The linear 3–cover of the Hermitean forms graph on 81 vertices

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Abstract

The Hermitean forms graphs Her(n,s) are a series of linear distance-regular graphs. The graph Her(2,3) has the coset graph of the shortened ternary Golay code as an antipodal distance-regular cover. We give a new construction for this linear 3–cover of Her(2,3) and show that it is unique.

1 Distance-regular graphs

A connected graph Γ is called distance-regular if, for any two vertices x and y and any integers i and j, the number of vertices at distance i from x and at distance j from y is a constant $p_{ij}(l)$ depending only on i,j and $l := d(x,y)$, but not on the particular choice of x and y. The numbers $p_{ij}(l)$ are called the **intersection numbers** of Γ. Distance-regular graphs are a very interesting type of graphs because of their nice structure and their connections to other fields of combinatorics. An extensive treatment of the area can be found in the book by Brouwer, Cohen and Neumaier [2].

Let Γ be a distance-regular graph of diameter d, and suppose x and y are two vertices of Γ at distance l. Then all the intersection numbers $p_{ij}(l)$ of Γ are determined by the numbers b_l and c_l of neighbours of x at distance $l+1$ and $l-1$ from y, respectively. Together, these numbers form the **intersection array** $(b_0,\ldots,b_{d-1};c_1,\ldots,c_d)$ of Γ. By a_l we denote the number of neighbours of x at distance l from y, and we use k for the

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valency of Γ (note that $k = b_0$). We summarize the properties of the intersection numbers important for us (for proofs see e.g. [2], Chapter 4).

Result 1.1 *Let Γ be a distance-regular graph of diameter d with intersection array $(b_0, \ldots, b_{d-1}; c_1, \ldots, c_d)$. Then*

(i) $k = a_i + b_i + c_i$ *for* $i = 0, \ldots, d$, *where* $c_0 = b_d := 0$.

(ii) $1 = c_1 \leq c_2 \leq \ldots \leq c_d \leq k$.

(iii) $k = b_0 \geq b_1 \geq \ldots \geq b_d$.

(iv) the $p_{ij}(l)$ can be computed recursively from the intersection array using

$$p_{i+1,j}(l) = \frac{1}{c_{i+1}}(p_{i,j-1}(l)b_{j-1} + p_{i,j}(l)(a_j - a_i) + p_{i,j+1}(l)c_{j+1} - p_{i-1,j}(l)b_{i-1}).$$

□

Distance-regular graphs are either primitive or imprimitive, and the imprimitive graphs are antipodal or bipartite (or maybe both). We are interested in antipodal distance-regular graphs because they give rise to smaller primitive graphs by a simple process called folding.

A graph Γ of diameter d is called **antipodal** if $d(x,y) = d(x,z) = d$ always implies $d(y,z) = d$ for any three vertices x, y, z of Γ, that is, the relation of being at distance d (or 0) is an equivalence relation on the vertices of Γ. The equivalence classes of this relation are called **fibres** or **antipodal classes** of Γ. The **folded graph** $\widetilde{\Gamma}$ of Γ is the graph having the fibres of Γ as vertices, two of them being adjacent in $\widetilde{\Gamma}$ if and only if they contain adjacent vertices in Γ. An antipodal graph is also called a **cover** of its folded graph. The following properties of antipodal distance-regular graphs and their folded graphs are well-known (see e.g. [2]).

Result 1.2 *Let Γ be an antipodal distance-regular graph with diameter D and intersection array $(B_0, \ldots, B_{D-1}; C_1, \ldots, C_D)$.*

(i) Any two fibres F_1 and F_2 of Γ contain the same number of vertices r, called the **index** *of Γ.*

(ii) The parameters of Γ satisfy $B_i = C_{d-i}$ for $i \neq \lfloor \frac{d}{2} \rfloor$ and $B_i = (r-1)C_{d-i}$ for $i = \lfloor \frac{d}{2} \rfloor$.

(iii) The folded graph $\tilde{\Gamma}$ has diameter $d := \lfloor \frac{D-1}{2} \rfloor$ and intersection array

$$(B_0, \ldots, B_{d-1}; C_1, \ldots, C_{d-1}, \gamma C_d), \quad \text{where } \gamma = \begin{cases} r & \text{if } D = 2d, \\ 1 & \text{if } D = 2d+1. \end{cases}$$

Conversely, if Δ is a distance-regular graph of diameter d and intersection array $(b_0, \ldots, b_{d-1}; c_1, \ldots, c_d)$, and Γ is an antipodal distance-regular cover of Δ, then one of the following holds

(a) $D = 2d$ and Γ has intersection array

$$(b_0, \ldots, b_{d-1}, \frac{r-1}{r} c_d, c_{d-1}, \ldots, c_1; c_1, \ldots, c_{d-1}, \frac{1}{r} c_d, b_{d-1}, \ldots, b_0),$$

(b) $D = 2d+1$ and Γ has intersection array

$$(b_0, \ldots, b_{d-1}, t(r-1), c_d, \ldots, c_1; c_1, \ldots, c_d, t, b_{d-1}, \ldots, b_0)$$

for some integer t satisfying $b_{d-1} \leq t(r-1) \leq c_d$ and $c_d \leq t \leq b_{d-1}$.

□

2 Linear graphs

In this paper we study a special type of distance-regular graphs. The vertices of a **linear graph** Γ are the elements of a vector space V over a finite field $F = GF(q)$, i.e. $V(\Gamma) = V = GF(q)^n$ for a prime power q and an integer $n \geq 1$. The adjacency relation in Γ satisfies

$$x \sim y \quad \text{in} \quad \Gamma \qquad \Longleftrightarrow \qquad \alpha x + b \sim \alpha y + b \quad \text{for any} \quad \alpha \in F \setminus \{0\}, b \in V.$$

A linear graph has the elementary abelian group $GF(q)^n$ as a sharply transitive automorphism group, so that linear graphs are a special case of Cayley graphs. If we take D to be the subset of elements of V adjacent to the zero vector $\underline{0} \in V$, D can be used to define adjacency in Γ by

$$x \sim y \qquad \Longleftrightarrow \qquad x - y \in D.$$

D obviously satisfies $\underline{0} \notin D$ and $D = \alpha D$ for all $\alpha \in F \setminus \{0\}$, so that D is the union of a number of 1-dimensional subspaces of $GF(q)^n$ with $\underline{0}$ excluded. Conversely, any subset D of $GF(q)^n$ satisfying $\underline{0} \notin D$ and $D = \alpha D$ for all $\alpha \in GF(q) \setminus \{0\}$ defines a linear graph $\Gamma(D)$ on $GF(q)^n$.

Linear graphs can be represented in a particularly nice way using affine and projective geometries. If Γ is a linear graph with vertex set $V =$

$GF(q)^n$, we can identify the vertices of Γ with the points of an affine geometry $\mathcal{A} \cong \mathrm{AG}(n,q)$. Then the subset D of V defining Γ consists of a set of lines through the point $\underline{0}$ in $\mathcal{A}$ (because $D = \alpha D$), and D does not contain $\underline{0}$.

Now let $\mathcal{P}$ be the hyperplane at infinity for $\mathcal{A}$, i.e. the points of $\mathcal{P}$ correspond to the parallel classes of lines of $\mathcal{A} \cong \mathrm{AG}(n,q)$. We have $\mathcal{P} \cong \mathrm{PG}(n-1,q)$, and in the projective geometry $\mathrm{PG}(n,q)$ formed by $\mathcal{A}$ and $\mathcal{P}$ together, each line L of $\mathcal{A}$ meets $\mathcal{P}$ in exactly one point. Thus the set Ω of points of $\mathcal{P}$ corresponding to the lines contained in D can be used to define adjacency in Γ by

$$x \sim y \qquad \Longleftrightarrow \qquad \text{the line } \overline{xy} \text{ in } \mathcal{A} \text{ meets } \mathcal{P} \text{ in a point of } \Omega.$$

Any set Ω of points of $\mathrm{PG}(n-1,q)$ thus defines a linear graph with the points of $\mathrm{AG}(n,q)$ as vertices.

We now look at the distance graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_d$ of a linear graph Γ of diameter d. The **distance graph** Γ_i has the same vertex set as Γ, and two vertices x and y of Γ are adjacent in Γ_i if and only if they have distance i in Γ. Defining

$$D^{(j)} := \left\{ \sum_{l=1}^{j} d_l : d_l \in D \right\} \setminus \{\underline{0}\} \qquad \text{for } j = 1, \ldots, d$$

we see that $x \sim y$ in Γ_i is equivalent to $x - y \in C^{(i)} := D^{(i)} \setminus \bigcup_{j=0}^{i-1} D^{(j)}$, where $D^{(0)} := \{\underline{0}\}$. Thus, each of the distance graphs Γ_i of Γ is again a linear graph defined by $C^{(i)}$ (note that $C^{(i)} = \alpha C^{(i)}$ and $\underline{0} \notin C^{(i)}$ for any $\alpha \in GF(q) \setminus \{0\}$, $i = 1, \ldots, d$).

Moreover, if Γ is connected, any two vertices x and y of Γ are at some distance i, and as $C^{(i)} \cap C^{(j)} = \emptyset$ for $i \neq j$, the sets $C^{(0)} := \{\underline{0}\}, C^{(1)} = D, \ldots, C^{(d)}$ partition the points of $\mathrm{AG}(n,q)$. As all $C^{(i)}$ are sets of 1-dimensional subspaces of $\mathrm{AG}(n,q)$, we can define Ω_i to be the set of points of $\mathrm{PG}(n-1,q)$ corresponding to the lines contained in $C^{(i)}$, for $i = 1, \ldots, d$. Then the Ω_i partition the points of $\mathcal{P} \cong \mathrm{PG}(n-1,q)$.

The following observation will help us to give a criterion for which partitions $(\Omega_1, \ldots, \Omega_d)$ of $\mathcal{P}$ belong to a linear graph. Let x and y be two vertices at distance i in the linear graph Γ, α the point where the line $\overline{xy}$ meets $\mathcal{P}$ (that is, $\alpha \in \Omega_i$) and suppose z is a third vertex of Γ.

Observation 2.1 *If z is on the line $\overline{xy}$, then $d(x,z) = d(y,z) = i$. Otherwise x, y and z determine a plane E in $PG(n, q)$ which meets $\mathcal{P} \cong PG(n-1,q)$ in a line L. Then L also contains the points $\beta := \overline{xz} \cap \mathcal{P}$ and $\gamma := \overline{yz} \cap \mathcal{P}$ and, if $\beta \in \Omega_j$, $\gamma \in \Omega_l$, then $d(x,z) = j$ and $d(y,z) = l$.*

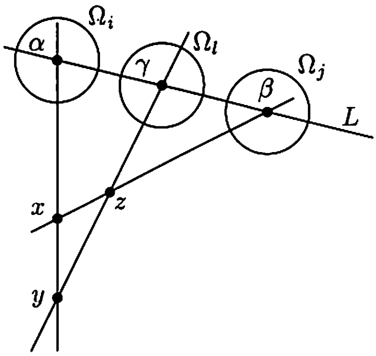

Fig. 1 Observation 2.1

Thus, if we want to find out for which j, l there exist vertices z with $d(x,z) = j$ and $d(y,z) = l$, we have to look at the intersections of lines L through α with the sets $\Omega_1, \ldots, \Omega_d$. To make statements easier to read, we introduce the following notation. A line L in PG($n-1,q$) has **structure** $[m_1, m_2, \ldots, m_d]$ (with respect to a partition $(\Omega_1, \ldots, \Omega_d)$ of $\mathcal{P}$) if L meets Ω_j in m_j points for $j = 1, \ldots, d$. Note that $\sum_{j=1}^{d} m_j = |L| = q+1$.

Lemma 2.2 *A partition $\Pi = (\Omega_1, \ldots, \Omega_d)$ of PG($n-1,q$) belongs to a linear graph Γ if and only if the following hold for $i = 2, \ldots, d$*

(i) for any $\alpha \in \Omega_i$ there is a line L through α which meets Ω_1 and Ω_{i-1}. For $i = 2$, there is a line L through α which meets Ω_1 in at least two (distinct) points.

(ii) if L is any line through $\alpha \in \Omega_i$, and L has structure $[m_1, \ldots, m_d]$, then

a) if $j + l < i$, $j \neq l$, then $m_j \neq 0$ implies $m_l = 0$.

b) if $j < \frac{i}{2}$, then $m_j \leq 1$.

Proof:

$\Longrightarrow$ If $(\Omega_1, \ldots, \Omega_d)$ belongs to a linear graph Γ, then (i) and (ii) hold because

(i) means that for two vertices x and y at distance i, there is a vertex z such that $d(x,z) = i-1$, $d(y,z) = 1$ (for $i = 2$, there is a vertex z such that $d(x,z) = d(y,z) = 1$).

(ii) means that, for two vertices x and y at distance i, there is no vertex z such that $d(x,z) = j$, $d(y,z) = l$ and $j + l < i$.

$\Longleftarrow$ Suppose $(\Omega_1, \ldots, \Omega_d)$ is a partition of $\mathcal{P} \cong \mathrm{PG}(n-1,q)$ satisfying (i) and (ii). Let Γ be the linear graph defined by the set Ω_1, that is, $x \sim y$ in Γ if and only if the line $\overline{xy}$ meets $\mathcal{P}$ in Ω_1. We have to show that $\Omega_2, \ldots, \Omega_d$ are the subsets of $\mathcal{P}$ corresponding to the distance graphs $\Gamma_2, \ldots, \Gamma_d$ of Γ. We have to prove

$$d(x,y) = i \quad \text{in } \Gamma \qquad \Longleftrightarrow \qquad \text{the line } \overline{xy} \text{ meets } \mathcal{P} \text{ in } \Omega_i. \qquad (*)$$

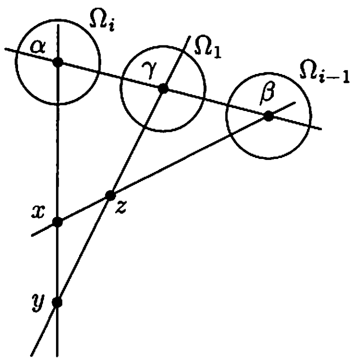

Fig. 2 Lemma 2.2

We use induction over i. For $i = 1$, the statement is clear. As induction hypothesis, suppose that statement $(*)$ is true for all $j < i$.

$\Leftarrow$ Suppose x and y are vertices of Γ such that $\overline{xy}$ meets $\mathcal{P}$ in $\alpha \in \Omega_i$. By induction hypothesis we know that $d(x,y) \geq i$. By (i), there is a line through α containing distinct points $\beta \in \Omega_{i-1}$ and $\gamma \in \Omega_1$ (note that β and γ can be chosen as distinct points also for $i = 2$). Then, in the plane E generated by x, y and β, the lines $\overline{x\beta}$ and $\overline{y\gamma}$ meet in some point z. Thus, we have $d(x,z) = i - 1$ and $d(y,z) = 1$, so that $d(x,y) \leq d(x,z) + d(y,z) = i$.

$\Rightarrow$ Suppose x and y are vertices of Γ at distance i. Then there is a vertex z with $d(x,z) = i - 1$, $d(y,z) = 1$. By induction hypothesis we know that the line $\overline{xz}$ meets $\mathcal{P}$ in some point $\beta \in \Omega_{i-1}$, and $\overline{yz}$ meets $\mathcal{P}$ in some point $\gamma \in \Omega_1$. The points x, y and z generate a plane E which meets $\mathcal{P}$ in a line L. Obviously, L contains the points β, γ and $\alpha := \overline{xy} \cap \mathcal{P}$. We have to show that α is in Ω_i.

By induction hypothesis, α cannot be in $\Omega_1, \ldots, \Omega_{i-1}$. Moreover, as L contains $\beta \in \Omega_{i-1}$ and $\gamma \in \Omega_1$, L cannot contain points from $\Omega_{i+1}, \ldots, \Omega_d$ by (ii). Thus α must be in Ω_i.

□

Our next step is to show how the intersection numbers $p_{ij}(l)$ for $i, j, l \in \{0, \ldots, d\}$ of a linear distance-regular graph Γ can be determined from the partition $\Pi = (\Omega_1, \ldots, \Omega_d)$ belonging to Γ. This correspondence was given by Godsil (personal communication); it will give us a criterion for which partitions of PG($n-1$,q) correspond to a (linear) distance-regular graph.

Theorem 2.3 *Suppose Γ is a linear distance-regular graph and $\Pi = (\Omega_1, \ldots, \Omega_d)$ the corresponding partition of PG($n-1$,q). Then obviously*

$$p_{ij}(0) = \delta_{ij}(q-1)|\Omega_i| \qquad \text{for} \quad i, j = 1, \ldots, d.$$

For $l \neq 0$ choose a point $\alpha \in \Omega_l$, and let $\mathcal{L}$ be the set of lines through α in PG($n-1$,q). Then

$$p_{ij}(l) = \begin{cases} \sum_{L\in\mathcal{L}} (|\Omega_l \cap L| - 1)(|\Omega_l \cap L| - 2) + q - 2 & \text{for } i = j = l, \\ \sum_{L\in\mathcal{L}} (|\Omega_i \cap L|)(|\Omega_l \cap L| - 1) & \text{for } i \neq l, j = l, \\ \sum_{L\in\mathcal{L}} (|\Omega_i \cap L|)(|\Omega_i \cap L| - 1) & \text{for } i = j \neq l, \\ \sum_{L\in\mathcal{L}} (|\Omega_i \cap L|)(|\Omega_j \cap L|) & \text{for } i \neq j, \ \ i, j \neq l. \end{cases}$$

Proof: Use Observation 2.1: If x and y are vertices of Γ at distance l, so that the line $\overline{xy}$ meets $\mathcal{P}$ in the point $\alpha \in \Omega_l$, then any pair of points (β, γ) with $\beta \in \Omega_i$, $\gamma \in \Omega_j$ and α, β, γ pairwise distinct on some line L through α gives a vertex z of Γ with $d(x, z) = i$ and $d(y, z) = j$. For $i = j = l$, the $q-2$ points $z \neq x, y$ on the line $\overline{xy}$ have distance l from both x and y.

□

Now suppose $\Pi = (\Omega_1, \ldots, \Omega_d)$ is any partition of PG($n-1$,q). For $\alpha \in \Omega_l$, let $\mathcal{L}$ denote the set of lines through α in PG($n-1$,q). Then we define

$$p_{ij}(\alpha) := \begin{cases} \sum_{L\in\mathcal{L}} (|\Omega_l \cap L| - 1)(|\Omega_l \cap L| - 2) + q - 2 & \text{for } i = j = l, \\ \sum_{L\in\mathcal{L}} (|\Omega_i \cap L|)(|\Omega_l \cap L| - 1) & \text{for } i \neq l, j = l, \\ \sum_{L\in\mathcal{L}} (|\Omega_i \cap L|)(|\Omega_i \cap L| - 1) & \text{for } i = j \neq l, \\ \sum_{L\in\mathcal{L}} (|\Omega_i \cap L|)(|\Omega_j \cap L|) & \text{for } i \neq j, \ \ i, j \neq l. \end{cases}$$

Corollary 2.4 *A partition $\Pi = (\Omega_1, \ldots, \Omega_d)$ of PG($n-1$,q) corresponds to a linear distance-regular graph Γ of diameter d if and only if*

(i) for all $i, j, l = 1, \ldots, d$, the numbers $p_{ij}(\alpha)$ are equal for all $\alpha \in \Omega_l$ (so that the intersection numbers $p_{ij}(l)$ are well-defined).

(ii) $p_{1,i-1}(i) \neq 0$ for $i = 1, \ldots, d$.

(iii) $i + j < l$ implies $p_{ij}(l) = 0$.

□

As one would expect, there is a close relationship between a linear graph and the corresponding partition of PG($n-1$,q). For example, the eigenvalues of a linear distance-regular graph Γ defined by partition $(\Omega_1, \ldots, \Omega_d)$ are

$$q|H \cap \Omega_1| - |\Omega_1|,$$

where H runs over the hyperplanes of PG($n-1$,q) (see Godsil [4], 12.9.3 p. 246). This is in fact true for the more general structure of linear association schemes. Also, a linear distance-regular graph is antipodal if and only if the cell Ω_d is a subspace (see Godsil [5], where Godsil investigated linear antipodal distance-regular graphs of diameter 3 and obtained very strong results). Calderbank and Kantor [3] studied linear strongly regular graphs and their relationship to coding theory, and gave a list of all known examples.

3 Hermitean forms graphs

We investigate a special series of linear distance-regular graphs. The vertex set of the **Hermitean forms graph** Her(n,s) is the set of Hermitean forms on the vector space $(\mathrm{GF}(s^2))^n$ (where s is a prime power and $n \geq 1$), that is, the set of all mappings $f : (\mathrm{GF}(s^2))^n \to (\mathrm{GF}(s^2))^n$ such that f is linear in y and $f(y,x) = \overline{f(x,y)}$ for all $x, y \in (\mathrm{GF}(s^2))^n$. Two vertices x and y are adjacent if their difference has rank 1. It can be shown that the Hermitean forms graph Her(n,s) is distance-regular of diameter n, in fact we have (see [2], 9.5 C) $d(x,y) =$ rank $(x-y)$. The parameters of Her(n,s) are

$$\begin{aligned} v &= s^{n^2}, \\ b_i &= \frac{1}{s+1}(s^{2n} - s^{2i}) \qquad \text{for} \quad i = 0, \ldots, n-1, \\ c_i &= \frac{1}{s+1}(s^{i-1}(s^i - (-1)^i)) \qquad \text{for} \quad i = 1, \ldots, n. \end{aligned}$$

The vertex set of the Hermitean forms graph Her(n,s) is an n^2– dimensional vector space over GF(s), and from

$$\text{rank } (x-y) = \text{ rank } ((\alpha x + b) - (\alpha y + b)) \quad \text{for} \quad \alpha \neq 0$$

we see that Her(n,s) is a linear graph.

As van Bon and Brouwer [1] determined all distance-regular antipodal covers of the Hermitean forms graphs Her(n,s) for $n \geq 3$ and the covers of diameter 5 for $n = 2$, it remains to find the covers of diameter 4 of the graphs Her(2,s). We know that the only cover of Her(2,2) is a unique 2–cover called the Wells graph (see [2], 9.2 E), and for Her(2,3) there exists a

3–cover coming from the shortened ternary Golay code (see [2], 11.3 H). No other antipodal distance-regular covers for the graphs Her(2,s) are known. The aim of this paper is to give a new construction of the 3–cover of Her(2,3) using the representation of Her(2,3) and its cover as linear graphs. First, we have a closer look at the graphs Her(2,s) in general.

A Hermitean forms graph Her(2,s) is distance-regular of diameter 2 (such graphs are called **strongly regular**), it has s^4 vertices and intersection array $((s-1)(s^2+1),\ (s-1)s^2\ ;\ 1,\ s(s-1))$. An antipodal distance-regular r–cover Γ of Her(2,s) of diameter 4 has intersection array

$$((s-1)(s^2+1),(s-1)s^2,\ \tfrac{r-1}{r}s(s-1),1;\ 1,\tfrac{s(s-1)}{r},(s-1)s^2,(s-1)(s^2+1)).$$

Some existence conditions on the parameters of such a cover are given in [8]. For example, if $s = p^t$ for $p = 2$ or 3, then t must be 1. Now look at Her(2,s) as a linear graph. We restrict ourselves to the case of s odd from now on. Note that, for $s = 2^t$, the only cover is the Wells graph, in which case $t = 1$.

We identify the vertices of Her(2,s), that is, the Hermitean 2×2–matrices over GF(s^2), with the vectors in AG(4,s) as follows. Represent a vertex $u = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$ of Her(2,s) by the vector (a,c,b_1,b_2) in $(\mathrm{GF}(s))^4$, where $b = b_1 + \xi b_2$ with $b_1, b_2 \in \mathrm{GF}(s)$ and ξ an element of $\mathrm{GF}(s^2)\backslash\mathrm{GF}(s)$ (so that $\mathrm{GF}(s^2) = \mathrm{GF}(s)(\xi)$). Then the vectors of AG(4,s) adjacent to $\underline{0}$ (corresponding to matrices of rank 1) are precisely the vectors $p = (p_1,p_2,p_3,p_4)$ satisfying $p_1p_2 - {p_3}^2 - {p_4}^2 = 0$, and the corresponding set Ω_1 of points in PG(3,s) forms a non-singular elliptic quadric. We know that in PG(3,s) with s odd, all elliptic quadrics are projectively equivalent (see e.g. Hirschfeld [6], 5.2.4), and are precisely the ovoids in PG(3,s) (see e.g. Hirschfeld [7],16.1.7). An **ovoid** $\mathcal{K}$ in a projective geometry $\mathcal{P}$ is a set of points of $\mathcal{P}$ such that

(i) $|\mathcal{K} \cap L| \leq 2$ for any line L in $\mathcal{P}$,

(ii) for any point $p \in \mathcal{K}$ the tangent lines through p (i.e. the lines which meet $\mathcal{K}$ only in p) form a hyperplane H_p.

Thus, the Hermitean forms graph Her(2,s) (for s odd) corresponds to a partition (Ω_1,Ω_2) of PG(3,s), where Ω_1 is an ovoid in PG(3,s), and conversely any ovoid in PG(3,s), for s odd, determines the Hermitean forms graph Her(2,s). In Calderbank and Kantor [3] the Hermitean forms graphs Her(2,s) appear as series TF3.

4 Construction

We present a new method for constructing a linear s–cover of Her(2,s). This construction uses the representation of Her(2,s) and of a linear s–cover of Her(2,s) as partitions of projective spaces. We need some information about projective spaces and ovoids in PG(3,s) which can be found in Hirschfeld [6] and [7]. The idea of the construction is to extend the partition (Ψ_1, Ψ_2) of PG(3,s) corresponding to the Hermitean forms graph Her(2,s) to a partition $(\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ of PG(4,s) defining a linear s–cover of Her(2,s). The cover then has AG(5,s) as vertex set.

We show how the construction works for $s = 3$. The Hermitean forms graph Her(2,3) has 81 vertices and intersection array $(20, 18; 1, 6)$; a 3-cover of this graph must have 243 vertices and intersection array $(20, 18, 4, 1; 1, 2, 18, 20)$. As everything becomes considerably more complicated for larger values of s, we do not know whether the construction method can be generalized. Note that the next possible value for s is 7 (see [8]).

Let $\mathcal{P} \cong$ PG(3,3) be a hyperplane of PG(4,3), and denote the affine geometry AG(4,3) we get by removing $\mathcal{P}$ from PG(4,3) by $\mathcal{A}$. Moreover, choose an ovoid $\mathcal{K}$ in $\mathcal{P}$, so that the partition $(\mathcal{K}, \mathcal{P} \setminus \mathcal{K})$ defines a Hermitean forms graph Her(2,3) on the points of $\mathcal{A}$.

If x is a point of $\mathcal{P}$ outside $\mathcal{K}$, we know (see e.g. Hirschfeld [7]) that x is on 3 lines meeting $\mathcal{K}$ in two points (the **bisecants**) and on 4 lines meeting $\mathcal{K}$ in exactly one point (the **tangents**). We call the 6 points of $\mathcal{K}$ on the bisecants the **bisecant points** of x and the 4 points on the tangents the **tangent points** of x. Analogously, a plane E of $\mathcal{P}$ meeting $\mathcal{K}$ in exactly one point will be called a **tangent plane**; any other plane of $\mathcal{P}$ meets $\mathcal{K}$ in an oval and is called an **oval plane**. Note that any point x in $\mathcal{P} \setminus \mathcal{K}$ is contained in 4 tangent planes and 9 oval planes. Also, two distinct ovals can have at most two points in common.

We can now begin to construct the desired partition $\Pi = (\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ of PG(4,3) corresponding to a linear 3-cover of Her(2,3). Such a partition should have $|\Omega_1| = 10$, $|\Omega_2| = 90$, $|\Omega_3| = 20$ and $|\Omega_4| = 1$. Our first step is to partition the ovoid $\mathcal{K}$ into two parts $\mathcal{K}_1, \mathcal{K}_3$ such that

(∗) $\mathcal{K}_1$ and $\mathcal{K}_3$ have the same size and neither $\mathcal{K}_1$ nor $\mathcal{K}_3$ contains an oval.

To show that such a partition exists, we need some preliminaries.

Lemma 4.1 *Let x be a point of $\mathcal{P}$ outside the ovoid $\mathcal{K}$. Suppose A, B and C are the bisecants to $\mathcal{K}$ through x, and t_1, t_2, t_3 and t_4 are the tangent points of x.*

(i) Of the 9 oval planes containing x, three are generated by two of the three bisecants A, B, C. The other six oval planes each contain one of

the bisecants and two of the tangent points of x. Naming the bisecant points on A a_1, a_2, on B b_1, b_2, and on C c_1, c_2, we can (by choosing the names of t_1, t_2, t_3, t_4 appropriately) always assume these ovals to be

$$\begin{array}{lll} a_1\ a_2\ t_1\ t_2, & a_1\ a_2\ t_3\ t_4, & b_1\ b_2\ t_1\ t_3, \\ b_1\ b_2\ t_2\ t_4, & c_1\ c_2\ t_1\ t_4, & c_1\ c_2\ t_2\ t_3. \end{array}$$

Note that no two of these ovals have two tangent points in common.

(ii) *The four tangent points of x form an oval, i.e. they lie in a common plane.*

(iii) *No two distinct points of $\mathcal{P} \setminus \mathcal{K}$ can have three tangent points in common.*

Proof:

(i) It is clear that any two bisecants through x generate a plane meeting $\mathcal{K}$ in an oval. Now any bisecant through x is in two more (oval) planes, and as the four planes containing a line L in PG(3,3) partition PG(3,3)$\setminus L$, and thus in particular partition the points of $\mathcal{K} \setminus L$, we can assume w.l.o.g. that $a_1a_2t_1t_2$ and $a_1a_2t_3t_4$ form an oval.

If we can show that no two of the ovals corresponding to planes through x can have two tangent points in common, the assertion follows by choosing the names for t_1, t_2, t_3 and t_4 appropriately. So suppose on the contrary that the plane P_1 contains a_1, a_2, t_1 and t_2, and that b_1, b_2, t_1, t_2 are contained in a plane P_2. Then P_1 and P_2 both contain the point x, and they also have the tangent points t_1 and t_2 in common. But x, t_1 and t_2 are not collinear, so they cannot be contained in two distinct planes.

(ii) We may assume w.l.o.g. that the planes containing x meet $\mathcal{K}$ in ovals as given in (i). Then the plane E determined by the three tangent points t_1, t_2, t_3 of x meets $\mathcal{K}$ in an oval O consisting of t_1, t_2, t_3 and some fourth point of $\mathcal{K}$. This fourth point cannot be one of the bisecant points a_1, a_2, b_1, b_2, c_1 or c_2, because then O would have exactly three points in common with one of the ovals of (i). The only possibility is t_4, so the four tangent points for x lie in a common plane.

(iii) Let x and y be distinct points of $\mathcal{P} \setminus \mathcal{K}$ both having tangent points t_1, t_2, t_3. Then by (ii), x and y also have the fourth tangent point t_4 in common. Let P_1, P_2, P_3, P_4 be the tangent planes for the common tangent points t_1, t_2, t_3, t_4 of x and y. Then the line $\overline{xy}$ is contained in P_1, P_2, P_3, P_4. Any line L in $\mathcal{P}$ is contained in four planes which partition $\mathcal{P} \setminus L$. But if $L = \overline{xy}$ is contained in the four tangent planes $P_1, \ldots, P_4$, there is no plane on L containing the points of $\mathcal{K} \setminus \{t_1, \ldots, t_4\}$. Contradiction.

□

Note that (ii) and (iii) above are quite obvious if we consider the polarity α corresponding to the ovoid $\mathcal{K}$. Then the polar plane E of x with respect to α contains the tangent points t_1, t_2, t_3 and t_4 of x, and x is the pole of E with respect to α.

Before showing that a partition $(\mathcal{K}_1, \mathcal{K}_3)$ of the ovoid $\mathcal{K}$ having property $(*)$ exists we prove that such a partition has another interesting property. This property will help us to construct such a partition and will also be useful later. To make it easier to state this property, we introduce the following notation. For a partition $(\mathcal{K}_1, \mathcal{K}_3)$ and a point x of $\mathcal{P}$ not in $\mathcal{K}$, a bisecant through x could have either

(i) two points in $\mathcal{K}_1$,

(ii) two points in $\mathcal{K}_3$,

(iii) one point in $\mathcal{K}_1$ and one point in $\mathcal{K}_3$.

We say that a bisecant has **type 1/1**, **type 3/3** or **type 1/3** (with respect to $(\mathcal{K}_1, \mathcal{K}_3)$) according to whether its two points in $\mathcal{K}$ are as in (i), (ii) or (iii), respectively. Also, if in the following an oval plane in $\mathcal{P}$ is called P_i, then the corresponding oval will be denoted by O_i, and vice versa. Note that, for a partition $(\mathcal{K}_1, \mathcal{K}_3)$ having property $(*)$, a point x outside $\mathcal{K}$ cannot be on two bisecants of type 1/1 or on two bisecants of type 3/3.

Lemma 4.2 *Let $(\mathcal{K}_1, \mathcal{K}_3)$ be a partition of an ovoid $\mathcal{K}$ having property $(*)$ and A a bisecant of $\mathcal{K}$ of type 1/1. Then A contains two points x, y of $\mathcal{P} \backslash \mathcal{K}$. One of them is on bisecants of types 1/1, 1/3, 1/3, and the other one is on bisecants of types 1/1, 1/3, 3/3. Analogously, for a bisecant A' of type 3/3, one of the two points of A' outside $\mathcal{K}$ is on bisecants of types 1/3, 1/3, 3/3, and the other one is on bisecants of types 1/1, 1/3, 3/3. Moreover, the tangent points for x are the bisecant points of y (the ones not on A) and vice versa.*

Proof: Suppose x is a point of A in $\mathcal{P} \backslash \mathcal{K}$. Denote the points where A meets $\mathcal{K}$ by a_1 and a_2. Moreover, let B and C be the other two bisecants through x with bisecant points b_1, b_2 and c_1, c_2, respectively, and denote the tangent points for x by d, e, f and g.

Now let $y \in \mathcal{P} \backslash \mathcal{K}$ be the fourth point on A. Then y cannot be on a bisecant with either b_1, b_2, c_1 or c_2 (otherwise, there would be an oval having three points in common with one of the ovals $a_1a_2b_1b_2$ or $a_1a_2c_1c_2$). Therefore, b_1, b_2, c_1 and c_2 must be the tangent points of y.

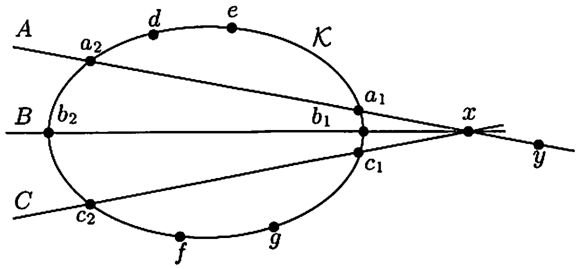

Fig. 3 Lemma 4.2

As x cannot be on two bisecants of type 1/1 or on two bisecants of type 3/3, there are two possibilities.

(i) Suppose both B and C are bisecants of type 1/3. Then we may assume w.l.o.g. that

$$\mathcal{K}_1 = \{a_1, a_2, b_1, c_1, d\} \qquad \text{and} \qquad \mathcal{K}_3 = \{b_2, c_2, e, f, g\}.$$

As d, e, f and g are the bisecant points of y, this implies that y must be on one bisecant of type 1/3 and one bisecant of type 3/3.

(ii) Suppose B is a bisecant of type 1/3 and C has type 3/3. Then we may assume w.l.o.g. that

$$\mathcal{K}_1 = \{a_1, a_2, b_1, d, e\} \qquad \text{and} \qquad \mathcal{K}_3 = \{b_2, c_1, c_2, f, g\}.$$

Again, d, e, f and g are the bisecant points of y, and y cannot be on a further bisecant of type 1/1. Thus, y must be on two bisecants of type 1/3.

The proof for a bisecant A' of type 3/3 works analogously. □

Corollary 4.3 *Let $(\mathcal{K}_1, \mathcal{K}_3)$ be a partition of an ovoid $\mathcal{K}$ having property $(*)$. Then the* 30 *points of $\mathcal{P}\backslash\mathcal{K}$ can be divided into three classes according to the types of the three bisecants to $\mathcal{K}$ they are on:*

(I) points being on bisecants of types 1/1, 1/3 *and* 1/3,

(II) points being on bisecants of types 1/1, 1/3 *and* 3/3,

(III) points being on bisecants of types 1/3, 1/3 *and* 3/3.

In particular, no point x of $\mathcal{P} \setminus \mathcal{K}$ is on three bisecants of type 1/3. *Each of the three classes contains* 10 *points.*

Proof: There are 10 bisecants to $\mathcal{K}$ of type $1/1$. By Lemma 4.2, each of them contains precisely one point in class (I) and one point in class (II) above. Similarly, the 10 bisecants of type $3/3$ yield 10 points in class (II) and 10 points in class (III). Therefore, each of the three classes of points above contains at least 10 points. Now $\mathcal{P} \setminus \mathcal{K}$ consists of 30 points, so that each class must contain precisely 10 points, and there cannot be any points being on three bisecants of type $1/3$. □

Lemma 4.4 *There exist* 72 *partitions* $(\mathcal{K}_1, \mathcal{K}_3)$ *of the ovoid* $\mathcal{K}$ *having property* $(*)$.

Proof: We construct all possible partitions of $\mathcal{K}$ having property $(*)$. To do so, we use the above results to construct all the ovals contained in $\mathcal{K}$ (there are 30 of them).

Choose any point x of $\mathcal{P} \setminus \mathcal{K}$, let A, B and C be the three bisecants through x, and a, a', b, b', c, c' the corresponding bisecant points. Moreover, name the four tangent points of x d, e, f and g. By Lemma 4.1, we may assume w.l.o.g. that $\mathcal{K}$ contains the following ovals.

(O_1)	$a\ a'\ d\ e$	(O_2)	$a\ a'\ f\ g$	(O_3)	$a\ a'\ b\ b'$
(O_4)	$a\ a'\ c\ c'$	(O_5)	$b\ b'\ c\ c'$	(O_6)	$b\ b'\ d\ f$
(O_7)	$b\ b'\ e\ g$	(O_8)	$c\ c'\ d\ g$	(O_9)	$c\ c'\ e\ f.$

To find more ovals, look at the lines $\overline{da}$ and $\overline{da'}$. Each of them is contained in four planes (P_1 being one of them). Let P_{10} be the plane generated by d, a and f. The fourth point of the corresponding oval O_{10} cannot be a', e or g (because then O_{10} would have three points in common with O_1 or O_2), nor can it be b or b' (look at O_6). Thus it must be c or c', and we are free to call this point c. Then the oval O_{11} determined by d, a' and f must have fourth point c'.

Using the same arguments as above, we find that the fourth point of the oval O_{12} containing a, d and g has to be b or b', and we name this point b. This implies that the oval O_{13} determined by a', d and g has b' as its fourth point. Now we know three out of the four planes containing the lines $\overline{da}$ and $\overline{da'}$, so that the remaining planes (or rather the corresponding ovals) are easy to find.

(O_{10})	$a\ d\ f\ c$	(O_{11})	$a'\ d\ f\ c'$	(O_{12})	$a\ d\ g\ b$
(O_{13})	$a'\ d\ g\ b'$	(O_{14})	$a\ d\ b'\ c'$	(O_{15})	$a'\ d\ b\ c.$

Moreover, the lines $\overline{dc}$ and $\overline{dc'}$ are each contained in three of the planes corresponding to the above ovals. This gives us

(O_{16}) $d\ c\ b'\ e$ $\qquad$ (O_{17}) $d\ c'\ b\ e.$

Next, look at the lines $\overline{ac}$ and $\overline{ac'}$. The plane P_4 contains both of them, and $\overline{ac}$ is in P_{10}, $\overline{ac'}$ is in P_{14}, so that there are two more planes on each of them. The plane P_{18} generated by a, c and b cannot contain a', c', b', d, f or g (look at the ovals O_4, O_3, O_{10} and O_{12}), thus the fourth point of the corresponding oval must be e. Using the same arguments, we find that the plane P_{19} determined by a, c' and b must contain f. We also find the ovals determined by the remaining planes on the lines $\overline{ac}$ and $\overline{ac'}$.

(O_{18})	$a\ c\ b\ e$	(O_{19})	$a\ c'\ b\ f$
(O_{20})	$a\ c\ b'\ g$	(O_{21})	$a\ c'\ e\ g.$

Using the lines $\overline{a'c}$ and $\overline{a'c'}$ in the same manner, we find the ovals

(O_{22})	$a'\ c\ b'\ f$	(O_{23})	$a'\ c'\ b\ g$
(O_{24})	$a'\ c\ e\ g$	(O_{25})	$a'\ c'\ b'\ e.$

Now it is easy to find the remaining 5 ovals in $\mathcal{K}$:

(O_{26})	$d\ e\ f\ g$	(O_{27})	$a\ b'\ e\ f$	(O_{28})	$a'\ b\ e\ f$
(O_{29})	$b\ c\ f\ g$	(O_{30})	$b'\ c'\ f\ g.$		

Knowing all the ovals contained in $\mathcal{K}$, we can now check whether some partition of $\mathcal{K}$ satisfies $(*)$ or not. We choose three points of $\mathcal{K}$ and find all possibilities for partitions where these three points are in $\mathcal{K}_1$. So let a, a' and b be in $\mathcal{K}_1$. Then b' has to be in $\mathcal{K}_3$ (otherwise $O_3 \subset \mathcal{K}_1$).

1.) Suppose $c \in \mathcal{K}_1$. Then $c' \in \mathcal{K}_3$ (O_4), $e \in \mathcal{K}_3$ (O_{18}), $d \in \mathcal{K}_3$ (O_{15}).

   a.) Suppose $f \in \mathcal{K}_1$, $g \in \mathcal{K}_3$. This gives the partition
   $\mathcal{K}_1 = \{a, a', b, c, f\}$, $\quad \mathcal{K}_3 = \{b', c', d, e, g\}$.

   b.) Suppose $g \in \mathcal{K}_1$, $f \in \mathcal{K}_3$. This gives the partition
   $\mathcal{K}_1 = \{a, a', b, c, g\}$, $\quad \mathcal{K}_3 = \{b', c', d, e, f\}$.

2.) Suppose $c' \in \mathcal{K}_1$. Then $c \in \mathcal{K}_3$ (O_4), $f \in \mathcal{K}_3$ (O_{19}), $g \in \mathcal{K}_3$ (O_{23}).

   a.) Suppose $d \in \mathcal{K}_1$, $e \in \mathcal{K}_3$. This gives the partition
   $\mathcal{K}_1 = \{a, a', b, c', d\}$, $\quad \mathcal{K}_3 = \{b', c, e, f, g\}$.

   b.) Suppose $e \in \mathcal{K}_1$, $d \in \mathcal{K}_3$. This gives the partition
   $\mathcal{K}_1 = \{a, a', b, c', e\}$, $\quad \mathcal{K}_3 = \{b', c, d, f, g\}$.

3.) Suppose $c' \in \mathcal{K}_1$. Then $c \in \mathcal{K}_3$ (O_4), $f \in \mathcal{K}_3$ (O_{19}), $g \in \mathcal{K}_3$ (O_{23}).

   a.) Suppose $d \in \mathcal{K}_1$, $e \in \mathcal{K}_3$. This gives the partition
   $\mathcal{K}_1 = \{a, a', b, c', d\}$, $\quad \mathcal{K}_3 = \{b', c, e, f, g\}$.

   b.) Suppose $e \in \mathcal{K}_1$, $d \in \mathcal{K}_3$. This gives the partition
   $\mathcal{K}_1 = \{a, a', b, c', e\}$, $\quad \mathcal{K}_3 = \{b', c, d, f, g\}$.

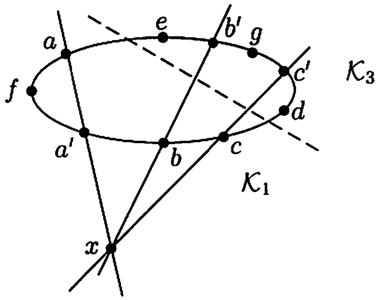

Fig. 4 Partition as in 1.a)

4.) Suppose $d \in \mathcal{K}_1$. Then $e \in \mathcal{K}_3$ (O_1), $g \in \mathcal{K}_3$ (O_{12}), $c \in \mathcal{K}_3$ (O_{15}).

a.) Suppose $c' \in \mathcal{K}_1$, $f \in \mathcal{K}_3$. This gives the same partition as in 2.a).

b.) Suppose $f \in \mathcal{K}_1$, $c' \in \mathcal{K}_3$. This gives the partition
$\mathcal{K}_1 = \{a, a', b, d, f\}$, $\mathcal{K}_3 = \{b', c, c', e, g\}$.

5.) Suppose $e \in \mathcal{K}_1$. Then $d \in \mathcal{K}_3$ (O_1), $c \in \mathcal{K}_3$ (O_{18}), $f \in \mathcal{K}_3$ (O_{28}).

a.) Suppose $c' \in \mathcal{K}_1$, $g \in \mathcal{K}_3$. This gives the same partition as in 2.b).

b.) Suppose $g \in \mathcal{K}_1$, $c' \in \mathcal{K}_3$. This gives the partition
$\mathcal{K}_1 = \{a, a', b, e, g\}$, $\mathcal{K}_3 = \{b', c, c', d, f\}$.

6.) Suppose $f \in \mathcal{K}_1$. Then $g \in \mathcal{K}_3$ (O_2), $c' \in \mathcal{K}_3$ (O_{19}), $e \in \mathcal{K}_3$ (O_{28}).

a.) Suppose $c \in \mathcal{K}_1$, $d \in \mathcal{K}_3$. This gives the same partition as in 1.a).

b.) Suppose $d \in \mathcal{K}_1$, $c \in \mathcal{K}_3$. This gives the same partition as in 3.b).

7.) Suppose $g \in \mathcal{K}_1$. Then $f \in \mathcal{K}_3$ (O_2), $d \in \mathcal{K}_3$ (O_{12}), $c' \in \mathcal{K}_3$ (O_{23}).

a.) Suppose $c \in \mathcal{K}_1$, $e \in \mathcal{K}_3$. This gives the same partition as in 1.b).

b.) Suppose $e \in \mathcal{K}_1$, $c \in \mathcal{K}_3$. This gives the same partition as in 4.b).

Thus, for any triple k_1, k_2, k_3 of distinct points from $\mathcal{K}$ there are six distinct partitions $(\mathcal{K}_1, \mathcal{K}_3)$ with $k_1, k_2, k_3 \in \mathcal{K}_1$ satisfying $(*)$. As we know that all triples of (distinct) points from $\mathcal{K}$ are projectively equivalent ($PGO_-(4, q)$

is triply transitive on $\mathcal{K}$, see [7]), and each partition can be derived from 10 distinct triples, we have a total of

$$\binom{10}{3} \cdot 6 \cdot \frac{1}{10} = 72$$

distinct partitions satisfying (∗). □

Our next step is to extend the partition $(\mathcal{K}_1, \mathcal{K}_3, \mathcal{P} \setminus \mathcal{K})$ of $\mathcal{P}$, where $(\mathcal{K}_1, \mathcal{K}_3)$ has property (∗), to a partition $\Pi = (\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ of PG(4,3). From now on we denote $\mathcal{K} = \{k_1, \ldots, k_{10}\}$ and $\mathcal{K}_1 = \{k_1, \ldots, k_5\}$, $\mathcal{K}_3 = \{k_6, \ldots, k_{10}\}$.

A main criterion for how to partition the points of PG(4,3) is the fact that a linear 3–cover Γ of Her(2,3) must have intersection number $p_{11}(2) = c_2 = 2$. This means, using Theorem 2.3, that any point in cell Ω_2 must be on exactly one line in PG(4,3) having two points in cell Ω_1. The parts $\mathcal{K}_1$ and $\mathcal{K}_3$ of the ovoid $\mathcal{K}$ are going to be subsets of the cells Ω_1 and Ω_3, respectively, and all points of $\mathcal{P} \setminus \mathcal{K}$ will be in Ω_2. We know from Corollary 4.3 that all points of $\mathcal{P} \setminus \mathcal{K}$ are in one of three classes according to what types of bisecants to $\mathcal{K}$ they are on, and the only points which are not on a bisecant of type 1/1 are the 10 points of class (III). We call these points $x_1, \ldots, x_{10}$. Our problem now is to choose the other points of Ω_1 in such a way that each of the x_i is on exactly one line having two points in this new part of Ω_1.

As before, we denote the zero vector in $\mathcal{A} \cong \mathrm{AG}(4,3)$ by $\underline{0}$, and D is the set of points $\neq \underline{0}$ of $\mathcal{A}$ which are on lines $\overline{\underline{0}k}$ for some $k \in \mathcal{K}$ (i.e. the vertices of Her(2,3) adjacent to $\underline{0}$), that is, if the line $\overline{\underline{0}k_l}$ contains points a_l and b_l for $l = 1, \ldots, 10$, then $D = \{a_l, b_l : l = 1, \ldots, 10\}$.

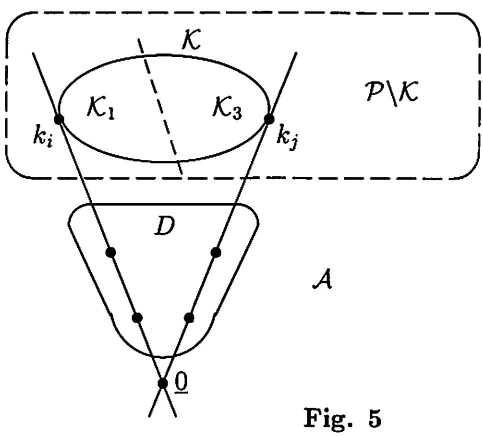

Fig. 5

We want to partition D into two parts D_1 and D_3 such that the 10 points $x_1, \ldots, x_{10}$ from class (III) are on exactly one line having two points in D_1. We use the following observation.

Observation 4.5 *Let x be a point of $\mathcal{P} \setminus \mathcal{K}$ and L a bisecant to $\mathcal{K}$ through x. Then, in the plane generated by L and $\underline{0}$, there are two lines through x meeting D in two points each. More precisely, if L contains the points k_i and k_j of $\mathcal{K}$, then x is*

(i) either on the lines $\overline{a_i a_j}$ and $\overline{b_i b_j}$

(ii) or on the lines $\overline{a_i b_j}$ and $\overline{b_i a_j}$.

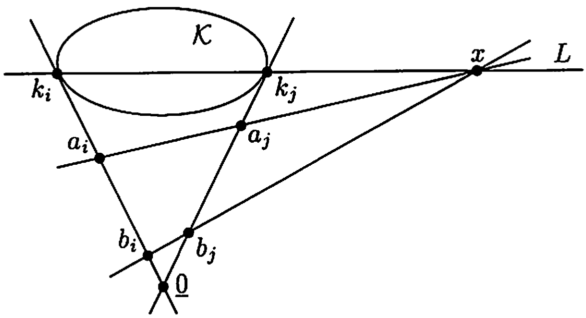

Fig. 6 Observation 4.5

Conversely, any line of A through x meeting D in two points corresponds to a bisecant (within $\mathcal{P}$) through x.

If x is on a tangent to $\mathcal{K}$ in the point k_l, then the lines $\overline{xa_l}$ and $\overline{xb_l}$ meet D only in a_l and b_l respectively, but not in any other point. Conversely, any line of $\mathcal{A}$ through x meeting D in exactly one point corresponds to a tangent to $\mathcal{K}$ through x.

Note that D has the property that the only lines of PG(4,3) meeting D in more than two points are the lines $\overline{\underline{0}k}$ for some $k \in \mathcal{K}$ (which are contained in $D \cup \{\underline{0}\} \cup \mathcal{K}$).

Each of the points $x_1, \ldots, x_{10}$ is on one bisecant of type 3/3. We can assume without loss of generality that

x_1 is on bisecant $\overline{k_6 k_7}$, x_2 is on bisecant $\overline{k_6 k_8}$,
x_3 is on bisecant $\overline{k_6 k_9}$, x_4 is on bisecant $\overline{k_6 k_{10}}$,
x_5 is on bisecant $\overline{k_7 k_8}$, x_6 is on bisecant $\overline{k_7 k_9}$,
x_7 is on bisecant $\overline{k_7 k_{10}}$, x_8 is on bisecant $\overline{k_8 k_9}$,
x_9 is on bisecant $\overline{k_8 k_{10}}$, x_{10} is on bisecant $\overline{k_9 k_{10}}$.

Moreover, using the above observation, we may also assume that

$$\begin{array}{llll} x_1 \text{ is on lines} & \overline{a_6a_7} & \text{and} & \overline{b_6b_7}, \\ x_2 \text{ is on lines} & \overline{a_6a_8} & \text{and} & \overline{b_6b_8}, \\ x_3 \text{ is on lines} & \overline{a_6a_9} & \text{and} & \overline{b_6b_9}, \\ x_4 \text{ is on lines} & \overline{a_6a_{10}} & \text{and} & \overline{b_6b_{10}}. \end{array}$$

We define $D_1 := \{a_6, a_7, a_8, a_9, a_{10}\}$, $D_3 := \{a_1, a_2, a_3, a_4, a_5, b_1, \ldots, b_{10}\}$. Then we know that each of the points $x_1, \ldots, x_4$ is on (precisely) one line meeting D_1 in two points. We still have to show this for $x_5, \ldots, x_{10}$.

Lemma 4.6 *With x_1, x_2 and x_5 as above,*

(i) x_1, x_2 and x_5 are on a common line in $\mathcal{P}$,

(ii) x_5 is on lines $\overline{a_7a_8}$ and $\overline{b_7b_8}$.

Proof:

(i) Look at the plane $P_1 \subset \mathcal{P}$ generated by k_6, k_7 and k_8. P_1 contains x_1, x_2 and x_5 and the remaining points y_1, y_2, y_5 of the bisecants $\overline{k_6k_7}, \overline{k_6k_8}$ and $\overline{k_7k_8}$ respectively (as in Lemma 4.2). Suppose the line $\overline{x_1x_2}$ does not contain x_5. Then, as any two lines meet in P_1, $\overline{x_1x_2}$ has to meet the line $\overline{k_7k_8}$ in y_5. It follows that $\overline{x_1y_2}$ meets $\overline{k_7k_8}$ in x_5 and $\overline{y_1y_2}$ contains y_5. Thus, if x_1, x_2 and x_5 are not on a common line, then y_1, y_2 and y_5 must be collinear. We want to show that this is not possible.

So suppose y_1, y_2 and y_5 are on a common line L.

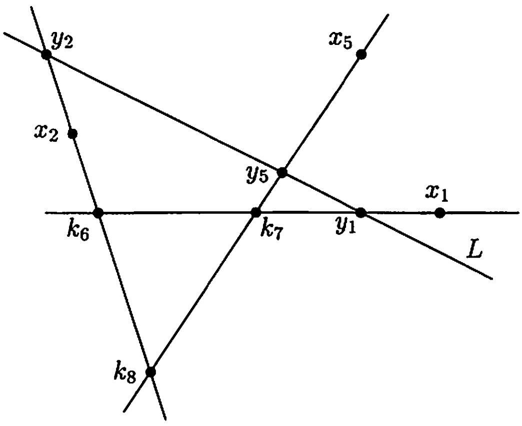

Fig. 7 Lemma 4.6

The plane P_1 contains k_6, k_7 and k_8, so the fourth point of the corresponding oval must be in $\mathcal{K}_1$. Let this point be k_1. Now there are two possibilities.

(a) If L does not meet $\mathcal{K}$, then L is contained in two oval planes (one of them being P_1) and two tangent planes. Thus, y_1, y_2 and y_5 have two tangent points in common, and at least one of these tangent points must be in $\mathcal{K}_1$ (otherwise the second oval plane P_2 corresponds to an oval contained in $\mathcal{K}_1$). Call this tangent point k_2.
As we saw in Lemma 4.2, the lines $\overline{x_1 k_2}$, $\overline{x_2 k_2}$ and $\overline{x_5 k_2}$ all have to be bisecants of type 1/3 to $\mathcal{K}$. Now

$$\begin{aligned} \overline{x_1 k_2} \cap \mathcal{K}_3 &\in \{k_8, k_9, k_{10}\}, \\ \overline{x_2 k_2} \cap \mathcal{K}_3 &\in \{k_7, k_9, k_{10}\}, \\ \overline{x_5 k_2} \cap \mathcal{K}_3 &\in \{k_6, k_9, k_{10}\}. \end{aligned}$$

Thus, at least one of these bisecants contains a point from $\{k_6, k_7, k_8\}$, so that k_2 is on (at least) one of the lines $\overline{x_1 k_8}$, $\overline{x_2 k_7}$, $\overline{x_5 k_6}$. But that implies that $k_2 \in P_1$, which contradicts our assumption that k_2 is in one of the tangent planes containing L.

(b) If L meets $\mathcal{K}$, then $L \cap \mathcal{K}$ must be a point of P_1, and thus must be k_1. Then k_1 is a common tangent point for y_1, y_2 and y_5, and L is contained in one tangent plane (the tangent plane for k_1) and three oval planes (one of them being P_1). Moreover, by Lemma 4.2, the lines $\overline{x_1 k_1}, \overline{x_2 k_1}$ and $\overline{x_5 k_1}$ must be bisecants to $\mathcal{K}$, and as k_1, k_6, k_7, k_8 form an oval, we must have

$$\begin{aligned} &x_1 \text{ is on bisecant } \overline{k_1 k_8}, \\ &x_2 \text{ is on bisecant } \overline{k_1 k_7}, \\ &x_5 \text{ is on bisecant } \overline{k_1 k_6}. \end{aligned}$$

Now x_1, x_2 and x_5 each must be on a further bisecant of type 1/3. As each of these three bisecants must contain one of the points k_9, k_{10}, either k_9 or k_{10} must be a bisecant point for (at least) two of the points x_1, x_2 and x_5. We may assume without loss of generality that k_9 is on bisecants with the points x_1 and x_2. Let k_2 be the point of $\mathcal{K}_1$ on the line $\overline{x_1 k_9}$. Then y_1 has tangent points k_1, k_2, k_8 and k_9, and we may assume w.l.o.g. that y_1 is on bisecants $\overline{k_3 k_4}$ and $\overline{k_5 k_{10}}$.

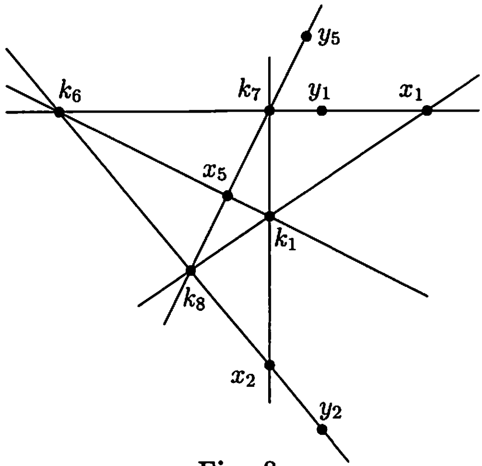

Fig. 8

Consider the plane P_2 generated by k_1, k_3 and k_4. The fourth point of the corresponding oval O_2 must be a tangent point of y_1, and it must be in $\mathcal{K}_3$. If O_2 contains k_8, then x_1 and y_1 are in P_2 (because $x_1 \in \overline{k_1k_8}$, $y_1 \in \overline{k_3k_4}$), so that $\overline{x_1y_1} = \overline{k_6k_7}$ is in P_2, contradiction. Therefore, we must have

$$(O_2) \quad k_1k_3k_4k_9$$

Moreover, consider the plane P_3 generated by k_2, k_3 and k_4. The corresponding oval O_3 must have a tangent point of y_1 contained in $\mathcal{K}_3$ as fourth point, too. Obviously, this point cannot be k_9, so that

$$(O_3) \quad k_2k_3k_4k_8.$$

Now we saw above that $\overline{x_2k_9}$ is a bisecant of type 1/3. Obviously, $\overline{x_2k_9}$ cannot meet $\mathcal{K}_1$ in k_1 or k_2. If $\overline{x_2k_9}$ contained k_3, then $k_1k_3k_7k_9$ would be an oval (generated by two bisecants through x_2) having three points in common with O_2. Analogously, k_4 cannot be contained in $\overline{x_2k_9}$. Therefore, the only possibility is that x_2 is on the bisecant k_5k_9. This means that x_2 has tangent points k_2, k_3, k_4 and k_{10}. These four points form an oval, which has three points in common with O_8, contradiction.

Thus, as none of the two cases is possible, y_1, y_2 and y_5 cannot be on a common line, and we must have that x_1, x_2 and x_5 are collinear.

(ii) Suppose x_5 is not on the line $\overline{a_7a_8}$. Then x_5 must be on $\overline{a_7b_8}$ and $\overline{b_7a_8}$, and y_5 has to be on the lines $\overline{a_7a_8}$ and $\overline{b_7b_8}$. By (i), x_1, x_2 and

x_5 are on a line L. This line together with the point a_6 generates a plane E which also contains the points a_7 and a_8, and the line $\overline{a_7a_8}$. Now $\overline{a_7a_8}$ meets $\mathcal{P}$ in some point z, and as $E \cap \mathcal{P} = L$, z must be a point of L. But if y_5 is on the line $\overline{a_7a_8}$, we must have $z = y_5$, so that y_5 would have to be on L, which is not possible. It follows that x_5 is on the lines $\overline{a_7a_8}$ and $\overline{b_7b_8}$.

□

Using the same arguments as in the above lemma for the other triangles $k_ik_jk_l$ of points in $\mathcal{K}_3$, we can show that

$$\begin{array}{lll} x_6 \text{ is on lines} & \overline{a_7a_9} & \text{and} \quad \overline{b_7b_9}, \\ x_7 \text{ is on lines} & \overline{a_7a_{10}} & \text{and} \quad \overline{b_7b_{10}}, \\ x_8 \text{ is on lines} & \overline{a_8a_9} & \text{and} \quad \overline{b_8b_9}, \\ x_9 \text{ is on lines} & \overline{a_8a_{10}} & \text{and} \quad \overline{b_8b_{10}}, \\ x_{10} \text{ is on lines} & \overline{a_9a_{10}} & \text{and} \quad \overline{b_9b_{10}}. \end{array}$$

We define the partition $\Pi = (\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ of PG(4,3) as follows:

$$\begin{array}{lll} \Omega_1 & := & D_1 \cup \mathcal{K}_1 = \{a_6, \ldots, a_{10}, k_1, \ldots, k_5\} \\ \Omega_3 & := & D_3 \cup \mathcal{K}_3 = \{a_1, \ldots, a_5, b_1, \ldots, b_{10}, k_6, \ldots, k_{10}\} \\ \Omega_4 & := & \{\underline{0}\} \\ \Omega_2 & := & PG(4,3) \setminus (D \cup \mathcal{K} \cup \{\underline{0}\}). \end{array}$$

It still requires some work to show that this partition defines a distance-regular antipodal 3–cover of Her(2,3).

Lemma 4.7 *Any point from Ω_2 is on exactly one line meeting Ω_1 in two points.*

Proof: There are three types of lines having two points in Ω_1:

a) lines containing two points from $\mathcal{K}_1$, i.e. bisecants of type 1/1,

b) lines containing two points from D_1,

c) lines containing one point from $\mathcal{K}_1$ and one point from D_1.

Note that no line can have more than two points in Ω_1 because such a line would imply that there is a line in $\mathcal{P}$ meeting the ovoid $\mathcal{K}$ in more than two points. The lines from a) are lines of $\mathcal{P}$, whereas the lines from b) and c) are lines of $\mathcal{A}$, i.e. they contain exactly one point from $\mathcal{P}$ and three points from $\mathcal{A}$.

(i) Any point $z \in \Omega_2 \cap \mathcal{P} = \mathcal{P} \setminus \mathcal{K}$ is on exactly one line having two points in Ω_1: Note that a point $z \in \mathcal{P} \setminus \mathcal{K}$ cannot be on a line from c). Moreover, we have seen in Corollary 4.3 that z must belong to one of three classes according to the types of bisecants z is on. If z is from class (I) or (II), then z is on (exactly) one line from a). We have chosen D_1 in such a way that each of the points $x_1, \dots, x_{10}$ from class (III) is on exactly one line from b). As there are $\frac{5 \cdot 4}{2} = 10$ lines as in b), and each of them meets $\mathcal{P}$ in exactly one point, it is clear that the statement of the lemma holds for any point z from $\mathcal{P} \setminus \mathcal{K}$.

(ii) Any point $z \in \Omega_2 \cap \mathcal{A}$ is on exactly one line having two points in Ω_1: We know that z cannot be on a line from a). Each of the 10 lines from b) contains exactly one point from $\{x_1, \dots, x_{10}\}$ and one point of $\Omega_2 \cap \mathcal{A}$. The 25 lines from c) each contain two points in $\Omega_2 \cap \mathcal{A}$. Thus, as $|\Omega_2 \cap \mathcal{A}| = 60$, it suffices to show that no point $z \in \Omega_2 \cap \mathcal{A}$ is on two lines from b) and/or c).

1.) Suppose z is on two lines $\overline{a_p a_q}$ and $\overline{a_r a_s}$ from b). Then the point $x := \overline{0z} \cap \mathcal{P}$ has to be on two bisecants of type 3/3 (namely $\overline{k_p k_q}$ and $\overline{k_r k_s}$), which is not possible.

2.) Suppose z is on two lines $\overline{k_i a_p}$ and $\overline{k_j a_q}$ from c). Then the plane P defined by these two lines meets $\mathcal{P}$ in a line L_1. As L_1 contains k_i, $k_j \in \mathcal{K}_1$, it is a bisecant to $\mathcal{K}$ of type 1/1. In the plane P, $L_2 := \overline{a_p a_q}$ meets L_1 in a point $x \in \mathcal{P} \setminus \mathcal{K}$. In fact, by our choice of $D_1 \ni a_p, a_q$, the point x must be one of the points $x_1, \dots, x_{10}$ from class (III) of Corollary 4.3, but then x cannot be on a bisecant of type 1/1. Contradiction.

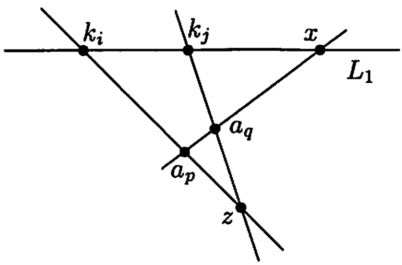

Fig. 9 Lemma 4.7

3.) Suppose z is on a line $\overline{a_q a_r}$ from b) and on a line $\overline{k_i a_p}$ from c), i.e. $i \in \{1, \dots, 5\}$ and $p, q, r \in \{6, \dots, 10\}$. Then these two lines generate a plane P_1, which meets $\mathcal{P}$ in a line $L_1 \ni k_i$. Moreover, L_1 contains the points $x_j := \overline{a_q a_r} \cap \mathcal{P}$, $x_k := \overline{a_p a_q} \cap \mathcal{P}$ and

$x_l := \overline{a_p a_r} \cap \mathcal{P}$.

As x_j, x_k, x_l are points from class (III) of Corollary 4.3, L_l is a line as we found in Lemma 4.6 (i). We show that such a line cannot contain a point $k_i \in \mathcal{K}_1$. For convenience, we substitute:

$$\begin{array}{lll} k_1 := k_i, & & \\ k_6 := k_p, & k_7 := k_q, & k_8 := k_r, \\ x_1 := x_k, & x_2 := x_l, & x_3 := x_j. \end{array}$$

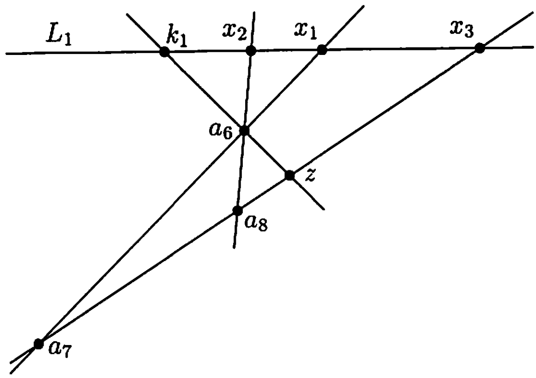

Fig. 10 Lemma 4.7

Then x_1, x_2, x_3 are points from class (III) of Corollary 4.3, and the bisecants of type 3/3 through them are $\overline{x_1 k_6 k_7}$, $\overline{x_2 k_6 k_8}$, and $\overline{x_3 k_7 k_8}$. The line L_1 consists of the points x_1, x_2 and x_3 and $k_1 \in \mathcal{K}_1$. Then $k_1 k_6 k_7 k_8$ form an oval and, as k_1 is a common tangent point for x_1, x_2 and x_3,

k_8 is a tangent point for x_1,
k_7 is a tangent point for x_2,
k_6 is a tangent point for x_3.

Now x_1, x_2, x_3 are each on two bisecants of type 1/3, and as the only remaining points from $\mathcal{K}_3$ are k_9 and k_{10}, we must have that $\overline{x_i k_9}$ and $\overline{x_i k_{10}}$ meet $\mathcal{K}_1$ for $i = 1, 2, 3$. Let $k_2 := \overline{x_1 k_9} \cap \mathcal{K}_1$ and $k_3 := \overline{x_1 k_{10}} \cap \mathcal{K}_1$, then $k_2 k_3 k_9 k_{10}$ is an oval. Now $\overline{x_2 k_9} \cap \mathcal{K}_1$ cannot be k_2. Also, we cannot have $k_3 = \overline{x_2 k_9} \cap \mathcal{K}_1$ (because then the plane determined by x_1, k_9 and k_{10} would contain x_2, and also k_1). Thus, we can assume $k_4 = \overline{x_2 k_9} \cap \mathcal{K}_1$ and $k_5 = \overline{x_2 k_{10}} \cap \mathcal{K}_1$. But $\overline{x_3 k_9}$ also has to contain a point from $\mathcal{K}_1$, and this point cannot be k_1, k_2, k_3, k_4 or k_5. Contradiction.

□

Theorem 4.8 *The partition* $\Pi = (\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ *as defined above gives rise to a linear distance-regular graph* Γ *on the points of* $AG(5,3)$. *The intersection numbers of* Γ *are equal to the intersection numbers of a 3-cover of* $Her(2,3)$.

Proof:
Let Γ be the graph having the points of AG(5,3) as vertex set and adjacency defined by

$$x \sim y \text{ in } \Gamma \quad \Longleftrightarrow \quad \overline{xy} \quad \text{meets PG(4,3) in a point of } \Omega_1.$$

We use Theorem 2.3 to find all the intersection numbers $p_{ij}(l)$ for Γ. If we find the $p_{ij}(l)$ to be the same as for a 3–cover of Her (2,3), then the properties (ii) and (iii) of Corollary 2.4 are clear, so that Γ is a linear distance-regular graph.
Note that the intersection numbers of a 3–cover of Her(2,3) can be calculated from the intersection array $(20,18,4,1;1,2,18,20)$ of such a cover using the recurrence relation given in 1.1.

It is easy to see that in Γ

$$\begin{aligned} p_{11}(0) &= 2\cdot|\Omega_1| = 20, & p_{22}(0) &= 2\cdot|\Omega_2| = 180,\\ p_{33}(0) &= 2\cdot|\Omega_3| = 40, & p_{44}(0) &= 2\cdot|\Omega_4| = 2. \end{aligned}$$

We now look at $x \in \Omega_l$, for $l = 1,\ldots,4$ and at the structure of the lines through x.

(i) $\Omega_4 = \{\underline{0}\}$.
Through the point $\underline{0}$ there are 10 lines of structure [1, 0, 2, 1] and 30 lines of structure [0, 3, 0, 1].
That gives $p_{11}(4) = p_{12}(4) = p_{23}(4) = 0$. Obviously, as $|\Omega_4| = 1$, we have $p_{14}(4) = p_{24}(4) = p_{34}(4) = 0$ and $p_{44}(4) = 1$. Also, we find

$$\begin{aligned} p_{13}(4) &= 10\cdot 1\cdot 2 = 20,\\ p_{33}(4) &= 10\cdot 2\cdot 1 = 20,\\ p_{22}(4) &= 30\cdot 3\cdot 2 = 180. \end{aligned}$$

(ii) Let $x \in \Omega_1$. Through x there are

1	line of structure	[1, 0, 2, 1]	(the line through $\underline{0} \in \Omega_4$),
9	lines of structure	[2, 2, 0, 0]	(lines through the other points of Ω_1),
18	lines of structure	[1, 2, 1, 0]	(lines through points of Ω_3),
12	lines of structure	[1, 3, 0, 0].	

That gives $p_{14}(1) = p_{24}(1) = p_{44}(1) = 0$, $p_{34}(1) = 2$, $p_{13}(1) = 0$,

$p_{11}(1) = 1$, and

$$\begin{aligned} p_{33}(1) &= 1 \cdot 2 = 2, \\ p_{22}(1) &= 9 \cdot 2 + 18 \cdot 2 + 12 \cdot 3 \cdot 2 = 126, \\ p_{12}(1) &= 9 \cdot 2 = 18, \\ p_{23}(1) &= 18 \cdot 2 = 36. \end{aligned}$$

(iii) Let $x \in \Omega_3$. Then x is on

1	line of structure	[1, 0, 2, 1]	(the line through Ω_4),
9	lines of structure	[1, 2, 1, 0]	(lines through points of Ω_1),
18	lines of structure	[0, 2, 2, 0]	(lines through other points of Ω_3),
12	lines of structure	[0, 3, 1, 0].	

That gives $p_{14}(3) = p_{34}(3) = 1$, $p_{24}(3) = p_{44}(3) = p_{11}(3) = 0$, $p_{13}(3) = 1$, $p_{33}(3) = 1$, and

$$\begin{aligned} p_{22}(3) &= 9 \cdot 2 + 18 \cdot 2 + 12 \cdot 3 \cdot 2 = 126, \\ p_{12}(3) &= 9 \cdot 2 = 18, \\ p_{23}(3) &= 18 \cdot 2 = 36. \end{aligned}$$

(iv) Let $x \in \Omega_2$. We know that x is on

1	line of structure [0, 3, 0, 1]	(the line through Ω_4),
1	line of structure [2, 2, 0, 0]	(exactly one line meeting Ω_1 in two points).

We don't know yet what structure the other lines through x have.

(a) Suppose $x \in \mathcal{P}$. Then x is on three bisecants and four tangents to $\mathcal{K}$, and by Lemma 4.3, x is in one of three classes according to the types of the three bisecants. By observation 4.5, each bisecant and each tangent gives rise to two lines of $\mathcal{A}$ through x:

- A bisecant of type 1/1 means that x is on two lines of $\mathcal{A}$ having structure [0, 2, 2, 0].

- A bisecant of type 1/3 means that x is on one line of structure [1, 2, 1, 0] and one line of structure [0, 2, 2, 0] in $\mathcal{A}$.

- A bisecant of type 3/3 means that x is
  either on two lines of structure [1, 2, 1, 0],
  or on one line of structure [2, 2, 0, 0] and

one line of structure [0, 2, 2, 0]

depending on whether x is in class (II) or (III) of Lemma 4.3 (by construction of D_1).

- A tangent to a point in $\mathcal{K}_1$ means that x is on two lines meeting D_3 in exactly one point each.

- A tangent to a point in $\mathcal{K}_3$ means that x is on one line meeting D_1 in exactly one point, and one line meeting D_3 in exactly one point.

We see that any tangent through x gives rise to one line of structure [1,3,0,0] and two lines of structure [0,3,1,0] (one of these three lines is the tangent itself), so that any point $x \in \mathcal{P} \cap \Omega_2$ is on 4 lines of structure [1,3,0,0] and 8 lines of structure [0,3,1,0]. Looking at the three classes of points from $\Omega_2 \cap \mathcal{P}$ separately, we find the structures of the lines through x meeting $\Omega_1 \cup \Omega_3$ in two points. Note that from Lemma 4.7 we know that any point in Ω_2 is on exactly one line meeting Ω_1 in two points.

(I) If x is on bisecants of type 1/1, 1/3, 1/3, then x is on lines of structure

[2, 2, 0, 0],	[1, 2, 1, 0] $\cdot$ 2	(the bisecants),
[0, 2, 2, 0] $\cdot$ 4,	[1, 2, 1, 0] $\cdot$ 2	(in $\mathcal{A}$, coming from the bisecants).

(II) If x is on bisecants of type 1/1, 1/3 and 3/3, then x is on lines of structure

[2, 2, 0, 0], [1, 2, 1, 0], [0, 2, 2, 0] (the bisecants),
[0, 2, 2, 0] $\cdot$ 3, [1, 2, 1, 0] $\cdot$ 3 (in $\mathcal{A}$, coming from the bisecants).

(III) If x is on bisecants of type 1/3, 1/3 and 3/3, then x is on lines of structure

[0, 2, 2, 0], [1, 2, 1, 0] $\cdot$ 2 (the bisecants),
[0, 2, 2, 0] $\cdot$ 3, [1, 2, 1, 0] $\cdot$ 2, [2, 2, 0, 0] (in $\mathcal{A}$, coming from the bisecants).

In each of the three cases, the lines through x meeting $\Omega_1 \cup \Omega_3$ in two points are 1 line of structure [2, 2, 0, 0], 4 lines of structure [0, 2, 2, 0], and 4 lines of structure [1, 2, 1, 0]. The remaining lines through x then have all points in Ω_2, so that x is on 18 lines of structure [0, 4, 0, 0].

(b) Now suppose $x \in \mathcal{A}$. Let z be the point where the line $\overline{0x}$ meets $\mathcal{P}$. Each line through x which meets $\Omega_1 \cup \Omega_3$ in one or two points corresponds to a tangent or a bisecant to $\mathcal{K}$ through z. Vice versa, any bisecant or tangent through z gives rise to three lines through x meeting $\Omega_1 \cup \Omega_3$ in two points or one point respectively.

- If z is on a bisecant of type 1/1, the three lines through x have structure $[1, 2, 1, 0] \cdot 2, \quad [0, 2, 2, 0]$.

- If z is on a bisecant of type 1/3, the three lines through x have structure

  either (α) $[0, 2, 2, 0] \cdot 2, \quad [2, 2, 0, 0]$
  or (β) $[1, 2, 1, 0] \cdot 2, \quad [0, 2, 2, 0]$.

- If z is on a bisecant of type 3/3, the three lines through x have structure

  either (α) $[0, 2, 2, 0] \cdot 2, \quad [2, 2, 0, 0]$
  or (β) $[1, 2, 1, 0] \cdot 2, \quad [0, 2, 2, 0]$.

- Any tangent through z gives rise to lines through x having structure $[1, 3, 0, 0], \quad [0, 3, 1, 0] \cdot 2$.

In Lemma 4.7 we showed that any point x from Ω_2 is on exactly one line of structure $[2, 2, 0, 0]$. Thus, only one of the bisecants through z can give rise to lines as in (α). This helps us to determine the structure of the lines through x. We find that in each of the three cases

(I) z is in class (I),
(II) z is in class (II),
(III) z is in class (III),

x is on	1	line of structure	$[2, 2, 0, 0]$,
	1	line of structure	$[0, 3, 0, 1]$,
	4	lines of structure	$[0, 2, 2, 0]$,
	4	lines of structure	$[1, 2, 1, 0]$,
	4	lines of structure	$[1, 3, 0, 0]$,
	8	lines of structure	$[0, 3, 1, 0]$,
	18	lines of structure	$[0, 4, 0, 0]$.

Thus, for any point x in Ω_2, the structure of the lines through x is the same, and we get $p_{11}(2) = 2$, $p_{14}(2) = p_{34}(2) = p_{44}(2) = 0$,

$p_{24}(2) = 1 \cdot 2 = 2$, $p_{13}(2) = 4$, and

$$\begin{aligned} p_{12}(2) &= 1 \cdot 2 + 4 \cdot 1 + 4 \cdot 2 = 14, \\ p_{23}(2) &= 4 \cdot 2 + 4 \cdot 1 + 8 \cdot 2 = 28, \\ p_{33}(2) &= 4 \cdot 2 = 8, \\ p_{22}(2) &= 1 \cdot 2 + 4 \cdot 2 + 8 \cdot 2 + 18 \cdot 3 \cdot 2 + 1 = 135. \end{aligned}$$

As we found all the $p_{ij}(l)$ to be equal to the intersection numbers of a 3-cover of Her(2,3), Corollary 2.4 implies that the partition $\Pi = (\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ indeed defines a linear distance-regular graph. □

Theorem 4.9 *The graph Γ defined by the partition $\Pi = (\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ is a linear 3-cover of Her(2,3).*

Proof: We give a covering map $\gamma : V(\Gamma) = AG(5,3) \to AG(4,3) = V(\mathrm{Her}(2,3))$ using the coordinatization of PG(5,3):

$$PG(5,3) = \{ \begin{pmatrix} \dot{x} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \text{ GF(3) } : \dot{x}, x_0, \ldots, x_4 \in \text{ GF(3) not all } = 0 \}.$$

Let the partition $\Pi = (\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ corresponding to Γ be defined on the hyperplane

$$\overline{\mathcal{P}} = \{ x \in PG(5,3) : \dot{x} = 0, x_0, \ldots, x_4 \in \text{ GF(3) not all } = 0 \} \cong PG(4,3)$$

of PG(5,3), so that the vertices of Γ are the points of

$$\overline{\mathcal{A}} = \{ x \in PG(5,3) : \dot{x} \neq 0, x_0, \ldots, x_4 \in \text{ GF(3) } \} \cong AG(5,3).$$

Moreover, let $\mathcal{P} \cong$ PG(3,3) be the hyperplane

$$\mathcal{P} = \{ \begin{pmatrix} 0 \\ 0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \text{ GF(3) } : x_1, \ldots, x_4 \in \text{ GF(3) not all } = 0 \}$$

of $\overline{\mathcal{P}}$ containing the ovoid $\mathcal{K}$, so that the partition $\sigma = (\mathcal{K}, \mathcal{P} \setminus \mathcal{K})$ of $\mathcal{P}$ corresponds to the graph Her(2,3) having the points of $\mathcal{A} \cong$ AG(4,3) as vertex set, where

$$\mathcal{A} = \{ (0, 1, x_1, x_2, x_3, x_4)^T \cdot \text{ GF(3) } : x_1, \ldots, x_4 \in \text{GF(3) } \}.$$

As all ovoids in PG(3,3) are projectively equivalent, and are equivalent to the elliptic quadrics in PG(3,3), we can without loss of generality take the ovoid $\mathcal{K}$ to be

$$\mathcal{K} = \{ \begin{pmatrix} 0 \\ 0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \mathrm{GF}(3) \in \mathcal{P} : {x_1}^2 + {x_2}^2 + {x_3}^2 - {x_4}^2 = 0\}.$$

We see that the set D of points on lines $\overline{\underline{0}k}$ for $k \in \mathcal{K}$ (where $\underline{0} = (0,1,0,0,0,0)^T \cdot \mathrm{GF}(3) \in \mathcal{A}$) used in the construction of Π above then is

$$D = \{x \in \mathcal{A} : {x_1}^2 + {x_2}^2 + {x_3}^2 - {x_4}^2 = 0; x_1, \ldots, x_4 \in \mathrm{GF}(3) \text{ not all} = 0\}.$$

Look at the mapping

$$\gamma : \mathrm{PG}(5,3) \setminus \{\underline{0}\} \quad \longrightarrow \quad \overline{\mathcal{P}} \cong \mathrm{PG}(4,3)$$

$$\begin{pmatrix} \dot{x} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \mathrm{GF}(3) \longrightarrow \begin{pmatrix} 0 \\ \dot{x} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \mathrm{GF}(3)$$

Then γ has the following properties:

(i) γ maps $\overline{\mathcal{A}} \cong \mathrm{AG}(5,3)$ onto $\mathcal{A} \cong \mathrm{AG}(4,3)$, so that γ maps the vertices of Γ to the vertices of Her(2,3). Each point $x = (0,1,x_1,x_2,x_3,x_4)^T$ in $\mathcal{A}$ has three preimages under γ, namely the points $x^{(i)} = (1,i,x_1,x_2,x_3,x_4)^T$ for $i = 0,1,2$. These three preimages of x have mutual distance 4 in Γ because they are on a common line $\overline{x^{(0)}x^{(1)}x^{(2)}}$ which meets $\overline{\mathcal{P}}$ in $\Omega_4 = \{\underline{0}\}$.

(ii) γ maps $\overline{\mathcal{P}} \setminus \{\underline{0}\}$ onto $\mathcal{P} \cong \mathrm{PG}(3,3)$, so that the partition Π is mapped to a partition $\gamma(\Pi)$ of $\mathcal{P}$. We see that $\gamma(\Omega_1 \cup \Omega_3) = \gamma(D \cup \mathcal{K}) = \mathcal{K}$ and $\gamma(\Omega_2) = \mathcal{P} \setminus \mathcal{K}$, so that $\gamma(\Pi) = \sigma$.

(iii) If $x^{(i)} \sim y^{(j)}$ in Γ, then $\gamma(x^{(i)}) = x \sim y = \gamma(y^{(j)})$ in Her(2,3) because if the line $\overline{x^{(i)}y^{(j)}}$ of $\overline{\mathcal{A}}$ meets $\overline{\mathcal{P}}$ in Ω_1, then the line $\overline{xy}$ of $\mathcal{A}$ meets $\mathcal{P}$ in $\mathcal{K}$.

Thus we know that γ is a graph homomorphism mapping Γ to Her(2,3). It remains to show that γ is a local isomorphism. Suppose $x^{(i)} \in V(\Gamma)$ has neighbours $y_1{}^{(j_1)}, \ldots, y_{20}{}^{(j_{20})}$, and let $x := \gamma(x^{(i)})$, $y_l := \gamma(y_l{}^{(j_l)})$ for $l = 1, \ldots, 20$. Then we know that $y_1, \ldots, y_{20}$ are the neighbours of x. As Γ has intersection number $p_{11}(1) = 1$, we can assume that $y_{2l-1}{}^{(j_{2l-1})} \sim y_{2l}{}^{(j_{2l})}$ for $l = 1, \ldots, 10$, and there are no other edges between the neighbours $y_1{}^{(j_1)}, \ldots, y_{20}{}^{(j_{20})}$ of $x^{(i)}$. Then we know that $y_{2l-1} \sim y_{2l}$ for $l = 1, \ldots, 10$ in Her(2,3). As Her(2,3) also has intersection number $p_{11}(1) = 1$, there are no other edges between the neighbours $y_1, \ldots, y_{20}$ of x. Thus,

$$\gamma : \{x^{(i)}, y_1{}^{(j_1)}, \ldots, y_{20}{}^{(j_{20})}\} \quad \longrightarrow \quad \{x, y_1, \ldots, y_{20}\}$$

is an isomorphism. Note that $p_{11}(1) = 1$ for Γ (and for Her(2,3) respectively) is true if and only if there is no line in $\overline{\mathcal{P}}$ meeting Ω_1 in more than two points (and no line in $\mathcal{P}$ meeting $\mathcal{K}$ in more than two points respectively). If the points $x^{(i)}, y_{2l-1}{}^{(j_{2l-1})}$ and $y_{2l}{}^{(j_{2l})}$ form a triangle in Γ, then they are on a common line in $\overline{\mathcal{A}}$ meeting $\overline{\mathcal{P}}$ in Ω_1. Analogously, if x, y_{2l-1} and y_{2l} form a triangle in Her(2,3), then they are on a common line in $\mathcal{A}$ meeting $\mathcal{P}$ in a point of $\mathcal{K}$.

□

Theorem 4.10 *The linear 3–cover of Her(2,3) is unique.*

Proof: The proof has two parts. In the first part, we show that all partitions $\Pi = (\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ constructed as above are projectively equivalent, so that they all yield the same graph. In the second part we prove that for any linear 3–cover of Her(2,3), the corresponding partition of PG(4,3) can be constructed by the above method.

Part I: We look again at our construction of the partition Π and show that wherever we chose one out of several possibilities, all possible choices are projectively equivalent.

(i) As all ovoids in PG(3,3) are projectively equivalent, it doesn't matter which ovoid $\mathcal{K}$ we start with.

(ii) In Lemma 4.4 we found 72 distinct partitions $(\mathcal{K}_1, \mathcal{K}_3)$ of $\mathcal{K}$ having property (∗). We show that all these partitions are projectively equivalent. As $PGO_-(4,3)$ is triply transitive on the points of an ovoid $\mathcal{K}$, we can choose any 3–triple of points of $\mathcal{K}$ to be in $\mathcal{K}_1$ and it suffices to prove that the six possible partitions of $\mathcal{K}$ having a given triple of points in $\mathcal{K}_1$ are projectively equivalent. We use the coordinatization of the points of PG(4,3):

$$x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \mathrm{GF}(3) \qquad \text{for } x_0, \ldots, x_4 \in \mathrm{GF}(3) \text{ not all } = 0.$$

Again, let $\mathcal{P} = \{x \in \text{PG}(4,3) : x_0 = 0\}$ and $\mathcal{K} = \{x \in \mathcal{P} : {x_1}^2 + {x_2}^2 + {x_3}^2 - {x_4}^2 = 0\}$. Then $\mathcal{K}$ consists of the following points:

$$\begin{array}{ll} k_1 = (0,0,0,1,1) \cdot \text{GF}(3)\ , & k_6 = (0,0,1,0,2) \cdot \text{GF}(3)\ , \\ k_2 = (0,0,0,1,2) \cdot \text{GF}(3)\ , & k_7 = (0,1,0,0,1) \cdot \text{GF}(3)\ , \\ k_3 = (0,0,1,0,1) \cdot \text{GF}(3)\ , & k_8 = (0,1,2,1,0) \cdot \text{GF}(3)\ , \\ k_4 = (0,1,0,0,2) \cdot \text{GF}(3)\ , & k_9 = (0,1,2,2,0) \cdot \text{GF}(3)\ , \\ k_5 = (0,1,1,1,0) \cdot \text{GF}(3)\ , & k_{10} = (0,1,1,2,0) \cdot \text{GF}(3)\ . \end{array}$$

Note that if we set

$$\begin{array}{lllll} a = k_1, & a' = k_2, & b = k_3, & c = k_4, & f = k_5, \\ b' = k_6, & c' = k_7, & d = k_8, & e = k_9, & g = k_{10}, \end{array}$$

then the elements of $\mathcal{K}$ form ovals as given in the proof of Lemma 4.4, and the partitions having property $(*)$ and satisfying $k_1, k_2, k_3 \in \mathcal{K}_1$ are

$$\begin{array}{lll} (1) & \mathcal{K}_1 = \{k_1,k_2,k_3,k_4,k_5\} & \mathcal{K}_3 = \{k_6,k_7,k_8,k_9,k_{10}\}, \\ (2) & \mathcal{K}_1 = \{k_1,k_2,k_3,k_4,k_{10}\} & \mathcal{K}_3 = \{k_5,k_6,k_7,k_8,k_9\}, \\ (3) & \mathcal{K}_1 = \{k_1,k_2,k_3,k_7,k_8\} & \mathcal{K}_3 = \{k_4,k_5,k_6,k_9,k_{10}\}, \\ (4) & \mathcal{K}_1 = \{k_1,k_2,k_3,k_7,k_9\} & \mathcal{K}_3 = \{k_4,k_5,k_6,k_8,k_{10}\}, \\ (5) & \mathcal{K}_1 = \{k_1,k_2,k_3,k_5,k_8\} & \mathcal{K}_3 = \{k_4,k_6,k_7,k_9,k_{10}\}, \\ (6) & \mathcal{K}_1 = \{k_1,k_2,k_3,k_9,k_{10}\} & \mathcal{K}_3 = \{k_4,k_5,k_6,k_7,k_8\}. \end{array}$$

We define E_{ij} to be the 5×5 matrix having ij-entry 1 and all other entries 0 and give elements of $\text{PGO}_-(4,3)$ mapping partition (1) to (2), ..., (6):

$$\begin{array}{lllll} (1) & \to & (2): & \beta_1 = E_{11} + 2E_{23} + 2E_{32} + E_{44} + E_{55}, \\ (1) & \to & (3): & \beta_2 = E_{11} + 2E_{23} + E_{24} + E_{25} + E_{32} \\ & & & \qquad +E_{43} + 2E_{44} + E_{45} + 2E_{53} + 2E_{54}, \\ (1) & \to & (4): & \beta_3 = E_{11} + E_{22} + 2E_{33} + E_{34} + E_{35} \\ & & & \qquad +E_{43} + 2E_{44} + E_{45} + 2E_{53} + 2E_{54}, \\ (1) & \to & (5): & \beta_4 = E_{11} + 2E_{23} + E_{24} + E_{25} + 2E_{32} \\ & & & \qquad +E_{34} + E_{35} + E_{42} + E_{43} + 2E_{45}, \\ (1) & \to & (6): & \beta_5 = E_{11} + 2E_{23} + E_{32} + E_{34} + E_{35} \\ & & & \qquad +2E_{42} + 2E_{44} + E_{45} + E_{52} + 2E_{54} \end{array}$$

(iii) Instead of the point $\underline{0} = (1,0,0,0,0)^T \in \text{AG}(4,3)$ we could have chosen any point z from $\mathcal{A}$. All these choices are projectively equivalent because, for any hyperplane H of a projective geometry $\mathcal{I}$ and two points $a, a' \notin H$, there is always a collineation of $\mathcal{I}$ fixing H and mapping a to a'.

(iv) When partitioning the set D into two parts, we have two choices for D_1:

(a) $D_1 = \{a_6, \ldots, a_{10}\}$,

(b) $D_1 = \{b_6, \ldots, b_{10}\}$.

These two choices are equivalent:
Let $k_l = (0, x_1, x_2, x_3, x_4)^T \cdot$ GF(3) be a point of $\mathcal{K}$. Then the two points a_l and b_l on the line $\overline{\underline{0}k_l}$ are $(1, x_1, x_2, x_3, x_4) \cdot$ GF(3) and $(1, 2x_1, 2x_2, 2x_3, 2x_4) \cdot$ GF(3). The collineation of PG(4,3) exchanging them (and thus exchanging the two choices for D_1) is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Thus, all partitions $\Pi = (\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ found by the construction presented above are projectively equivalent, and correspond to isomorphic graphs.

Part II: Suppose Γ is any linear 3-cover of Her(2,3). Then Γ has the points of $\overline{\mathcal{A}} \cong$ AG(5,3) as vertices, and corresponds to some partition $\tau = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)$ of $\overline{\mathcal{P}} \cong$ PG(4,3). We want to show that the partition τ can be found by our construction above. We use the intersection numbers $p_{ij}(l)$ of Γ and Theorem 2.3, and the fact that Γ is linear.

(i) As $p_{44}(4) = 1$, Ψ_4 must consist of a single point. We call this point $\underline{z}$. Also, from $p_{11}(0) = k_1 = 20$, $p_{22}(0) = k_2 = 180$ and $p_{33}(0) = k_3 = 40$, it is clear that $|\Psi_1| = 10$, $|\Psi_2| = 90$ and $|\Psi_3| = 20$.

(ii) From $p_{11}(4) = p_{12}(4) = p_{23}(4) = 0$, we see that there are no lines through $\underline{z}$

- meeting Ψ_1 in at least two points,
- meeting Ψ_1 and Ψ_2 in at least one point each,
- meeting Ψ_2 and Ψ_3 in at least one point each.

Moreover, as $p_{13}(4) = 20$, $p_{33}(4) = 20$ and $p_{22}(4) = 180$, the 40 lines through $\underline{z}$ are of two kinds:

(a) 10 lines meeting Ψ_1 in one point, Ψ_3 in two points, i.e. lines of structure $[1, 0, 2, 1]$,

(b) 30 lines meeting Ψ_2 in three points, i.e. lines of structure $[0, 3, 0, 1]$.

(iii) Denote $C := \Psi_1 \cup \Psi_3$. From $p_{11}(1) = 1$, $p_{11}(3) = 0$, $p_{13}(3) = 1$ and $p_{33}(3) = 1$ it follows that the only lines containing three points of C are the lines from (ii) (a) above (containing $\underline{z}$). All other lines can meet C in at most two points. Thus any point $x \in C$ is on

(a) one line $\overline{\underline{z}x}$ which contains two more points of C,

(b) 27 lines $\overline{xy}$ for some $y \in C \setminus \{\overline{\underline{z}x}\}$. These lines meet C in x and y and Ψ_2 in two points.

(c) 12 lines meeting C only in x, and Ψ_2 in three points.

(iv) Let x be a point in Ψ_2. From the $p_{ij}(2)$ we can see what structure the lines through x must have:

(a) one line containing $\underline{z}$; this line has structure $[0, 3, 0, 1]$ (one of the lines of (ii)(b), note that $p_{24}(2) = 2$),

(b) one line having structure $[2, 2, 0, 0]$, i.e. meeting Ψ_1 in two points ($p_{11}(2) = 2$),

(c) 4 lines having structure $[1, 2, 1, 0]$ ($p_{13}(2) = 4$),

(d) 4 lines having structure $[1, 3, 0, 0]$ ($p_{12}(2) = 14$, the lines from (b) and (c) give already $2 + 4 = 6$ vertices w with $d(u, w) = 1$, $d(v, w) = 2$ for vertices u, v with $d(u, v) = 2$),

(e) 4 lines of structure $[0, 2, 2, 0]$ ($p_{33}(2) = 8$),

(f) 8 lines of structure $[0, 3, 1, 0]$ ($p_{23}(2) = 28$, use also the lines from (c) and (e)),

(g) 18 lines of structure $[0, 4, 0, 0]$.

(v) Next we want to show that there is a hyperplane H in $\mathcal{P} \cong \mathrm{PG}(4,3)$ which contains 5 points from Ψ_1 and 5 points from Ψ_3. Choose any point $a \in \Psi_2$. We know that a is on exactly one line L_1 of structure $[2, 2, 0, 0]$. Let p_1, p_2 be the points of L_1 in Ψ_1. Moreover, let q_1 and r_1 be the two points of Ψ_3 on the line $\overline{p_1\underline{z}}$, and q_2 and r_2 be the two points of Ψ_3 on $\overline{p_2\underline{z}}$. Moreover, choose one of the four lines of structure $[1, 2, 1, 0]$ through a. Such a line cannot contain q_i, r_i for $i = 1, 2$, because in the plane generated by L_1 and $\underline{z}$ the lines $\overline{aq_1}$ and $\overline{ar_1}$ meet $\overline{p_2\underline{z}}$ in q_2 or r_2 (so that these lines have structure $[0, 2, 2, 0]$). Thus, the line L_2 of structure $[1, 2, 1, 0]$ we chose contains a point $p_3 \in \Psi_1$ and $q_4 \in \Psi_3$, and no two of the points p_1, p_2, p_3, q_4 are on a common line with $\underline{z}$.

Then, L_1 and L_2 together generate a plane E which does not contain $\underline{z}$. Moreover, the set $\{p_1, p_2, p_3, q_4\}$ is an oval in E (there is no line in E containing more than two points from C, because $\underline{z} \notin E$). As the maximal cardinality of a set of points in a projective plane PG(2,3) no two of which are collinear is 4, this also implies that E cannot contain any more points from C. The plane E is contained in four hyperplanes $H_1, \ldots, H_4$ of $\overline{\mathcal{P}}$, and these hyperplanes partition the points of $\overline{\mathcal{P}} \setminus E$. In particular, they partition $C \setminus \{p_1, p_2, p_3, q_4\}$. Let H_1 be the hyperplane generated by E and the point $\underline{z}$. Then H_1 contains at least 12 points from C, namely the points on the lines $\overline{p_i \underline{z}}$ for $i = 1, 2, 3$ and $\overline{q_4 \underline{z}}$. Any hyperplane not containing $\underline{z}$ must meet each of the 10 lines of structure $[1, 0, 2, 1]$ through $\underline{z}$ in exactly one point, so that $|H_i \cap C| = 10$ for $i = 2, 3, 4$. Now $|C| = 30$, so that

$$\begin{aligned} 30 &= |E \cap C| + (|H_1 \cap C| - |E \cap C|) + 3 \cdot (10 - |E \cap C|) \\ &= 30 + |H_1 \cap C| - 3 \cdot |E \cap C|, \end{aligned}$$

which implies $|H_1 \cap C| = 12$. It remains to show that at least one (in fact, all) of the hyperplanes H_2, H_3, H_4 contains 5 points from Ψ_1 and 5 points from Ψ_3. We know that H_1 contains 4 points from Ψ_1 (namely p_1, p_2, p_3 and the point $p_4 := \overline{q_4 \underline{z}} \cap \Psi_1$). Thus, $H_2 \cup H_3 \cup H_4$ contains p_1, p_2, p_3 and the 6 remaining points $p_5, \ldots, p_{10}$ from Ψ_1. This means that at least one of H_2, H_3, H_4 contains at least two of the points $p_5, \ldots, p_{10}$. Suppose H_2 contains 3 (or more) of these points, i.e. a total of 6 (or more) points from Ψ_1. Then there are (at least) $\frac{6 \cdot 5}{2} = 15$ lines in H_2 containing two points in Ψ_1. Each of these lines has its other two points in $H_2 \cap \Psi_2$. As no point in Ψ_2 is on more than one line having two points in Ψ_1, the 30 points in $H_2 \cap \Psi_2$ all are on exactly one of these 15 lines. Note that this already shows $|H_2 \cap \Psi_1| \leq 6$. Then, there are 4 points in Ψ_1 not in H_2. Any two of them are on a common line, and each of these lines meets H_2 in a point of $H_2 \cap \Psi_2$. But then these would be points being on two lines of structure $[2, 2, 0, 0]$, which is not possible. Thus, H_2, H_3, H_4 cannot contain more than 5 points from Ψ_1, so that each of them contains exactly 5 points from Ψ_1, and 5 points from Ψ_3.

(vi) The set $H_i \cap C$ for $i = 2, 3, 4$ must be an ovoid, because any set of $q^2 + 1$ points in PG(3,q) no three of which are collinear form an ovoid.

This concludes the proof.

□

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