Integral Sum Numbers of Graphs*

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ABSTRACT. The sum graph of a set S of positive integers is the graph $G^+(S)$ having S as its vertex set, with two distinct vertices adjacent whenever their sum is in S. If S is allowed to be a subset of all integers, a graph so obtained is called an integral sum graph. The integral sum number of a given graph G is the smallest number of isolated vertices which when added to G result in an integral sum graph. In this paper, we find the integral sum numbers of caterpillars, cycles, wheels, and complete bipartite graphs.

1 Introduction

The *sum graph* of a subset S of $N = \{1, 2, 3, \cdots\}$ is the graph $G^+(S)$ whose vertex set $V = S$ and whose edge set $E = \{uv | u \neq v \text{ in } S \text{ and } u + v \in S\}$. A *sum graph* is a graph that is isomorphic to the sum graph of some subset of N. This concept was introduced in [4], where some basic properties of sum graphs were presented. Given any graph G with n vertices v_i and m edges, it is trivial that the union $G \cup mK_1$ of G with m isolated vertices is a sum graph. This fact follows at once by labeling each v_i with 10^i and the m isolated vertices with $10^i + 10^j$ whenever $v_iv_j \in E$. From this, we can define the *sum number* $\sigma(G)$ of G as the smallest nonnegative integer m such that $G \cup mK_1$ is a sum graph. Note that G is a sum graph if and only if $\sigma(G) = 0$.

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Harary [5] gave

$$\sigma(C_n) = \begin{cases} 3, & \text{if } n = 4, \\ 2, & \text{if } n \neq 4. \end{cases}$$

Ellingham [3] proved that $\sigma(T) = 1$ for any nontrivial tree T. Bergstrand et al. [2] found that $\sigma(K_n) = 2n - 3$ for $n \geq 4$. Hartsfield and Smyth [6] demonstrated that $\sigma(K_{m,n}) = \lceil \frac{3m+n-3}{2} \rceil$ for $2 \leq m \leq n$. The following example shows that their solution is only a good upper bound rather than an exact value of $\sigma(K_{m,n})$. If

$$A = \{1,5\},\ B = \{90,91,92,95,96,100\},\ I = \{93,97,101,105\},$$

then $G^+(A \cup B \cup I) = K_{2,6} \cup 4K_1$ and so $\sigma(K_{2,6}) \leq 4 < 5 = \lceil \frac{3\cdot 2+6-3}{2} \rceil$.

Suppose S is a subset of the set Z of all integers. The *integral sum graph* $G^+(S)$ is defined as the sum graph, the difference being that $S \subseteq Z$, instead of $S \subseteq N$. The *integral sum number* $\zeta(G)$ is the smallest nonnegative integer m such that $G \cup mK_1$ is an *integral sum graph* (i.e., $G \cup mK_1$ is isomorphic to $G^+(S)$ for some $S \subseteq Z$). These concepts were introduced by Harary [5], who also raised some unsolved problems.

The main results presented in this paper are the integral sum numbers of caterpillars, cycles, wheels, and complete bipartite graphs.

2 Caterpillars, cycles, and wheels

The purpose of this section is to prove that every caterpillar, every n-cycle C_n with $n \neq 4$, or every n-wheel W_n with $n \neq 3$ is an integral sum graph. Note that a sum graph is an integral sum graph. Therefore we have

Lemma 1 *$0 \leq \zeta(G) \leq \sigma(G)$ for any graph G.*

Lemma 1, together with Ellingham's result, shows that $0 \leq \zeta(T) \leq 1$ for any tree T. A star $K_{1,n}$ is an example where $\zeta(K_{1,n}) = 0 < 1 = \sigma(K_{1,n})$, since $G^+(\{1, n+1, n+2, \cdots, 2n, 2n+1\}) = K_{1,n} \cup K_1$ and $G^+(\{0, n+1, n+2, \cdots, 2n\}) = K_{1,n}$. Harary [5] showed that any P_n is an integral sum graph. He then conjectured that any integral sum tree is a caterpillar. Chen [1] gave an infinite number of integral sum trees that are not caterpillars. Our first result is to show that every caterpillar is an integral sum graph. We conjecture that any tree is an integral sum graph.

For technical reasons, we define a more general concept than an integral sum graph as follows. Suppose x is a vertex of a graph $G = (V, E)$. G is said to be a *$(*)$-sum graph with respect to x* if Conditions (S1) and (S2) hold.

(S1) There is a one-to-one function f from V to Z such that G is an integral sum graph of $\{f(u)\colon u \in V\}$.

(S2) $f(x) > |f(u)| > 0 > f(v)$ for all $u \in V - \{x\}$ and some $v \in V$.

Lemma 2 *If $G = (V, E)$ is a $(*)$-sum graph with respect to x, then $G^* = (V^*, E^*)$ is a $(*)$-sum graph with respect to x_k, where $V^* = V \cup \{x_1, x_2, \cdots, x_k\}$ and $E^* = E \cup \{xx_1, xx_2, \cdots, xx_k\}$ for some $k \geq 1$.*

Proof: Suppose $f(y) = \min\{f(v) \colon v \in V \text{ and } f(v) < 0\}$. Define f^* on V^* by

$$f^*(u) = \begin{cases} -f(u), & \text{if } u \in V, \\ if(x) - f(y), & \text{if } u = x_i \text{ and } 1 \leq i \leq k, \end{cases}$$

see Figure 1 for the construction of G^* and f^*. We shall show that f^*, with respect to x_k, satisfies Conditions (S1) and (S2). Note that

$$f^*(x_k) > f^*(x_{k-1}) > \cdots > f^*(x_1) > |f^*(u)| > 0 > f^*(x) \text{ for all } u \in V.$$

Then f^* is a one-to-one function from V^* to Z and Condition (S2) holds for f^* with respect to x_k. So, we only need to show that G^* is the integral sum graph of $\{f^*(u) \colon u \in V^*\}$.

First note that $f^*(y) > f^*(u)$ for all $u \in V - \{y\}$. Therefore, for $u \neq v$ in V and $w \in V^* - V$, $f^*(w) \geq f(x) - f(y) > -2f(y) = 2f^*(y) > f^*(u) + f^*(v)$, i.e., $f^*(u) + f^*(v) \neq f^*(w)$. Next, suppose $f^*(u) + f^*(v) = f^*(w)$ for some $u \in V^* - V, v \in V^* - \{x\}$, and $w \in V^*$. If $w \in V$, then $|f^*(v)| = f^*(u) - f^*(w) \geq f(x) + f^*(y) - f^*(w) \geq f(x)$. If $w \in V^* - V$, since $w \neq u$, $|f^*(v)| = |f^*(w) - f^*(u)| \geq |f(x)|$. In any case, $v \in V^* - V$, which implies $w \in V^* - V$ too. So, $if(x) - f(y) + jf(x) - f(y) = pf(x) - f(y)$ for some $1 \leq i, j, p \leq k$. Hence $(i + j - p)f(x) = f(y)$, which implies $|f(y)| \geq |f(x)|$, a contradiction. Therefore $f^*(u) + f^*(v) \neq f^*(w)$ for $u \in V^* - V, v \in V^* - \{x\}$, and $w \in V^*$. Thus G^* is an integral sum graph of $\{f^*(u) \colon u \in V^*\}$. □

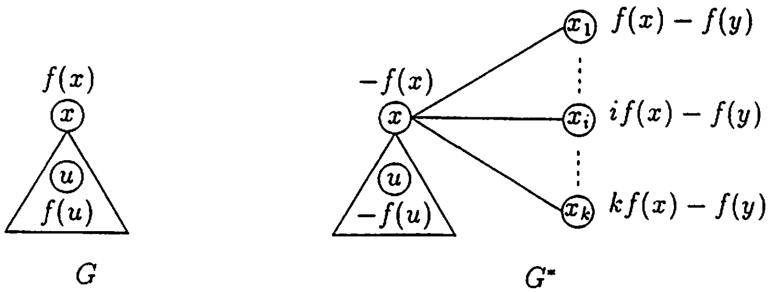

Figure 1. The construction of G^* and f^*

Theorem 3 *Every caterpillar is an integral sum graph.*

Proof: Suppose $T_r = (V_r, E_r)$ is a caterpillar with a *spine* of length r as shown in Figure 2, i.e.,

$$V_r = \{u_{i,j} \colon 0 \leq i \leq r \text{ and } 1 \leq j \leq n_i\} \qquad \text{and}$$

$$E_r = \{u_{i,n_i}u_{i+1,j}\colon\ 0 \le i \le r-1 \text{ and } 1 \le j \le n_{i+1}\},$$

where $n_0 = 1 \le n_i$ for $1 \le i \le r$ and $2 \le n_1$ for $r \ge 2$. Note that, if we consider T_{r-1} as G in Lemma 2 and $u_{r-1,n_{r-1}}$ as x and n_r as k, then T_r is G^*.

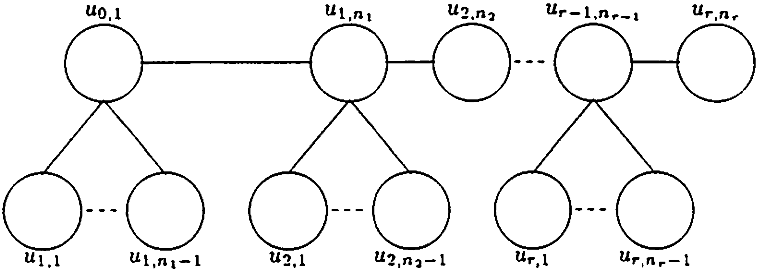

Figure 2. A caterpillar T_r

For $r = 1$, $T_1 = G^+(\{0, n_1+1, n_1+2, \cdots, 2n_1\})$ is an integral sum graph. For $r = 2$, define

$$f_2(u_{i,j}) = \begin{cases} 1, & \text{if } i = 0 \text{ and } j = 1, \\ j - 2n_1, & \text{if } i = 1 \text{ and } 1 \le j \le n_1, \\ 1 - n_1, & \text{if } i = 2 \text{ and } j = 1, \\ 1 + (j-1)n_1, & \text{if } i = 2 \text{ and } 2 \le j \le n_2, \end{cases}$$

see Figure 3. Since $n_1 \ge 2$, it is straightforward to check that T_2 is an integral sum graph. Moreover, for the case of $n_2 \ge 3$, T_2 is a $(*)$-sum graph with respect to u_{2,n_2}. Hence, by induction and Lemma 2, T_r is an integral sum graph for $r \ge 3$ and $n_2 \ge 3$.

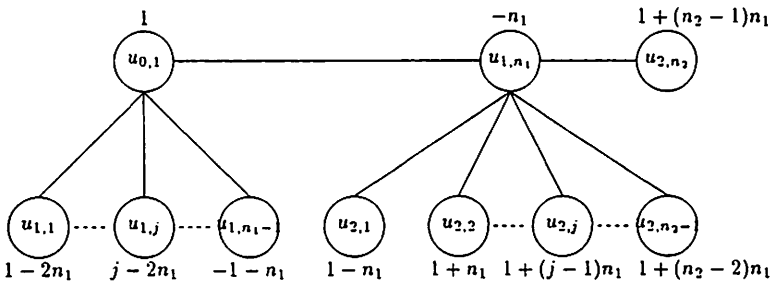

Figure 3. A labeling f_2 for T_2

For $r = 3$ with $n_2 \le 2 < 3 \le n_1$, define

$$f_3(u_{i,j}) = \begin{cases} 3, & \text{if } i = 0 \text{ and } j = 1, \\ 3j - 4, & \text{if } i = 1 \text{ and } 1 \le j \le n_1, \\ -3n_1 + 7, & \text{if } i = 2 \text{ and } j = 1 < 2 = n_2, \\ -3n_1 + 3, & \text{if } i = 2 \text{ and } j = n_2, \\ 3jn_1 - 3j + 2, & \text{if } i = 3 \text{ and } 1 \le j \le n_3 \end{cases}$$

(see Figure 4). Since $n_1 \ge 3$, a direct check shows that T_3 is a $(*)$-sum graph with respect to u_{3,n_3}. Hence, by induction and Lemma 2, T_r is an integral sum graph for $r \ge 3$ and $n_2 \le 2 < 3 \le n_1$.

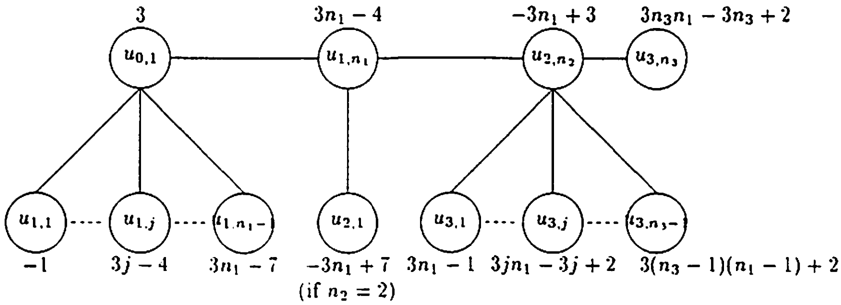

Figure 4. A labeling f_3 for T_3 with $n_2 \le 2 < 3 \le n_1$

For $r = 3$ with $n_2 \le 2 = n_1$, define

$$f_4(u_{i,j}) = \begin{cases} n_2 + 1, & \text{if } i = 0 \text{ and } j = 1, \\ n_2, & \text{if } i = 1 \text{ and } j = 1, \\ -1, & \text{if } i = 1 \text{ and } j = 2, \\ n_2 + 2, & \text{if } i = 2 \text{ and } j = 1 < 2 = n_2, \\ 2n_2 + 1, & \text{if } i = 2 \text{ and } j = n_2 \end{cases}$$

(see Figure 5). It is easy to check that T_2 is a $(*)$-sum graph with respect to u_{2,n_2}. Hence, by induction and Lemma 2, T_r is an integral sum graph for $r \ge 3$ and $n_2 \le 2 = n_1$. □

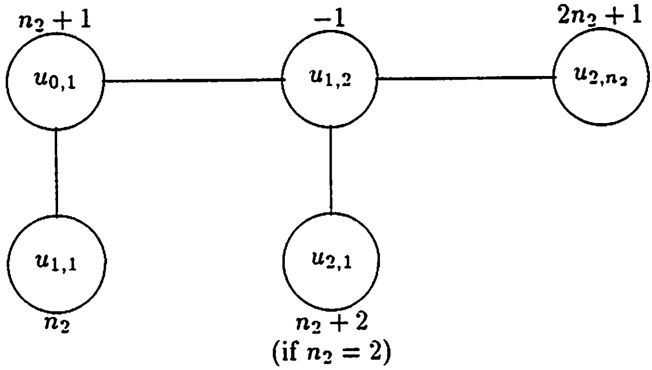

Figure 5. A labeling f_4 for T_2 with $n_2 \leq 2 = n_1$

Next, we study the integral sum numbers of n-cycles C_n and n-wheels W_n, which are obtained from C_n by adding a new vertex adjacent to each of the vertices in C_n. Harary [5] showed that $\zeta(C_4) = 3$ and $\zeta(W_3) = \zeta(K_4) = 5$. Except for these two cases, however, all other cycles and wheels are integral sum graphs.

Theorem 4 *The n-cycle C_n is an integral sum graph for $n \neq 4$.*

Proof: It is easy to confirm that

$$\begin{aligned}
C_3 &= G^+(\{-1,0,1\}),\\
C_5 &= G^+(\{1,2,-1,3,-2\}),\\
C_6 &= G^+(\{1,4,-3,-1,5,-4\}),\\
C_7 &= G^+(\{1,2,-5,7,-3,4,3\}),\\
C_8 &= G^+(\{1,5,2,7,-2,9,-8,6\}),\\
C_9 &= G^+(\{3,4,-1,5,-12,17,-7,10,7\}),\\
C_n &= G^+(\{a_1,a_2,\cdots,a_{n-4},-a_{n-5},a_{n-3},a_3-a_{n-3},a_{n-3}-a_1\})
\end{aligned}$$

for $n \geq 10$, where $a_1 = 1$, $a_2 = 2$, and $a_i = a_{i-1} + a_{i-2}$ for $3 \leq i \leq n$. □

Theorem 5 *The n-wheel W_n is an integral sum graph for $n \neq 3$.*

Proof: It is easy to confirm that

$$\begin{aligned}
W_4 &= G^+(\{1,-1,2,-2,0\}),\\
W_5 &= G^+(\{1,2,-2,3,-3,0\}),\\
W_6 &= G^+(\{1,2,3,-3,5,-4,0\}),\\
W_7 &= G^+(\{1,2,3,5,-5,8,-6,0\}),\\
W_8 &= G^+(\{1,2,-9,11,-7,7,4,3,0\}),\\
W_9 &= G^+(\{1,2,-10,12,-3,3,9,-8,8,0\}),\\
W_{10} &= G^+(\{3,4,-1,5,-22,27,-17,17,10,7,0\}),\\
W_n &= G^+(\{a_1,a_2,\cdots,a_{n-4},-a_{n-4},a_{n-3},a_3-a_{n-3},a_{n-3}-a_1,0\})
\end{aligned}$$

for $n \geq 11$, where $a_1 = 1$, $a_2 = 2$, and $a_i = a_{i-1} + a_{i-2}$ for $3 \leq i \leq n$. □

3 Complete bipartite graphs

As the example in Section 1 shows, Hartsfield and Smyth's solution is only a good upper bound rather an exact value for $\sigma(K_{m,n})$. This section gives complete solutions to $\sigma(K_{m,n})$ and $\zeta(K_{m,n})$. As shown in Section 2, $\zeta(K_{1,n}) = 0 < 1 = \sigma(K_{1,n})$. So, from now on, we consider only $K_{m,n}$ with $m \geq 2$ and $n \geq 2$.

In this section, we suppose that $S \subseteq Z$ is such that $G^+(S) = K_{m,n} \cup \zeta(K_{m,n})K_1$, where $A \subseteq S$, $B \subseteq S$, and $S - A - B$ corresponds to the partite set of m vertices, the partite set of n vertices, and the isolated vertices in $\zeta(K_{m,n})K_1$. Note that $0 \notin S$, otherwise $K_{m,n} \cup \zeta(K_{m,n})K_1$ has a vertex adjacent to all other vertices, which contradicts $m \geq 2$ and $n \geq 2$.

Lemma 6 *(i) If $a + b \in B$ for some $a \in A$ and $b \in B$, then $a' + b \in B$ for all $a' \in A$.*

(ii) If $a+b \in A$ for some $a \in A$ and $b \in B$, then $a+b' \in A$ for all $b' \in B$.

Proof: (i) $a' \in A$ and $b \in B$ imply $a' + b \in S$. Also, $a' \in A$ and $a + b \in B$ imply $a' + (a + b) \in S$ or $a + (a' + b) \in S$. Then, $a \in A$, $a' + b \in S$ and $a + (a' + b) \in S$ imply $a = a' + b$ or $a' + b \in B$. Since $a' + b \in B$ means (i) holds, we may assume

$$a = a' + b. \tag{1}$$

And so, $2a = a' + (a+b) \in S$. Next, $a \in A$ and $a+b \in B$ imply $a+(a+b) \in S$ or $2a+b \in S$. Then, $2a \in S$, $b \in B$, and $2a+b \in S$ imply $2a = b$ or $2a \in A$. We shall prove that neither is true.

Suppose $2a = b$. By (1), $-a = a'$. $a \in A$ and $3a = a + b \in B$ imply $4a \in S$. Then, $-a = a' \in A$, $4a \in S$, and $-a + 4a = 3a \in B$ imply $-a = 4a$ or $4a \in B$. Since $-a = 4a$ implies $0 = a \in S$ is impossible, we may assume $4a \in B$. $a \in A$ and $4a \in B$ imply $5a \in S$. But this implies $b + (a + b) = 5a \in S$ for $b \in B$ and $a + b \in B$, contradicting $b \neq a + b$ and $\{b, a + b\} \notin E(K_{m,n})$.

Suppose $2a \in A$. This, together with $b \in B$ and $a + b \in B$, implies $2a + b \in S$ and $3a + b \in S$. Then, $a \in A$, $2a + b \in S$, and $a + (2a + b) \in S$ imply $a = 2a + b$ or $2a + b \in B$. Since $a = 2a + b$ implies $0 = a + b \in B$ is impossible, we may assume $2a + b \in B$. $a' \in A$ and $2a + b \in B$ imply $a' + (2a + b) \in S$. But this and (1) imply $a + 2a = a' + (2a + b) \in S$ for $a \in A$ and $2a \in A$, contradicting $a \neq 2a$ and $\{a, 2a\} \notin E(K_{m,n})$.

(ii) The proof is similar to that of (i). □

Lemma 7 *If $|A| \le |B|$, then $a+b \in S-A$ for all $a \in A$ and $b \in B$.*

Proof: Suppose $a+b \in A$ for some $a \in A$ and $b \in B$. By Lemma 6 (ii), $a+b' \in A$ for all $b' \in B$, and so $|A| \ge |B|$. However, $a+0 \in A$ and $0 \notin B$, so $|A| > |B|$, which is impossible. □

Now, for convenience, we assume $2 \le |A| = m \le n = |B|$. Let $A = \{a_1, a_2, \cdots, a_m\}$ with $a_1 < a_2 < \cdots < a_m$ and $B_0 = S - A - B$. For any $b \in B$, by Lemma 7, we have $a_1 + b \in S - A$. If $a_1 + b \in B$, then $a_1 + (a_1 + b) = 2a_1 + b \in S - A$. Continuing this process, there must be a positive integer x such that $xa_1 + b \in B_0$. For any $b \in B$, let $k_b = \min\{x | xa_1 + b \in B_0 \text{ and } x \in N\}$ and $k = \max\{k_b | b \in B\}$. Then, B can be partitioned into $B_1, B_2, \cdots, B_k$, where $B_i = \{b \in B | k_b = i\}$.

Denote $X + Y = \{x + y | x \in X \text{ and } y \in Y\}$.

Lemma 8 *$A + B_i \subseteq B_{i-1}$ and $|B_{i-1}| \ge |B_i| + m - 1$ for $1 \le i \le k$.*

Proof: We shall prove $A + B_i \subseteq B_{i-1}$ by induction on i.

Suppose $a + b \notin B_0$ for some $a \in A$ and $b \in B_1 \subseteq B$. By Lemma 7, $a + b \in S - A$ and so $a + b \in B$. By Lemma 6 (i), $a \in A$, $b \in B$, and $a + b \in B$ imply $a_1 + b \in B$ and so $a_1 + b \notin B_0$, contradicts the fact that $b \in B_1$. Thus, $A + B_1 \subseteq B_0$.

Assume $A + B_{i-1} \subseteq B_{i-2}$. For any $b \in B_i$, $ia_1 + b \in B_0$ or $(i-1)a_1 + (a_1+b) \in B_0$, *i.e.*, $k_{a_1+b} \le i-1$. On the other hand, $k_{a_1+b}a_1 + (a_1+b) \in B_0$ or $(k_{a_1+b}+1)a_1 + b \in B_0$ implies $k_{a_1+b} + 1 \ge k_b = i$. So, $k_{a_1+b} = i-1$ and then $a_1 + b \in B_{i-1}$. For any $a \in A$, $a + (a_1 + b) \in B_{i-2}$ by the induction hypothesis. Consequently, $(i-2)a_1 + a + (a_1+b) = (i-1)a_1 + (a+b) \in B_0$, *i.e.*, $k_{a+b} \le i-1$. Also, $k_{a+b}a_1 + (a+b) \in B_0$ or $(k_{a+b}-1)a_1 + a + (a_1+b) \in B_0$ implies $k_{a+b} - 1 \ge k_{a+(a_1+b)} = i-2$. So, $k_{a+b} = i-1$ and then $a+b \in B_{i-1}$. Thus, $A + B_i \subseteq B_{i-1}$.

Moreover, for $1 \le i \le k$, let $B_i = \{b_1; b_2, \cdots, b_j\}$, where $b_1 < b_2 < \cdots < b_j$. We have

$$a_1 + b_1 < a_2 + b_1 < \cdots < a_m + b_1 < a_m + b_2 < \cdots < a_m + b_j,$$

which are $m + j - 1$ distinct element in B_{i-1}. So, $|B_{i-1}| \ge |B_i| + m - 1$. □

Lemma 9 *If m, n, h, $n_0, n_1, n_2, \cdots, n_h$ are positive integers such that $2 \le m \le n = \sum_{i=1}^{h} n_i$ and $n_{i-1} \ge n_i + m - 1$ for $1 \le i \le h$, then $K_{m,n} \cup n_0 K_1$ is a sum graph.*

Proof: Choose integers $d > h + 1$ and $c > \max\{2md, h + n_0 d\}$. Consider the following sets of positive integers

$$X = \{x_q \equiv 1 + (q-1)d | 1 \le q \le m\},$$
$$Y_i = \{y_{ij} \equiv c + h - i + jd | 1 \le j \le n_i\} \quad \text{for } 0 \le i \le h.$$

Note that

$$1 \le x_q < md < c < y_{ij} < 2c \quad for\ x_q \in X\ and\ y_{ij} \in Y_i. \tag{2}$$

So any $y_{ij} >$ any x_q, *i.e.*, $X \cap Y_i = \emptyset$ for $0 \le i \le h$. Also $Y_i \cap Y_{i'} = \emptyset$ for $0 \le i < i' \le h$, otherwise $c+h-i+jd = c+h-i'+j'd$ would imply $i'-i$ is a multiple of d, which contradicts $1 \le i'-i \le h < d$. Let

$$Y = Y_1 \cup \cdots \cup Y_h \text{ and } S = X \cup Y \cup Y_0.$$

Then, $|X| = m$, $|Y| = n$, $|S| = m+n+n_0$. For $x_q \in X$ and $y_{ij} \in Y_i$ with $1 \le i \le h$, $1 \le q \le m$ and $1 \le j \le n_i$ imply $1 \le j+q-1 \le n_i+m-1 \le n_{i-1}$, and so

$$x_q + y_{ij} = c + h - (i-1) + (j+q-1)d \in Y_{i-1} \subseteq S.$$

Also, for all other $(x,y) \in S \times S$, $x+y \notin S$. More precisely, for $x_p, x_q \in X$, $x_p + x_q = 2 + (p+q-2)d \notin X$ and $x_p + x_q < 2md < c <$ any y_{ij}. For $x_q \in X$ and $y_{0j} \in Y_0$, $x_q + y_{0j} = c+h+1+(q+j-1)d \neq$ any $c+h-i+j'd$ with $0 \le i \le h$, otherwise $i+1$ is a multiple of d, which contradicts $1 \le i+1 \le h+1 < d$. For $y_{ij}, y_{i'j'} \in Y, y_{ij} + y_{i'j'} > 2c$, and so $y_{ij} + y_{i'j'}$ is not in S. Thus $G^+(S) = K_{m,n} \cup n_0 K_1$. □

Theorem 10 *If $2 \le m \le n$, then $\zeta(K_{m,n}) = \sigma(K_{m,n}) = \lceil \frac{n}{p} + \frac{(p+1)(m-1)}{2} \rceil$, where $p = \lceil \sqrt{\frac{2n}{m-1} + \frac{1}{4}} - \frac{1}{2} \rceil$ is the unique positive integer such that $\frac{(p-1)p(m-1)}{2} < n \le \frac{p(p+1)(m-1)}{2}$.*

Proof: Let $n_0 = |B_0| = \zeta(K_{m,n})$ and p be a unique positive integer such that

$$1 + p(m-1) \le n_0 \le (p+1)(m-1). \tag{3}$$

From Lemma 8, we have $|B_i| \le n_0 - i(m-1)$ for $1 \le i \le k$. In particular, $1 \le |B_k| \le n_0 - k(m-1) \le (p+1-k)(m-1)$, and so $k \le p$. Next, $n = |B| = |\cup_{i=1}^k B_i| = \sum_{i=1}^k |B_i| \le \sum_{i=1}^k \{n_0 - i(m-1)\} \le \sum_{i=1}^p \{n_0 - i(m-1)\} = pn_0 - \frac{p(p+1)(m-1)}{2}$. Note that, by (3), each $n_i \equiv n_0 - i(m-1) \ge 1$ for $1 \le i \le p$.

Suppose $n \le pn_0 - \frac{p(p+1)(m-1)}{2} - p = \sum_{i=1}^p (n_i - 1)$. Choose h such that $\sum_{i=1}^{h-1}(n_i - 1) < n \le \sum_{i=1}^h (n_i - 1)$; and set $n'_i = n_i - 1$ for $0 \le i \le h-1$ and $n'_h = n - \sum_{i=1}^{h-1} n'_i$. Then m, n, h, $n'_0, \cdots, n'_h$ satisfy the conditions in Lemma 9. By Lemma 9, $K_{m,n} \cup n'_0 K_1$ is a sum graph, which contradicts $n'_0 < n_0 = \zeta(K_{m,n})$. So,

$$pn_0 - \frac{p(p+1)(m-1)}{2} - p < n \le pn_0 - \frac{p(p+1)(m-1)}{2}, \tag{4}$$

or

$$\frac{n}{p}+\frac{(p+1)(m-1)}{2}\leq n_0<\frac{n}{p}+\frac{(p+1)(m-1)}{2}+1;$$

and then $\zeta(K_{m,n})=\lceil\frac{n}{p}+\frac{(p+1)(m-1)}{2}\rceil$. By (3) and (4), we have

$$\frac{(p-1)p(m-1)}{2}<n\leq\frac{p(p+1)(m-1)}{2},$$

or

$$(p-\frac{1}{2})^2=p^2-p+\frac{1}{4}<\frac{2n}{m-1}+\frac{1}{4}\leq p^2+p+\frac{1}{4}=(p+\frac{1}{2})^2,$$

or $p=\lceil\sqrt{\frac{2n}{m-1}+\frac{1}{4}}-\frac{1}{2}\rceil$.

Finally, by applying Lemma 9 and by using m, n, $h=k$, $n_0=|B_0|, n_1=|B_1|,\cdots,n_k=|B_k|$, we have $\sigma(K_{m,n})\leq\zeta(K_{m,n})$. This together with Lemma 1 gives $\sigma(K_{m,n})=\zeta(K_{m,n})$. □

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