LOOPS WITH PARTITIONS AND MATCHINGS

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Abstract

An alternating circular list of distinct r-element subsets of some finite set X and distinct r-partitions of type τ is said to be a τ-loop if successive members of the list are orthogonal. We address the problem of finding complete τ-loops including all r-element subsets of X, for any fixed $|X|$ and type τ.

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1 Introduction

An r–partition of an n–element set X has type $\tau = \{[m_i, t_i] \mid i = 1, \ldots, s\}$ if it has m_i classes of cardinality t_i, $i = 1, \ldots, s$. We assume that class sizes are distinct. To exclude trivial types, we assume $1 < r < n$, where $n = \sum_{i=1}^{s} m_i t_i$ and $r = \sum_{i=1}^{s} m_i$. Let A be an r–element subset and let π be an r–partition of X. We say that A and π are *orthogonal* if A is a transversal of π, that is every class of π has an element in A.

An alternating circular list of distinct r–element subsets of X and distinct partitions of type τ is called a *τ-loop*, or a loop with partitions of type τ, if each r–element subset is orthogonal to both of its neighboring partitions. A τ–loop on X is said to be *complete* if it includes all r–element subsets of X. A τ–loop $(A_1, \pi_1, \ldots, A_m, \pi_m)$ with r–partitions is called *strong* if $|A_i \cap A_{i+1}| = r - 1$, for every $1 \leq i \leq m$.

For given $n \geq 3$ and non trivial type τ, we address the problem of finding *complete τ-loops*. An obvious necessary condition for the existence of a complete τ–loop is $N(\tau) \geq \binom{n}{r}$, where $N(\tau)$ is the number of partitions of type τ. We conjecture that this condition is sufficient: there exists a complete τ–loop for every non trivial type τ if and only if $N(\tau) \geq \binom{n}{r}$. In this paper the conjecture is supported with particular cases that are thought to have some interest for their own right.

Exceptional partition types violating the inequality $N(\tau) \geq \binom{n}{r}$, have been characterized in [5]. In Section 2 we include a proof, for the sake of completeness, and we verify the conjecture for small n. The constructions given for $n \leq 7$ illustrate some methods of construction or serve as basis cases in later recursions.

In Section 3 a related concept, the notion of complete r–loops is introduced. An *r-matching* is the union of r pairwise disjoint 2–element subsets of X. An r–element subset $A \subset X$ and an r–matching ν are orthogonal, if each pair of ν has an element in A. An alternating circular list of r–element subsets and r–matchings of X is called an *r-loop*, or a loop of X with r–matchings, if each subset is orthogonal to both of its neighboring matchings. A loop is *strong* if in addition consecutive r–element subsets share $r - 1$

common elements. In Section 3 we show that an n element set has a complete strong r–loop, for every $r > 1$ and $n \geq 2r + 1$ (Theorem 3.1). This result implies the existence of complete strong τ–loops, for every τ with classes of size 1 or 2 (Corollary 3.2). In Section 4 we use a different construction based on parity considerations, and we show the existence of complete τ–loops with partitions of the same type, for every $n \geq 2r \geq 8$ (Theorem 4.2).

We know the conjecture is true for 2–partitions, as well. We just state the corresponding result from the first version of the present paper without proof: *There exist complete strong τ–loops with 2–partitions of any non exceptional type.*

A complete τ–loop consisting of all r–element subsets of X connected by partitions of any type may be used to calculate the cardinality of a minimal generating set of idempotents of the semigroup $K(n,r)$ of all the total transformations of X with a range of at most r elements (see [2]). The semigroups $K(n,r)$, $1 \leq r \leq n$, are special cases of S_n–normal semigroups, the transformation semigroups on X that are invariant under conjugations by permutations of X. It was shown in [4] that an S_n–normal semigroup is idempotent–generated, but the minimum size of a generating set, the idempotent rank, remains unknown. Complete τ–loops of X with particular partition types should aid greatly in determining the idempotent rank of S_n–normal semigroups.

The existence of complete τ– or r–loops are special instances of a general problem of finding cycles in "highly symmetric" bipartite graphs that cover the smaller partite set. Let U be the set of all r–element sets of some underlying set X, and let V be the set of some objects of prescribed type associated to X (e.g. subsets, partitions, matchings, etc.). Consider the bipartite graph $G[U,V]$ on vertex set $U \cup V$ defined by a binary relation on $U \times V$ (e.g. inclusion, or $u \in U$ and $v \in V$ are orthogonal, etc.) The indicated examples result in bipartite graphs with a high degree of symmetry. The alternating circular chains of r–tuples and objects of X (e.g. τ–loops or r–loops) that include all r–tuples correspond to cycles covering every vertex in U. If $|U| = |V|$, then the complete loops become hamiltonian cycles in the bipartite graph $G[U,V]$.

Several problems in the literature have a similar form or have a very similar nature. For $1 < r < p < n = |X|$, let V be the set of all p–element subsets of X, and assume that the binary relation is defined by inclusion. Then $G = G[U, V]$ is the bipartite graph induced between two levels of the boolean lattice $\mathcal{B}_n$. It is an open problem to determine whether G has a cycle (or path) which includes every vertex of the smaller of the two partite sets. In particular, the famous Middle Layers Problem (for $p = r + 1$, $n = 2r + 1$) or the Antipodal Layers Problem (for $p = n - r$) are still unsolved (see [3], [6], and [7]). In the lack of general existence theorems for large cycles, most of the partial results are based on recursive constructions or algorithms exploiting the particular structure of the bipartite graphs. Our problem of finding τ–or r–loops has the same characteristics. However, the involved bipartite structures seem to be less restrictive compared to the subgraphs of the boolean lattice mentioned above, thus offering complete answer to some particular questions.

2 Exceptional types and small cases

A necessary condition for the existence of a complete τ–loop is $N(\tau) \geq \binom{n}{r}$, where

$$N(\tau) = \frac{n!}{\prod_{i=1}^{s} [m_i! \; (t_i!)^{m_i}]}$$

is the number of partitions of type $\tau = \{[m_i, t_i] \mid i = 1, \ldots, s\}$ (see e.g. [1]). Types violating this inequality, called here exceptional types, have been characterized in [5]. We include a short proof for the sake of completeness.

Proposition 2.1 *For $2 \leq r \leq n - 1$, $N(\tau) < \binom{n}{r}$ if and only if τ is one of the following types: $\{[2, 2]\}$, $\{[2, 3]\}$, $\{[3, 2]\}$ or $\{[1, t], [n-t, 1]\}$ with $t > (n + 1)/2$.*

Proof. By substituting $r = \sum_{i=1}^{s} m_i$ and $n = \sum_{i=1}^{s} m_i t_i$, we obtain

$$\frac{N(\tau)}{\binom{n}{r}} = \frac{r!(n-r)!}{\prod_{i=1}^{s} [m_i!(t_i!)^{m_i}]} = \binom{r}{m_1, m_2, \ldots, m_s} \frac{\left(\sum_{i=1}^{s} m_i(t_i - 1)\right)!}{\prod_{i=1}^{s} \prod_{j=2}^{t_i} j^{m_i}}.$$

Assume that $t_1 > t_2 > \ldots > t_s$. If $s \geq 2$, we have

$$\frac{N(\tau)}{\binom{n}{r}} \geq r \cdot \frac{\left(\sum\limits_{i=1}^{s} m_i(t_i-1)\right)!}{\prod\limits_{i=1}^{s}\prod\limits_{j=2}^{t_i} j^{m_i}} = r \cdot \frac{m_1!}{2^{m_1}} \left(\frac{m_1+1}{a_1} \cdot \frac{m_1+2}{a_2} \cdots \frac{m_1+k}{a_k} \right),$$

where $k = \sum\limits_{i=1}^{s} m_i(t_i-1) - m_1 = n - r - m_1$, and $a_1 \leq a_2 \leq \ldots \leq a_k$ is a monotone permutation of the factors of the denominator not counted in 2^{m_1}. If $t_2 \geq 2$, then $a_1 = 2$. Because $m_1!/2^{m_1-1} \geq 1$ and $(m_1+k)/a_k \geq 1$, we obtain $N(\tau)/\binom{n}{r} \geq 1$.

This argument shows that if $N(\tau) < \binom{n}{r}$, then either $t_2 < 2$ or $s < 2$.

The first case, $t_2 < 2$, clearly implies $t_2 = 1$ and $s = 2$. If $m_1 \geq 2$, then the sequence (a_k) starts with $a_1 = 2$, thus the right-hand side is at least 1. Hence we may assume that $m_1 = 1$. Because $r = n - t_1 + 1$, we have $N(\tau)/\binom{n}{r} = r/t_1 = (n - t_1 + 1)/t_1$. Consequently, for $s = 2$, $N(\tau) < \binom{n}{r}$ if and only if $t_1 > (n+1)/2$. The corresponding type is $\{[1, t_1], [n - t_1, 1]\}$ as stated.
Assume now that $s = 1$. In this case $r = m_1$, $n = rt_1$, thus we get

$$\frac{N(\tau)}{\binom{n}{r}} = \frac{(n-r)!}{(t_1!)^r} = \frac{r!}{2^r} \left(\prod_{k=1}^{n-2r} \frac{r+k}{a_k} \right).$$

If $r \geq 4$ then the right-hand side is at least 1. Thus we may assume that $r = 2$ or 3. If $r = 3$ and $t_1 \geq 3$, then

$$\frac{N(\tau)}{\binom{n}{r}} = \frac{\prod\limits_{i=1}^{2t_1-3} (t_1+i)}{(t_1!)^2} = \left(\frac{\prod\limits_{i=1}^{t_1-1} (t_1+i)}{t_1!} \right) \left(\prod_{i=1}^{t_1-3} \frac{2t_1+i}{i+2} \right) \frac{2t_1}{2t_1} > 1,$$

therefore $t_1 = 2$ follows. If $r = 2$ and $t_1 \geq 4$, then

$$\frac{N(\tau)}{\binom{n}{r}} = \frac{\prod\limits_{i=1}^{t_1-2} (t_1+i)}{t_1!} = \frac{(t_1+1)(t_1+2)}{2 \cdot 3 \cdot t_1} \left(\prod_{i=4}^{t_1-1} \frac{t_1-1+i}{i} \right) > 1,$$

therefore $t_1 = 2$ or 3 follows. It is easy to verify that $N(\tau)/\binom{n}{r} < 1$ holds for the obtained types $\{[2,2]\}$, $\{[2,3]\}$ and $\{[3,2]\}$. □

Trivial types ($r = 1$ or n), together with non trivial types characterized in Proposition 2.1 are called *exceptional*. We study the question whether there exist complete τ–loops, for every non exceptional type τ. As a first step we consider small cases that either will serve as basis cases in later recursions or just illustrate some techniques of construction.

Proposition 2.2 *There exist complete strong τ-loops for every non exceptional type τ and $n \leq 6$.*

Proof. Let $X = X_n = \{1, 2, \ldots, n\}$, where the elements are considered as mod n congruence classes (in particular, $n + 1 = 1$). The sequence of transversals and the sequence of partitions in a τ–loop $(A_1, \pi_1, \ldots, A_m, \pi_m)$ is denoted by $\mathcal{A}$ and Π, respectively.

For $\tau = \{[1, 2], [n-2, 1]\}$ and $n \geq 3$, $A_i = X \setminus \{i\}, \pi_i = \{i, i+1\}\{i+2\}\ldots\{i-1\}$, $i = 1, \ldots, n$, define a complete strong τ–loop. Thus, we may assume that either $m_1 \geq 2$ or $t_1 \geq 3$, where m_1 and t_1 is the number and size of the largest classes of the partition, respectively. In particular, $n \geq 5$.

Case $n = 5$. Let $\mathcal{A}_0$ and Π_0 be lists of doubletons and 2–matchings, respectively, defined as follows:

$$\begin{aligned}\mathcal{A}_0 = \quad & (\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\},\\ & \{1,3\},\{3,5\},\{5,2\},\{2,4\},\{4,1\}),\\ \Pi_0 = \quad & (\{1,3\}\{2,4\},\{2,4\}\{3,5\},\{3,5\}\{1,4\},\\ & \{1,4\}\{2,5\},\{3,5\}\{1,2\},\{1,5\}\{3,4\},\\ & \{2,3\}\{4,5\},\{4,5\}\{1,2\},\{1,2\}\{3,4\},\{2,4\}\{1,5\}).\end{aligned}$$

There are three types to consider. For $\tau = \{[1, 3], [1, 2]\}$, Π is obtained by extending each member of Π_0 to get a list of distinct partitions of type τ. For example,

$$\begin{aligned}\Pi = \quad & (\{2,4,5\}\{1,3\},\{1,2,4\}\{3,5\},\{2,3,5\}\{1,4\},\\ & \{1,3,4\}\{2,5),\{3,4,5\}\{1,2\},\{2,3,4\}\{1,5\},\\ & \{1,4,5\}\{2,3\},\{1,2,3\}\{4,5\},\{1,2,5\}\{3,4\},\{1,3,5\}\{2,4\}).\end{aligned}$$

A complete strong τ–loop is obtained by alternately taking elements from $\mathcal{A}_0$ and Π.

For $\tau = \{[2,2],[1,1]\}$, let $\mathcal{A} = \{X_5 \setminus A_i \mid A_i \in \mathcal{A}_0\}$, and define Π by extending every member of Π_0 with the uncovered element of X_5 as a singleton class.

For $\tau = \{[1,3],[2,1]\}$, a complete strong τ-loop is defined with

$$\begin{aligned}
\mathcal{A} = \quad & (\{1,2,3\},\{1,3,4\},\{3,4,5\},\{1,3,5\},\{1,2,5\},\\
& \{2,3,5\},\{2,3,4\},\{2,4,5\},\{1,4,5\},\{1,2,4\}),\\
\Pi = \quad & (\{2,4,5\}\{1\}\{3\},\{1,2,5\}\{3\}\{4\},\\
& \{1,2,4\}\{3\}\{5\},\{2,3,4\}\{1\}\{5\},\\
& \{1,3,4\}\{2\}\{5\},\{1,4,5\}\{2\}\{3\},\\
& \{1,3,5\}\{2\}\{4\},\{1,2,3\}\{4\}\{5\},\\
& \{2,3,5\}\{1\}\{4\},\{3,4,5\}\{1\}\{2\}).
\end{aligned}$$

Case $n = 6$. There are four types to consider. For $\tau = \{[2,2],[2,1]\}$ we obtain a complete strong τ-loop with

$$\begin{aligned}
\mathcal{A} = \quad & (\{1,4,5,6\},\{1,2,5,6\},\{1,2,3,6\},\{2,3,4,6\},\{2,4,5,6\},\\
& \{1,2,4,6\},\{1,3,4,6\},\{1,3,5,6\},\{2,3,5,6\},\{3,4,5,6\},\\
& \{2,3,4,5\},\{1,3,4,5\},\{1,2,3,5\},\{1,2,3,4\},\{1,2,4,5\}),\\
\Pi = \quad & (\{2,4\}\{3,5\}\{1\}\{6\},\{3,5\}\{4,1\}\{2\}\{6\},\{4,1\}\{5,2\}\{3\}\{6\},\\
& \{5,3\}\{1,2\}\{4\}\{6\},\{1,5\}\{3,4\}\{2\}\{6\},\{3,2\}\{5,4\}\{1\}\{6\},\\
& \{5,4\}\{2,1\}\{3\}\{6\},\{2,1\}\{4,3\}\{5\}\{6\},\{4,2\}\{1,5\}\{3\}\{6\},\\
& \{1,4\}\{2,6\}\{3\}\{5\},\{1,2\}\{3,6\}\{4\}\{5\},\{2,4\}\{1,6\}\{3\}\{5\},\\
& \{4,5\}\{3,6\}\{1\}\{2\},\{3,5\}\{1,6\}\{2\}\{4\},\{3,4\}\{2,6\}\{1\}\{5\}).
\end{aligned}$$

For $\tau = \{[1,3],[3,1]\}$:

$$\begin{aligned}
\mathcal{A} = \quad & (\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,4,5,6\},\{2,4,5,6\},\\
& \{2,3,4,5\},\{2,3,4,6\},\{2,3,5,6\},\{1,2,5,6\},\{1,3,5,6\},\\
& \{3,4,5,6\},\{1,3,4,5\},\{1,3,4,6\},\{1,2,3,6\},\{1,2,4,6\}),\\
\Pi = \quad & (\{4,5,6\}\{1\}\{2\}\{3\},\{3,4,6\}\{1\}\{2\}\{5\},\{2,3,6\}\{1\}\{4\}\{5\},\\
& \{1,2,3\}\{4\}\{5\}\{6\},\{1,3,6\}\{2\}\{4\}\{5\},\{1,5,6\}\{2\}\{3\}\{4\},\\
& \{1,4,5\}\{2\}\{3\}\{6\},\{1,3,4\}\{2\}\{5\}\{6\},\{2,3,4\}\{1\}\{5\}\{6\},\\
& \{1,2,4\}\{3\}\{5\}\{6\},\{1,2,6\}\{3\}\{4\}\{5\},\{2,5,6\}\{1\}\{3\}\{4\},\\
& \{2,4,5\}\{1\}\{3\}\{6\},\{3,4,5\}\{1\}\{2\}\{6\},\{3,5,6\}\{1\}\{2\}\{4\}).
\end{aligned}$$

For $\tau = \{[1,3],[1,2],[1,1]\}$:

$$\begin{aligned}\mathcal{A} = \quad & (\{3,4,6\},\{4,5,6\},\{1,5,6\},\{1,3,6\},\{3,5,6\},\\ & \{2,5,6\},\{2,4,6\},\{1,4,6\},\{1,2,6\},\{2,3,6\},\\ & \{2,3,5\},\{1,3,5\},\{1,3,4\},\{1,2,4\},\{2,4,5\},\\ & \{2,3,4\},\{1,2,3\},\{1,2,5\},\{1,4,5\},\{3,4,5\}),\\ \Pi = \quad & (\{2,3,5\}\{1,4\}\{6\},\{1,3,4\}\{2,5\}\{6\},\{3,4,5\}\{1,2\}\{6\},\\ & \{2,3,4\}\{1,5\}\{6\},\{1,4,5\}\{23\}\{6\},\{1,2,3\}\{4,5\}\{6\},\\ & \{1,2,5\}\{3,4\}\{6\},\{1,3,5\}\{2,4\}\{6\},\{2,4,5\}\{1,3\}\{6\},\\ & \{1,5,6\}\{2,4\}\{3\},\{1,2,6\}\{3,4\}\{5\},\{4,5,6\}\{1,2\}\{3\},\\ & \{2,3,6\}\{4,5\}\{1\},\{1,5,6\}\{3,4\}\{2\},\{3,5,6\}\{1,2\}\{4\},\\ & \{1,4,6\}\{2,5\}\{3\},\{3,5,6\}\{1,4\}\{2\},\{2,4,6\}\{3,5\}\{1\},\\ & \{1,3,6\}\{2,4\}\{5\},\{2,5,6\}\{1,3\}\{4\}).\end{aligned}$$

For $\tau = \{[1,4],[1,2]\}$:

$$\begin{aligned}\mathcal{A} = \quad & (\{3,4\},\{1,3\},\{1,4\},\{4,6\},\{1,6\},\\ & \{5,6\},\{3,5\},\{3,6\},\{2,6\},\{2,3\},\\ & \{1,2\},\{1,5\},\{2,5\},\{2,4\},\{4,5\}),\\ \Pi = \quad & (\{1,4,5,6\}\{2,3\},\{2,3,4,6\}\{1,5\},\{1,3,5,6\}\{2,4\},\\ & \{1,2,3,4\}\{5,6\},\{1,2,4,5\}\{3,6\},\{1,2,3,6\}\{4,5\},\\ & \{2,4,5,6\}\{1,3\},\{1,2,3,5\}\{4,6\},\{3,4,5,6\}\{1,2\},\\ & \{1,3,4,6\}\{2,5\},\{2,3,4,5\}\{1,6\},\{1,2,4,6\}\{3,5\},\\ & \{1,3,4,5\}\{2,6\},\{1,2,5,6\}\{3,4\},\{2,3,5,6\}\{1,4\}).\end{aligned}$$

□

3 Loops with r–matchings

Let $r \geq 1, n \geq 2r$, and let X be a set of n elements. W.l.o.g. we may assume that the underlying set X is either $X_n = \{1,\ldots,n\}$ or $X_{n-1}^0 = \{0,1,\ldots,n-1\}$, and the elements of X are considered as modulo n congruence classes.

Theorem 3.1 *There exist complete strong loops with r-matchings, for every* $n \geq 2r+1 \geq 3$.

Proof. Induction on r and n. The theorem is true for $r = 1$ and $n \geq 3$. Indeed, $A_i = \{i\}$ and $\mu_i = \{i, i+1\}$, for $1 \leq i \leq n$, define a μ–loop on X_n with 1–matchings.

Step A. Let $r \geq 2$, $n \geq 2r+2$, $p = \binom{n-1}{r}$ and $q = \binom{n-1}{r-1}$. Set $X = X^0_{n-1}$ and $X' = X_{n-1}$. Suppose that there exist complete strong loops with r–and $(r-1)$–matchings on $(n-1)$–element sets. Let $(A_1, \mu_1, \ldots, A_p, \mu_p)$ be a complete strong loop on X' with r–matchings such that $A_p = \{1, 2, \ldots, r\}$, $A_1 = \{2, \ldots, r, r+1\}$, and $\mu_p = \{\{i, (r+i)\} \mid 1 \leq i \leq r\}$. Let $(B'_1, \nu'_1, \ldots, B'_q, \nu'_q)$ be a complete strong loop with $(r-1)$–matchings on the set $\{b_1, \ldots, b_{n-1}\}$. Assume that $B'_q = \{b_1, \ldots, b_{r-1}\}$, and $B'_1 = \{b_2, \ldots, b_{r-1}, b_{r+1}\}$ if $r \geq 3$, and $B'_1 = \{b_3\}$ if $r = 2$.

Consider the natural extension of the bijection $\beta(b_i) = i$ from the subsets of $\{b_1, \ldots, b_{n-1}\}$ onto the power set $2^{X'}$. For every $1 \leq i \leq q$, let $B_i = \beta(B'_i) \cup \{0\}$, and let $\nu_i = \{\{\beta(x), \beta(y)\} \mid \{x, y\} \in \nu'_i\} \cup \{0, \beta(x_i)\}$, where $x_i \in \{b_1, \ldots, b_{n-1}\} \setminus \{b_r\}$ is an arbitrary element not covered by ν'_i. Notice that such an x_i exists, because ν'_i covers $2(r-1) < (n-1) - 2$ elements, by assumption. Observe that every r–element subset of X occurs exactly once in the list $(A_1, \ldots, A_p, B_q, B_{q-1}, \ldots, B_1)$, furthermore, $A_p \cap B_q = \{1, \ldots, r-1\}$ and $B_1 \cap A_1 = \{2, \ldots, r-1, r+1\}$. Furthermore, the list $(\mu_1, \ldots, \mu_{p-1}, \nu_{q-1}, \ldots, \nu_1)$ contains distinct r–matchings of X.

Define two variants of μ_p as follows: $\omega_1 = \{1, 2r\} \cup \{\{i, r+i\} \mid 2 \leq i \leq r-1\} \cup \{0, r\}$ and $\omega_2 = \{r+1, 2r\} \cup \{\{i, r+i\} \mid 2 \leq i \leq r-1\} \cup \{0, r\}$. Note that both ω_1 and ω_2 contain the pair $\{0, r\}$, hence they are different from any μ_i or ν_i. It is straightforward to verify that $(A_1, \mu_1, \ldots, \mu_{p-1}, A_p, \omega_1, B_q, \nu_{q-1}, B_{q-1}, \ldots, \nu_1, B_1, \omega_2)$ is a complete strong loop of X with r–matchings.

The circular lists $\mathcal{A}_0$ and Π_0 used in the proof of Proposition 2.2 define a complete strong loop with 2–matchings, for $n = 5$. Because the theorem is also true for $r = 1$ and $n \geq 3$, Step A provides the construction for $r = 2$ and every $n \geq 5$. In the same way, if there is a loop with r–matchings for fixed r and $n = 2r+1$, then Step A yields the required loops for every $n > 2r+1$. To conclude the proof of the theorem it is enough to construct loops, for every $r \geq 3$ and $n = 2r+1$.

Step B. Let $r \geq 2$, $n = 2r+3$, $p = \binom{2r+1}{r}$ and $q = \binom{2r+1}{r-1}$. Set $X = X_n$ and $X' = X \setminus \{n-1, n\} = X_{n-2}$. Suppose that there exist complete strong loops with r– and $(r-1)$–matchings on $(n-2)$–element sets. Let $(A'_1, \mu'_1, \ldots, A'_p, \mu'_p)$ be a complete strong loop with r–matchings on $A = \{a_1, \ldots, a_{2r+1}\}$. For every $1 \le i \le p$, let $A_i = A \setminus A'_i$. One may assume that $A_p = \{a_1, a_2, \ldots, a_{r+1}\}$, $A_1 = \{a_2, \ldots, a_{r+1}, a_{r+2}\}$, and $\mu'_p = \{\{a_i, a_{r+1+i}\} \mid 1 \le i \le r\}$.

Consider the natural extension of the bijection $\alpha(a_i) = i$ from 2^A onto $2^{X'}$. Define $\mu_i = \{\{\alpha(u_i), \alpha(v_i)\} \mid \{u_i, v_i\} \in \mu'_i\} \cup \{\alpha(x_i), n-1\}$, for every $1 \le i \le p$, where $x_i \in A$ is the unique element not covered by μ'_i. In particular, $\mu_p = \{\{i, r+1+i\} \mid 1 \le i \le r\} \cup \{r+1, n-1\}$. Note that $(A_1, \mu_1, \ldots, A_p, \mu_p)$ is a strong μ–loop including all $(r+1)$–element subsets of X not containing $n-1$ and n, furthermore, its $(r+1)$–matchings do not cover n.

Next we construct loops including $(r+1)$–element subsets of X that contain exactly one of $n-1$ and n. We start with defining a bijection $\gamma : A \longrightarrow X'$ which maps the loop $(A'_1, \mu'_1, \ldots, A'_p, \mu'_p)$ into a loop of X'. One may assume that $\gamma(A'_p) = \{1, 2, \ldots, r\}$ and $\gamma(A'_1) = \{2, \ldots, r, r+2\}$. For each $1 \le i \le p$, let $C_i = \gamma(A'_i) \cup \{n\}$, $\pi_i = \{\{\gamma(u_i), \gamma(v_i)\} \mid \{u_i, v_i\} \in \mu'_i\} \cup \{\gamma(x_i), n\}$, where $x_i \in A$ is the unique element of A not covered by μ'_i. Note that $n-1$ is not covered by any π_i. Let $D_i = \gamma(A'_i) \cup \{n-1\}$, $\rho_i = \{\{\gamma(u_i), \gamma(v_i)\} \mid \{u_i, v_i\} \in \mu'_i\} \cup \{n-1, n\}$. Clearly, $(C_1, \pi_1, \ldots, C_p, \pi_p)$ and $(D_1, \rho_1, \ldots, D_p, \rho_p)$ are strong loops that include all $(r+1)$–element subsets of X containing exactly one of $n-1$ and n.

Finally, let $(B'_1, \nu'_1, \ldots, B'_q, \nu'_q)$ be a complete strong loop with $(r-1)$–matchings on $B = \{b_1, \ldots, b_{2r+1}\}$. Assume that $B'_q = \{b_1, \ldots, b_{r-1}\}$ and $B'_1 = \{b_2, \ldots, b_{r-1}, b_{r+2}\}$ (if $r = 2$, then $B'_1 = \{b_4\}$). For each $1 \le i \le q$, the $(r-1)$–matching ν'_i covers $2(r-1) = (2r+1) - 3$ elements. Hence there exists a pair $x_i, y_i \in B$ not covered by ν'_i such that $\{x_i, y_i\} \ne \{b_r, b_{r+1}\}$. Consider the natural extension of the bijection $\beta(b_i) = i$ from 2^B onto $2^{X'}$. For every $1 \le i \le q$, let $B_i = \beta(B'_i) \cup \{n-1, n\}$, and let $\nu_i = \{\{\beta(x), \beta(y)\} \mid \{x, y\} \in \nu'_i\} \cup \{\beta(x_i), n-1\} \cup \{\beta(y_i), n\}$, where x_i and y_i are as above. Note that $(B_1, \nu_1, \ldots, B_q, \nu_q)$ is a strong loop that includes all $(r+1)$–element subsets of X containing both $n-1$ and n. Furthermore, by definition, no ν_i covers both r and $r+1$.

The $(r+1)$–matchings we have defined so far are distinct. Indeed, matchings from different loops are distinguished by their intersections on $\{n-1, n\}$. Now we combine the four μ–loops into one by removing $\mu_p, \pi_p, \rho_p, \nu_q$ and by joining the pieces with distinct variants of $\mu_p = \{\{i, r+1+i\} \mid 1 \le i \le r\} \cup \{r+1, n-1\}$. Set $\omega_0 = \{\{i, r+1+i\} \mid 2 \le i \le r-1\}$, and define

$$
\begin{aligned}
\omega_1 &= \{1, 2r+1\} \cup \{r, n\} \cup \{r+1, n-1\} \cup \omega_0 , \\
\omega_2 &= \{r+2, 2r+1\} \cup \{r+1, n\} \cup \{r, n-1\} \cup \omega_0 , \\
\omega_3 &= \{r+2, 1\} \cup \{r, n\} \cup \{r+1, n-1\} \cup \omega_0 , \\
\omega_4 &= \{r+2, 1\} \cup \{r+1, n\} \cup \{r, n-1\} \cup \omega_0 .
\end{aligned}
$$

These $(r+1)$–matchings differ from any ν_i on the set $\{r, r+1\}$, by the definition of ν_i. Furthermore, any ω_i is distinguished from any μ_j, π_j or ρ_j by its intersection on $\{n-1, n\}$. Clearly, ω_1 is orthogonal to $A_p = \{1, \dots, r+1\}$ and $D_p = \{1, \dots, r, n-1\}$, ω_2 is orthogonal to $C_1 = \{2, \dots, r, r+2, n\}$ and $A_1 = \{2, \dots, r+2\}$, ω_3 is orthogonal to $D_1 = \{2, \dots, r, r+2, n-1\}$ and $B_1 = \{2, \dots, r-1, r+2, n-1, n\}$, and ω_4 is orthogonal to $B_q = \{1, \dots, r-1, n-1, n\}$ and $C_p = \{1, \dots, r, n\}$. Consequently, $(A_1, \mu_1, \dots, A_p, \omega_1, D_p, \rho_{p-1}, \dots, \rho_1, D_1, \omega_3, B_1, \nu_1, \dots, B_q, \omega_4, C_p, \pi_{p-1}, \dots, \pi_1, C_1, \omega_2)$ is a complete strong loop with $(r+1)$–matchings for the $(2r+3)$–element set X. This concludes Step B and the proof of the theorem. □

As a corollary of Theorem 3.1 we obtain the following result on τ–loops.

Corollary 3.2 *There exist complete strong τ-loops with partitions of type $\{[m,2],[n-2m,1]\}$, for every $n \ge 2m+1$ and $m \ge 1$.*

Proof. From Theorem 3.1 immediately follows that there is a complete strong loop $(A_1, \mu_1, \dots, A_p, \mu_p)$ on X_n with m–matchings, for every $n > 2m$. Extend each μ_i with the uncovered $n-2m$ elements of X_n to a partition π_i of type $\tau = \{[m,2],[n-2m,1]\}$. The complement $\overline{A_i} = X_n \setminus A_i$ is orthogonal to π_i, and $(\overline{A_1}, \pi_1, \dots, \overline{A_p}, \pi_p)$ is obviously a complete strong τ–loop of X_n. □

4 Partitions with classes of size 1 or 2

For $m_1 \ge 0, m_2 \ge 1$, let $\tau = \{[m_1,1],[m_2,2]\}$ be a partition type. Set $n = m_1 + 2m_2$, $r = m_1 + m_2$, and $m = \binom{n}{r}$. Note that if $m_1 = 0$, then any partition of type τ becomes an r–matching. Classes of size 1 and 2 in a partition are called *singletons* and *doubletons*, respectively. The *parity* of a singleton $\{k\}$, for $k \in X_n$, is defined as the parity of the integer k. Based on parity considerations we give a recursive construction which extends to any partition with classes of size 1 or 2. This method resolves the missing case of $n = 2r$, and also offers a second proof for Corollary 3.2.

Let $(A_1, \pi_1, \ldots, A_m, \pi_m)$ be a complete τ–loop on X_n with partitions of type τ. We say that the loop satisfies the *parity condition* if the number of even singletons in π_i is $\lfloor m_1/2 \rfloor$, for every $1 \le i \le m$, and it satisfies the *boundary conditions* if $A_1 = \{1, 2, \ldots, r\}$ and $A_m = \{1, 2, \ldots, r-1, n\}$. Parity and boundary conditions will help us in pasting loops together in the recursive construction.

It is easy to see that there are no complete τ–loops satisfying the parity condition if $m_2 = 1$ and n is odd. Therefore we will assume that $m_2 \ge 2$, and $n \ge 6$. First we construct complete τ–loops with the parity and boundary conditions, for $m_2 = 2$ and every $n \ge 6$.

Lemma 4.1 *For every $r \ge 4$, there exist complete τ-loops with partitions of type $\{[r-2, 1], [2, 2]\}$ satisfying the parity and boundary conditions.*

Proof. The sequence of r–element subsets and the sequence of partitions in a τ–loop will be denoted by $\mathcal{A}$ and Π, respectively. Set $n = r + 2$. If $r = 4$, then the following lists define a complete loop on X_6 satisfying the parity and boundary conditions.

$$\begin{aligned}
\mathcal{A} = \quad & (\{1,2,3,4\},\{3,4,5,6\},\{2,3,4,6\},\{1,3,4,6\},\{1,2,5,6\},\\
& \{2,4,5,6\},\{1,2,4,5\},\{1,3,4,5\},\{1,4,5,6\},\{2,3,4,5\},\\
& \{1,2,3,5\},\{2,3,5,6\},\{1,3,5,6\},\{1,2,4,6\},\{1,2,3,6\}),\\
\Pi = \quad & (\{3\}\{4\}\{1,5\}\{2,6\},\{3\}\{4\}\{1,6\}\{2,5\},\{3\}\{4\}\{1,2\}\{5,6\},\\
& \{1\}\{6\}\{2,4\}\{3,5\},\{5\}\{6\}\{1,4\}\{2,3\},\{2\}\{5\}\{1,6\}\{3,4\},\\
& \{4\}\{5\}\{1,6\}\{2,3\},\{4\}\{5\}\{1,2\}\{3,6\},\{4\}\{5\}\{1,3\}\{2,6\},\\
& \{2\}\{3\}\{1,4\}\{5,6\},\{2\}\{3\}\{1,6\}\{4,5\},\{3\}\{6\}\{1,2\}\{4,5\},\\
& \{1\}\{6\}\{2,3\}\{4,5\},\{1\}\{6\}\{2,5\}\{3,4\},\{2\}\{3\}\{1,5\}\{4,6\}).
\end{aligned}$$

Suppose now that $r \ge 5$, that is $n = r + 2 \ge 7$. Let $X = X_n$, $X' = X_{n-1}$ and set $m' = \binom{n-1}{r-1}$. Assume that there exists a complete loop

$(A'_1, \pi'_1, \ldots, A_{m'}, \pi'_{m'})$ on X' with partitions of type $\{[r-1,1],[2,2]\}$ which satisfies the parity and boundary conditions. In particular, $A'_1 = \{1,2,\ldots,r-1\}$ and $A'_{m'} = \{1,\ldots,r-2,r+1\}$. For $i = 1,2,\ldots,m'$, define $A_i = A'_i \cup \{n\}$ and let π_i be the partition of X obtained by extending π'_i with the singleton $\{n\}$. Define $\pi_0 = \{1\}\{2\}\ldots\{r-2\}\{r-1,r+1\}\{r,n\}$.

Let $B_1 = \{1,2,\ldots,r\}$, and for $i = 2,\ldots,r+1$, let $B_i = X'\backslash\{i-1\}$. Define $\sigma_1 = \{3\}\{4\}\ldots\{r\}\{1,r+1\}\{2,n\}$, and $\sigma_{r+1} = \{1\}\{2\}\ldots\{r-2\}\{r,r+1\}\{r-1,n\}$. For $i = 2,\ldots,r$, let σ_i be the partition of X with two doubletons, $\{i-1,i\}$ and $\{r+1,n\}$ (and $r-2$ singletons).

Obviously $(B_1,\sigma_1,\ldots,B_{r+1},\sigma_{r+1},A_{m'},\pi_{m'-1},\ldots,\pi_1,A_1,\pi_0)$ is a complete τ–loop on X, and it satisfies the parity and boundary conditions. □

Theorem 4.2 *There exist complete τ-loops for every non exceptional partition type with classes of size* 1 *or* 2.

Proof. For $m_1 \geq 0, m_2 \geq 1$, let $\tau = \{[m_1,1],[m_2,2]\}$, $n = m_1 + 2m_2$, $r = m_1 + m_2$, and $m = \binom{n}{r}$. For $m_2 = 1$, complete loops are given in Proposition 2.2. For $m_2 = 2$, Lemma 4.1 implies the existence of complete loops. Hence the theorem is true for $m_2 = 1$ and 2.

If $m_2 \geq 3$, $\tau = \{[1,1],[3,2]\}$ is the only non exceptional type, for $n = 6$ and 7. In this case a complete τ–loop with the parity and boundary conditions is defined as follows.

$$\begin{aligned}
\mathcal{A} = \quad & (\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{2,3,4,5\},\{1,3,4,5\},\\
& \{2,3,4,6\},\{1,2,3,6\},\{1,2,4,6\},\{1,3,4,6\},\{3,4,5,6\},\\
& \{2,4,5,6\},\{1,3,5,6\},\{2,3,5,6\},\{1,2,5,6\},\{1,4,5,6\},\\
& \{1,3,5,7\},\{1,2,5,7\},\{1,4,5,7\},\{1,5,6,7\},\{3,5,6,7\},\\
& \{3,4,5,7\},\{2,5,6,7\},\{2,3,5,7\},\{2,4,5,7\},\{4,5,6,7\},\\
& \{1,2,4,7\},\{1,4,6,7\},\{1,3,6,7\},\{2,4,6,7\},\{3,4,6,7\},\\
& \{2,3,6,7\},\{1,2,6,7\},\{1,3,4,7\},\{2,3,4,7\},\{1,2,3,7\}),\\
\Pi = \quad & (\{3\}\{1,5\}\{2,6\}\{4,7\},\{3\}\{1,6\}\{2,7\}\{4,5\},\\
& \{5\}\{1,6\}\{2,7\}\{3,4\},\{5\}\{1,3\}\{2,7\}\{4,6\},\\
& \{3\}\{1,2\}\{4,7\}\{5,6\},\{3\}\{1,6\}\{2,5\}\{4,7\},\\
& \{3\}\{1,4\}\{2,5\}\{6,7\},\{1\}\{2,5\}\{3,4\}\{6,7\},
\end{aligned}$$

$\{1\}\{2,3\}\{4,5\}\{6,7\}, \{3\}\{1,5\}\{2,4\}\{6,7\},$
$\{5\}\{1,4\}\{2,3\}\{6,7\}, \{5\}\{1,2\}\{3,4\}\{6,7\},$
$\{3\}\{1,2\}\{4,5\}\{6,7\}, \{5\}\{1,3\}\{2,4\}\{6,7\},$
$\{1\}\{2,4\}\{3,5\}\{6,7\}, \{5\}\{1,2\}\{3,4\}\{6,7\},$
$\{5\}\{1,6\}\{2,3\}\{4,7\}, \{5\}\{1,6\}\{2,4\}\{3,7\},$
$\{5\}\{1,2\}\{4,6\}\{3,7\}, \{5\}\{1,3\}\{2,6\}\{4,7\},$
$\{5\}\{1,7\}\{2,3\}\{4,6\}, \{5\}\{1,7\}\{2,4\}\{3,6\},$
$\{5\}\{1,2\}\{3,6\}\{4,7\}, \{5\}\{1,7\}\{2,6\}\{3,4\},$
$\{5\}\{1,4\}\{2,6\}\{3,7\}, \{7\}\{1,6\}\{2,5\}\{3,4\},$
$\{7\}\{1,3\}\{2,6\}\{4,5\}, \{7\}\{1,2\}\{3,4\}\{5,6\},$
$\{7\}\{1,4\}\{2,3\}\{5,6\}, \{7\}\{1,6\}\{2,3\}\{4,5\},$
$\{7\}\{1,3\}\{2,4\}\{5,6\}, \{7\}\{1,3\}\{2,5\}\{4,6\},$
$\{7\}\{1,5\}\{2,4\}\{3,6\}, \{7\}\{1,2\}\{3,6\}\{4,5\},$
$\{7\}\{1,4\}\{2,5\}\{3,6\}).$

For the recursive constructions we need stronger parity and boundary conditions formulated in the following properties:
(1) the number of even singletons in π_i is $\lfloor m_1/2 \rfloor$, for $1 \le i \le m$,
(2) $A_1 = \{1,2,\ldots,r\}$, $A_m = \{1,2,\ldots,r-1,n\}$ if $m_1 > 0$, and $A_m = \{r-1,r,\ldots,n-3,n\}$ if $m_1 = 0$,
(3) if $\{1\}$ is a singleton in π_i then $\{n-1,n\}$ is a doubleton in π_i, for $m_1 = 1$, and $\{n\}$ is a singleton in π_i, for $m_1 = 2$.

Observe that properties (1)–(3) are satisfied by the loop above and by those constructed in Lemma 4.1, for $m_2 = 2$. Suppose now that $m_2 \ge 3$, $n \ge 8$, and assume that there exist τ–loops on X_k with properties (1)–(3), for all non exceptional types such that $m_2 \ge 2$ and $7 \le k < n$.

Case 1: $m_1 \ge 2$. Let $X = X_n$, $X' = X_{n-1}$, and set $k = \binom{n-1}{r}$, $l = \binom{n-1}{r-1}$. Let $(A_1, \pi'_1, \ldots, A_k, \pi'_k)$ be a complete loop on X' with partitions of type $\{[m_1+1,1],[m_2-1,2]\}$, and let $(B'_1, \sigma'_1, \ldots, B'_l, \sigma'_l)$ be a complete loop on X' with partitions of type $\{[m_1-1,1],[m_2,2]\}$. Also assume that both loops satisfy properties (1)–(3).

For $i = 1, \ldots, k-1$, let π_i be the partition obtained from π'_i by adjoining n to the least singleton class of the opposite parity (i.e., parity different from that of n). For each $i = 1, \ldots, l$, let $B_i =$

$B_i' \cup \{n\}$ and let σ_i be the partition obtained from σ_i' by adding $\{n\}$ as a singleton class. Note that π_i and σ_j are distinct partitions of X containing $\lfloor m_1/2 \rfloor$ even singletons, for every $1 \le i \le k-1$ and $1 \le j \le l-1$.

For $m_2 \ge 4$, let ρ_1 be a partition orthogonal to both $A_k = \{1,2,\ldots,r-1,n-1\}$ and $B_l = \{1,2,\ldots,r-2,n-1,n\}$ such that its singletons are $\{3\},\{4\},\ldots,\{m_1+2\}$ and it contains the doubleton $\{r-1,n\}$. Let ρ_2 be a partition orthogonal to both $A_1 = \{1,2,\ldots,r\}$ and $B_1 = \{1,2,\ldots,r-1,n\}$ such that its singletons are $\{3\},\{4\},\ldots,\{m_1+2\}$ and it contains the doubleton $\{r,n\}$. For $m_2 = 3$, ρ_1 and ρ_2 are defined similarily, but their singleton classes are $\{1\},\{2\},\ldots,\{m_1\}$, if $m_1 \ge 3$, and $\{2\},\{3\}$, if $m_1 = 2$. Note that ρ_j, $j = 1,2$, is different from π_i, for every $1 \le i \le k-1$. Indeed, if $\{x,n\}$ is a doubleton in π_i, then by definition, π_i has no smaller singleton than x withparity opposite to that of n, but this property is not true for ρ_j.

Obviously $(A_1,\pi_1,\ldots,\pi_{k-1},A_k,\rho_1, B_l,\sigma_{l-1},\ldots,\sigma_1,B_1,\rho_2)$ is a complete τ-loop on X, and it satisfies properties (1) and (2). If $m_1 = 2$, then $\{1\}$ is not a singleton of ρ_1 or ρ_2. Because in this case $n = 2m_2 + 2$ is even, $\{1\}$ is not a singleton in any π_i, $1 \le i \le k-1$. Finally, since $\{n\}$ is a singleton class in every σ, $1 \le j \le l-1$, the loop satisfies property (3) as well.

Case 2: $m_1 = 1$. Let $X = X_n$, $X' = X_{n-1}$, and set $k = \binom{n-1}{r}$, $l = \binom{n-1}{r-1}$. Let $(A_1,\pi_1',\ldots,A_k,\pi_k')$ be a complete loop on X' with partitions of type $\{[2,1],[m_2-1,2]\}$, and let $(B_1',\sigma_1',\ldots,B_l',\sigma_l')$ be a complete loop on X' with partitions of type $\{[m_2,2]\}$. Also assume that both loops satisfy properties (1)–(3).

For $i = 1,\ldots,k-1$, let π_i be the partition obtained from π_i' by adjoining n to the even singleton class. For each $i = 1,\ldots,l$, let $B_i = B_i' \cup \{n\}$ and let σ_i be the partition obtained from σ_i' by adding $\{n\}$ as a singleton class. Note that π_i and σ_j are distinct partitions of X with no even singleton, for every $1 \le i \le k-1$ and $1 \le j \le l-1$. Observe that $\{1\}$ is a singleton in π_i if and only if it is a singleton class in π_i'. Thus, by (3), $\{n-1\}$ is a singleton class in π' which implies that $\{n-1,n\}$ is a doubleton in π_i.

Let ρ_1 be partition of X with classes $\{1\}\{r,n\}\{2,r+1\}\{3,r+2\}\ldots\{r-1,n-1\}$. Set $x = r-1$ if r is even, and $x = r-2$ if r is odd, and define ρ_2 as a partition of X with classes $\{x\}\{1,n\}\{2,r\}\{3,r+1\}\ldots\{r-$

$3, n-4\}\{x-1, n-2\}\{n-1, n-3\}$. Then $(A_1, \pi_1, \ldots, \pi_{k-1}, A_k, \rho_2, B_l, \sigma_{l-1}, \ldots, \sigma_1, B_1, \rho_1)$ is a complete loop on X satisfying properties (1)–(3).

Case 3: $m_1 = 0$. Set $X = X_n$, $X' = X_{n-1}$, and $k = \binom{n-1}{r}$. Let $(A_1, \pi'_1, \ldots, A_k, \pi'_k)$ be a complete loop on X' with partitions of type $\{[1,1],[m_2-1,2]\}$ satisfying properties (1)–(3). In particular, in each partition the only singleton is odd, and if $\{1\}$ is a singleton of π', then $\{n-2, n-1\}$ is a doubleton in π'. For $i = 1, \ldots, k-1$, let π_i be the partition obtained from π'_i by adjoining n to the unique (odd) singleton class.

Define a permutation β of X' by $\beta(1) = 1, \beta(2) = n-1$, and $\beta(i) = i-1$, for $i = 3, \ldots, n-1$. Using β to denote its natural extension to the power set $2^{X'}$, define $B'_i = \beta(A_i)$ and $\sigma'_i = \beta(\pi'_i)$. For every $i = 1, \ldots, k$, let $B_i = X \setminus B'_i$, and let σ_i be the partition obtained from σ'_i by adjoining n to its singleton class. This singleton is either even or equal to $\{1\}$ which implies that $\pi_i \neq \sigma_j$, for every $1 \leq i, j \leq k-1$.

Let $\rho_1 = \{1, n\}\{2, n-2\}\{3, n-3\} \ldots \{r-1, r+1\}\{r, n-1\}$, and let $\rho_2 = \{1, n\}\{2, r+1\}\{3, r+2\} \ldots \{r-1, n-2\}\{r, n-1\}$. Note that, by property (2), $A_1 = \{1, 2, \ldots, r\}$, $A_k = \{1, 2, \ldots, r-1, n-1\}$. Thus $B_1 = \{r, r+1, \ldots, n-2, n\}$ and $B_k = \{r-1, r, \ldots, n-3, n\}$. Therefore, ρ_1 is orthogonal to A_k and B_1, and ρ_2 is orthogonal to B_k and A_1. Then $(A_1, \pi_1, \ldots, \pi_{k-1}, A_k, \rho_1, B_1, \sigma_1, \ldots, \sigma_{k-1}, B_k, \rho_2)$ is a complete loop on X satisfying properties (1)–(3). □

References

[1] I. Anderson, *Combinatorics of Finite Sets*, Oxford University Press, 1987.

[2] J.M. Howie, R.B. McFadden, Idempotent rank in finite full transformation semigroups, Proc. Royal Soc. Edinburgh, 114A (1990) 161–167.

[3] G. Hurlbert, The antipodal layers problem, Discrete Math. 128 (1994) 237-243.

[4] I. Levi, R.B. McFadden, S_n–normal semigroups, Proc. Edinburgh Math. Soc. 37 (1994) 471–476.

[5] I. Levi, R.B. McFadden, The shape of $\mathcal{D}$–classes in S_n–normal semigroups, to appear in Proc. of Tennessee Topology Conference, May 1996.

[6] C.D. Savage, Long cycles in the middle two levels of the Boolean lattice, Ars Combinatoria, 35–A (1993) 97–108.

[7] C.D. Savage and P. Winkler, Monotone Gray Codes and the Middle Levels Problem, Journal of Combinatorial Theory A, 70 (1995) 230-248.