 0 *fixed points if and only if* $v \equiv 0$ *or* 16 (mod 24).

3 BICYCLIC ANTIAUTOMORPHISMS WITH 1 FIXED POINT

LEMMA 4: *Let* (M,β) *be an* MTS(v) *admitting a bicyclic antiautomorphism with* 1 *fixed point, where* $v = 2N+1$, N *being the length of each of the cycles.*

Then $v \equiv 1, 3, 7, 9, 13, 15, 19$ *or* $21 \pmod{24}$.

Proof: Clearly, since $v = 2N + 1$, v must be odd. Also, we must have $v \equiv 0$ or $1 \pmod 3$. Thus $v \equiv 1, 3, 7, 9, 13, 15, 19$ or $21 \pmod{24}$. □

Phelps and Rosa [4] have shown that for $v \equiv 1, 3, 7, 9, 15$, or $19 \pmod{24}$ there exists an STS(v) admitting a 2-rotational automorphism; that is, an automorphism consisting of two cycles each of length $N = (v-1)/2$ and 1 fixed point. Then the corresponding MTS(v)'s admit a bicyclic antiautomorphism with 1 fixed point.

The only cases remaining are then $v \equiv 13$ or $21 \pmod{24}$.

LEMMA 5: *If* $v \equiv 13 \pmod{24}$, *there exists an* MTS(v) *which admits a bicyclic antiautomorphism with* 1 *fixed point.*

Proof: Let $v = 24k + 13$, $N = 12k + 6$.

For $k = 0$ the base triples are $(0_0, 0_1, 1_1)$, $(0_0, 1_1, 3_1)$, $(0_0, 3_1, 2_1)$, $(0_1, 0_0, 4_0)$, $(0_0, 2_0, 4_0)$ and $(0_1, 2_1, 4_1)$, along with the following and their reverses:
$(\infty, 0_0, 3_0), (\infty, 0_1, 3_1), (0_1, 1_0, 2_0)$.

For $k = 1$ the base triples are $(0_0, 1_1, 7_1)$, $(0_1, 9_0, 17_0)$, $(0_1, 11_0, 14_0)$, $(0_1, 14_0, 9_0)$, $(0_0, 3_0, 4_0)$, $(0_0, 4_0, 6_0)$, $(0_0, 6_0, 12_0)$, $(0_0, 8_0, 7_0)$, $(0_0, 11_0, 13_0)$ and $(0_1, 6_1, 12_1)$, along with the following and their reverses:
$(\infty, 0_0, 9_0), (\infty, 0_1, 9_1), (0_0, 0_1, 8_1), (0_0, 2_1, 6_1), (0_0, 3_1, 5_1), (0_0, 10_1, 17_1)$,
$(0_0, 11_1, 16_1), (0_0, 12_1, 15_1), (0_0, 13_1, 14_1)$.

For $k = 2$ the base triples are $(0_0, 9_0, 18_0)$, $(0_0, 13_0, 26_0)$, $(0_0, 18_0, 14_0)$ and $(0_0, 23_0, 16_0)$, along with the following and their reverses:
$(\infty, 0_0, 15_0), (\infty, 0_1, 15_1)$,
$(0_0, 0_1, 13_1), (0_0, 1_1, 12_1), \ldots, (0_0, 6_1, 7_1)$,
$(0_0, 15_1, 29_1), (0_0, 16_1, 28_1), \ldots, (0_0, 21_1, 23_1)$,
$(0_1, 16_0, 8_0), (0_0, 1_0, 3_0), (0_0, 5_0, 11_0), (0_0, 10_0, 20_0)$.

For $k = 3$ the base triples are $(0_0, 4_0, 12_0)$, $(0_0, 15_0, 30_0)$, $(0_0, 17_0, 25_0)$ and $(0_0, 19_0, 23_0)$, along with the following and their reverses:
$(\infty, 0_0, 21_0), (\infty, 0_1, 21_1)$,
$(0_0, 0_1, 19_1), (0_0, 1_1, 18_1), \ldots, (0_0, 9_1, 10_1)$,
$(0_0, 21_1, 41_1), (0_0, 22_1, 40_1), \ldots, (0_0, 30_1, 32_1)$,
$(0_1, 22_0, 11_0), (0_0, 5_0, 6_0), (0_0, 9_0, 16_0), (0_0, 10_0, 13_0), (0_0, 14_0, 28_0)$,
$(0_0, 18_0, 20_0)$.

For $k = 4$ the base triples are $(0_0, 15_0, 30_0)$, $(0_0, 25_0, 50_0)$, $(0_0, 30_0, 26_0)$ and $(0_0, 41_0, 28_0)$, along with the following and their reverses:
$(\infty, 0_0, 27_0), (\infty, 0_1, 27_1)$,
$(0_0, 0_1, 25_1), (0_0, 1_1, 24_1), \ldots, (0_0, 12_1, 13_1)$,
$(0_0, 27_1, 53_1), (0_0, 28_1, 52_1), \ldots, (0_0, 39_1, 41_1)$,

$(0_1, 28_0, 14_0), (0_0, 8_0, 11_0), (0_0, 9_0, 10_0), (0_0, 12_0, 19_0), (0_0, 16_0, 21_0),$
$(0_0, 17_0, 23_0), (0_0, 18_0, 36_0), (0_0, 20_0, 22_0).$

For $k = 5$ the base triples are $(0_0, 1_0, 2_0)$, $(0_0, 2_0, 6_0)$, $(0_0, 6_0, 3_0)$ and $(0_0, 31_0, 35_0)$, along with the following and their reverses:
$(\infty, 0_0, 33_0), (\infty, 0_1, 33_1),$
$(0_0, 0_1, 31_1), (0_0, 1_1, 30_1), \ldots, (0_0, 15_1, 16_1),$
$(0_0, 33_1, 65_1), (0_0, 34_1, 64_1), \ldots, (0_0, 48_1, 50_1),$
$(0_1, 34_0, 17_0), (0_0, 14_0, 27_0), (0_0, 15_0, 26_0), (0_0, 16_0, 24_0), (0_0, 18_0, 25_0),$
$(0_0, 19_0, 29_0), (0_0, 20_0, 32_0), (0_0, 21_0, 30_0), (0_0, 22_0, 44_0), (0_0, 23_0, 28_0).$

For $k = 6$ the base triples are $(0_0, 1_0, 2_0)$, $(0_0, 2_0, 6_0)$, $(0_0, 6_0, 3_0)$ and $(0_0, 37_0, 41_0)$, along with the following and their reverses:
$(\infty, 0_0, 39_0), (\infty, 0_1, 39_1),$
$(0_0, 0_1, 37_1), (0_0, 1_1, 36_1), \ldots, (0_0, 18_1, 19_1),$
$(0_0, 39_1, 77_1), (0_0, 40_1, 76_1), \ldots, (0_0, 57_1, 59_1),$
$(0_1, 40_0, 20_0), (0_0, 16_0, 27_0), (0_0, 17_0, 25_0), (0_0, 18_0, 33_0), (0_0, 19_0, 32_0),$
$(0_0, 21_0, 35_0), (0_0, 22_0, 31_0), (0_0, 23_0, 30_0), (0_0, 24_0, 36_0), (0_0, 26_0, 52_0),$
$(0_0, 28_0, 38_0), (0_0, 29_0, 34_0).$

For $k \geq 7$ the base triples are $(0_0, 1_0, 2_0)$, $(0_0, 2_0, 6_0)$, $(0_0, 6_0, 3_0)$ and $(0_0, (6k+1)_0, (6k+5)_0)$, along with the following and their reverses:
$(\infty, 0_0, (6k+3)_0), (\infty, 0_1, (6k+3)_1),$
$(0_0, 0_1, (6k+1)_1), (0_0, 1_1, (6k)_1), \ldots, (0_0, (3k)_1, (3k+1)_1),$
$(0_0, (6k+3)_1, (12k+5)_1), (0_0, (6k+4)_1, (12k+4)_1), \ldots,$
$(0_0, (9k+3)_1, (9k+5)_1),$
$(0_1, (6k+4)_0, (3k+2)_0), (0_0, (4k+2)_0, (8k+4)_0),$
$(0_0, (2k+6)_0, (4k-1)_0), (0_0, (2k+7)_0, (4k-2)_0), \ldots, (0_0, (3k-1)_0, (3k+6)_0),$
$(0_0, (4k+5)_0, (6k-1)_0), (0_0, (4k+6)_0, (6k-2)_0), \ldots, (0_0, (5k-2)_0, (5k+6)_0),$
$(0_0, (2k+4)_0, (4k+3)_0), (0_0, (2k+5)_0, (4k+1)_0), (0_0, (3k)_0, (5k+3)_0),$
$(0_0, (3k+1)_0, (5k+2)_0), (0_0, (3k+3)_0, (5k+5)_0), (0_0, (3k+4)_0, (5k+1)_0),$
$(0_0, (3k+5)_0, (5k)_0), (0_0, (4k)_0, (6k)_0), (0_0, (4k+4)_0, (6k+2)_0),$
$(0_0, (5k-1)_0, (5k+4)_0).$ □

LEMMA 6: *If* $v \equiv 21 \pmod{24}$, *there exists an* MTS(v) *which admits a bicyclic antiautomorphism with* 1 *fixed point.*

Proof: Let $v = 24k + 21$, $N = 12k + 10$.

For $k = 0$ the base triples are $(0_0, 0_1, 4_1)$, $(0_0, 1_1, 2_1)$, $(0_0, 2_1, 3_1)$, $(0_0, 3_1, 7_1)$, $(0_0, 4_1, 6_1)$, $(0_0, 5_1, 8_1)$, $(0_0, 7_1, 9_1)$, $(0_0, 8_1, 1_1)$, $(0_1, 0_0, 4_0)$ and $(0_1, 1_0, 5_0)$, along with the following and their reverses:
$(\infty, 0_0, 5_0), (\infty, 0_1, 5_1), (0_0, 2_0, 3_0).$

For $k = 1$ the base triples are $(0_0, 4_0, 12_0)$, $(0_0, 5_0, 10_0)$, $(0_0, 7_0, 15_0)$ and $(0_0, 9_0, 13_0)$, along with the following and their reverses:
$(\infty, 0_0, 11_0), (\infty, 0_1, 11_1),$
$(0_0, 0_1, 10_1), (0_0, 1_1, 9_1), \ldots, (0_0, 4_1, 6_1),$

$(0_0,12_1,21_1),(0_0,13_1,20_1),\ldots,(0_0,16_1,17_1)$,
$(0_1,11_0,17_0),(0_0,2_0,3_0)$.

For $k=2$ the base triples are $(0_0,4_0,12_0)$, $(0_0,11_0,22_0)$, $(0_0,13_0,21_0)$ and $(0_0,15_0,19_0)$, along with the following and their reverses:
$(\infty,0_0,17_0),(\infty,0_1,17_1)$,
$(0_0,0_1,16_1),(0_0,1_1,15_1),\ldots,(0_0,7_1,9_1)$,
$(0_0,18_1,33_1),(0_0,19_1,32_1),\ldots,(0_0,25_1,26_1)$,
$(0_1,17_0,26_0),(0_0,5_0,6_0),(0_0,7_0,10_0),(0_0,14_0,16_0)$.

For $k=3$ the base triples are $(0_0,4_0,20_0)$, $(0_0,13_0,26_0)$, $(0_0,15_0,31_0)$ and $(0_0,21_0,25_0)$, along with the following and their reverses:
$(\infty,0_0,23_0),(\infty,0_1,23_1)$,
$(0_0,0_1,22_1),(0_0,1_1,21_1),\ldots,(0_0,10_1,12_1)$,
$(0_0,24_1,45_1),(0_0,25_1,44_1),\ldots,(0_0,34_1,35_1)$,
$(0_1,23_0,35_0),(0_0,7_0,10_0),(0_0,8_0,14_0),(0_0,9_0,11_0),(0_0,17_0,22_0)$,
$(0_0,18_0,19_0)$.

For $k=4$ the base triples are $(0_0,4_0,12_0)$, $(0_0,23_0,46_0)$, $(0_0,25_0,33_0)$ and $(0_0,27_0,31_0)$, along with the following and their reverses:
$(\infty,0_0,29_0),(\infty,0_1,29_1)$,
$(0_0,0_1,28_1),(0_0,1_1,27_1),\ldots,(0_0,13_1,15_1)$,
$(0_0,30_1,57_1),(0_0,31_1,56_1),\ldots,(0_0,43_1,44_1)$,
$(0_1,29_0,44_0),(0_0,10_0,17_0),(0_0,11_0,16_0),(0_0,13_0,14_0)$,
$(0_0,18_0,21_0),(0_0,19_0,28_0),(0_0,20_0,26_0),(0_0,22_0,24_0)$.

For $k=5$ the base triples are $(0_0,4_0,12_0)$, $(0_0,29_0,58_0)$, $(0_0,31_0,39_0)$ and $(0_0,33_0,37_0)$, along with the following and their reverses:
$(\infty,0_0,35_0),(\infty,0_1,35_1)$,
$(0_0,0_1,34_1),(0_0,1_1,33_1),\ldots,(0_0,16_1,18_1)$,
$(0_0,36_1,69_1),(0_0,37_1,68_1),\ldots,(0_0,52_1,53_1)$,
$(0_1,35_0,53_0),(0_0,13_0,24_0),(0_0,14_0,21_0),(0_0,15_0,20_0),(0_0,16_0,19_0)$,
$(0_0,17_0,27_0),(0_0,22_0,23_0),(0_0,25_0,34_0),(0_0,26_0,32_0),(0_0,28_0,30_0)$.

For $k=6$ the base triples are $(0_0,1_0,2_0)$, $(0_0,2_0,6_0)$, $(0_0,6_0,3_0)$ and $(0_0,39_0,43_0)$, along with the following and their reverses:
$(\infty,0_0,41_0),(\infty,0_1,41_1)$,
$(0_0,0_1,40_1),(0_0,1_1,39_1),\ldots,(0_0,19_1,21_1)$,
$(0_0,42_1,81_1),(0_0,43_1,80_1),\ldots,(0_0,61_1,62_1)$,
$(0_1,41_0,62_0),(0_0,17_0,26_0),(0_0,18_0,33_0),(0_0,19_0,35_0),(0_0,20_0,31_0)$,
$(0_0,22_0,32_0),(0_0,23_0,30_0),(0_0,24_0,38_0),(0_0,25_0,37_0),(0_0,27_0,40_0)$,
$(0_0,28_0,36_0),(0_0,29_0,34_0)$.

For $k\geq 7$ the base triples are $(0_0,1_0,2_0)$, $(0_0,2_0,6_0)$, $(0_0,6_0,3_0)$ and $(0_0,(6k+3)_0,(6k+7)_0)$, along with the following and their reverses:
$(\infty,0_0,(6k+5)_0),(\infty,0_1,(6k+5)_1)$,
$(0_0,0_1,(6k+4)_1),(0_0,1_1,(6k+3)_1),\ldots,(0_0,(3k+1)_1,(3k+3)_1)$,
$(0_0,(6k+6)_1,(12k+9)_1),(0_0,(6k+7)_1,(12k+8)_1),\ldots$,
$(0_0,(9k+7)_1,(9k+8)_1)$,

$(0_1, (6k+5)_0, (9k+8)_0)$,
$(0_0, (2k+6)_0, (4k-1)_0), (0_0, (2k+7)_0, (4k-2)_0), \ldots, (0_0, (3k-1)_0, (3k+6)_0)$,
$(0_0, (4k+4)_0, (6k)_0), (0_0, (4k+5)_0, (6k-1)_0), \ldots, (0_0, (5k-2)_0, (5k+6)_0)$,
$(0_0, (2k+5)_0, (4k+2)_0), (0_0, (3k)_0, (5k+3)_0), (0_0, (3k+1)_0, (5k+5)_0)$,
$(0_0, (3k+2)_0, (5k+1)_0), (0_0, (3k+4)_0, (5k+2)_0), (0_0, (3k+5)_0, (5k)_0)$,
$(0_0, (4k)_0, (6k+2)_0), (0_0, (4k+1)_0, (6k+1)_0), (0_0, (4k+3)_0, (6k+4)_0)$,
$(0_0, (5k-1)_0, (5k+4)_0)$. □

By the previous three lemmas and the results of Phelps and Rosa we have the following theorem.

THEOREM 2: *There exists an* MTS(v) *admitting a bicyclic antiautomorphism with* 1 *fixed point if and only if* $v \equiv$ 1, 3, 7, 9, 13, 15, 19 *or* 21 (mod 24).

References

[1] N. P. Carnes, *Cyclic antiautomorphisms of Mendelsohn triple systems,* Discrete Math. 126 (1994), 29-45.

[2] T. P. Kirkman, *On a problem in combinations*, Cambridge and Dublin Math. J. 2 (1847), 191-204.

[3] N. S. Mendelsohn, *A natural generalization of Steiner triple systems,* Computers in Number Theory (Academic Press, New York, 1971), 323-338.

[4] K. T. Phelps and A. Rosa, *Steiner triple systems with rotational automorphisms*, Discrete Math. 33 (1981), 57-66.