# On the Group of Line Crossings on the 2-D Torus

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**Abstract**

We give a new and simple proof for the cyclic group of line crossings on the 2-D torus.

## 1 Introduction

Let $\mathbf{R}^2$ be the plane. For $(x_1, y_1), (x_2, y_2) \in \mathbf{R}^2$, define a relation $R$ on $\mathbf{R}^2$ by $(x_1, y_1) R (x_2, y_2)$ if and only if both $x_2 - x_1$ and $y_2 - y_1$ are integers. This relation is an equivalence relation. The equivalence class of $(x, y)$ is denoted by $[x, y]$. The (2-dimensional) torus is defined to be the set of all equivalence classes under $R$, i.e. $T = \mathbf{R}^2 / R$. Since each equivalence class has a unique representative in the square $[0,1) \times [0,1)$, $T$ can be identified with the square $[0,1) \times [0,1)$.

A line on the torus $T$ is the subset defined by $\bar{L} = \{[x,y] \in T | \, [x,y] \cap L \neq \emptyset\}$ where $L$ denotes a line on the plane. The slope of $\bar{L}$ is defined to be the slope of $L$. Now, let $S$ be a set of $n$ ordered pairs $(a_i, b_i)$ of integers with $\gcd(a_i, b_i) = 1$ and $a_i b_j \neq a_j b_i$ whenever $i \neq j$. For each $1 \leq i \leq n$, let $L_i$ be the line on the plane satisfying the linear equation $a_i x + b_i y = 0$. Figueroa and Salzberg [1] studied the set $J = \bigcap_{i=1}^{n} \bar{L}_i$ and showed that $J$ is a cyclic group under the addition module 1. As mentioned by Figueroa and Salzberg, this result can be used in dealing with the discrete limited angle model for computerized tomography proposed by Salzberg [2] in projective spaces and developed by Salzberg and Steffens [4] in an Euclidean geometry.

Figueroa and Salzberg [1] studied the group $J$ by using group theory. We shall give a new and simple proof in the next section. The method we will

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use is a generalization of the method used by Salzberg [3] for studying the intersection of two lines $\bar{L}_k$ and $\bar{L}_\ell$ with integral slopes $k$ and $\ell$, respectively.

# 2 The group of line crossings

All notations in the last section are inherited here. For $1 \le i, j \le n$, let $d_i$, $A_{i,j}$ be the determinant of the $2 \times 2$ matrix $A_{i,j} = \begin{bmatrix} a_i & b_i \\ a_j & b_j \end{bmatrix}$. Moreover, let $d$ be the greatest common divisor of all $d_{i,j}$, $1 \le i < j \le n$. We state and prove Figueroa and Salzberg's result in the following.

**Theorem 1** *If $n = 1$, then $J = \bar{L}_1$ is an infinite cyclic group. If $n \ge 2$, then $d > 0$ and $J$ is a cyclic group of order $d$ with a generator $\left[\frac{-b_1}{d}, \frac{a_1}{d}\right]$.*

**Proof.** If $n = 1$, then $J = \bar{L}_1$ is trivially an infinite cyclic group by definition.

We now consider $n \ge 2$. Since for $1 \le i \ne j \le n$, $a_i b_j \ne a_j b_i$ and $\gcd(a_i, b_i) = 1 = \gcd(a_j, b_j)$, we have $d_{i,j} \ne 0$ for all $1 \le i < j \le n$. So, $d \ne 0$.

Let $[x_0, y_0] \in J$. There are integers $c_1, \cdots, c_n$ so that $(x_0, y_0)$ is a common solution of linear system $a_i x + b_i y = c_i$, $1 \le i \le n$. So, $(x_0, y_0)$ is a common solution of linear system

$$\begin{aligned} a_i x + b_i y &= c_i \\ a_j x + b_j y &= c_j \end{aligned} \tag{1}$$

for all $1 \le i < j \le n$. This implies that both $x_0$ and $y_0$ are rational number. Write $x_0 = \frac{s}{t}$ and $y_0 = \frac{v}{u}$ with $\gcd(s, t) = 1 = \gcd(u, v)$ and $tu \ne 0$. Since $\left(\frac{s}{t}, \frac{v}{u}\right)$ is a common system of (1), $1 \le i < j \le n$, both $t$ and $u$ divide $d_{i,j}$ for all $1 \le i, j \le n$. So, $t|d$ and $u|d$. This implies that we may write $(x_0, y_0) = \left(\frac{s_0}{d}, \frac{t_0}{d}\right)$ for some integers $s_0$ and $t_0$. Therefore, every element of $J$ is of the form $\left[\frac{s}{d}, \frac{t}{d}\right]$ for some integers $s$ and $t$.

On the other hand, $(x_0, y_0)$ is a common solution of equations $a_1 x + b_1 y = c_1$ and $a_2 x + b_2 y = c_2$. So, $(x_0, y_0) = c_2\left(\frac{-b_1}{d_{12}}, \frac{a_1}{d_{12}}\right) + c_1\left(\frac{b_2}{d_{12}}, \frac{-a_2}{d_{12}}\right)$ by Cramer's rule. This implies that every element of $J$ can also be written as $s\left[\frac{-b_1}{d_{12}}, \frac{a_1}{d_{12}}\right] + t\left[\frac{b_2}{d_{12}}, \frac{-a_2}{d_{12}}\right]$ for some suitable integers $s$ and $t$. Since $\gcd(a_1, b_1) = 1$, there are integers $s_1$ and $t_1$ satisfying $s_1 a_1 + t_1 b_1 = 1$. Now, $\left[\frac{b_2}{d_{12}}, \frac{-a_2}{d_{12}}\right] + (s_1 a_2 + t_1 b_2)\left[\frac{-b_1}{d_{12}}, \frac{a_1}{d_{12}}\right] = (s_1 a_1 + t_1 b_1)\left[\frac{b_2}{d_{12}}, \frac{-a_2}{d_{12}}\right] + (s_1 a_2 + t_1 b_2)\left[\frac{-b_1}{d_{12}}, \frac{a_1}{d_{12}}\right] = \left[\frac{s_1(a_1 b_2 - a_2 b_1)}{d_{12}}, \frac{t_1(a_1 b_2 - a_2 b_1)}{d_{12}}\right] = [t_1, t_2] = [0, 0]$. Thus $\left[\frac{b_2}{d_{12}}, \frac{-a_2}{d_{12}}\right] = -(s_1 a_2 + t_1 b_2)\left[\frac{-b_1}{d_{12}}, \frac{a_1}{d_{12}}\right]$. Combining the results above, every element of $J$ is also of the form $\left[\frac{-b_1 u}{d_{12}}, \frac{a_1 u}{d_{12}}\right]$ for some integer $u$.

Notice that, from the definition, $[x_1, y_1] = [x_2, y_2]$ if and only if both $x_2 - x_1$ and $y_2 - y_1$ are integers. Since $\gcd(a_1, b_1) = 1$ and every element of $J$ can be written in two forms $\left[\frac{s}{d}, \frac{t}{d}\right]$ and $\left[\frac{-b_1 u}{d_{12}}, \frac{a_1 u}{d_{12}}\right]$, every element of $J$ must be of the form $r\left[\frac{-b_1}{d}, \frac{a_1}{d}\right]$ for some integer $r$.

Finally, if we can show that for any integer $r$, $a_i \frac{-b_1 r}{d} + b_i \frac{a_1 r}{d}$ is an integer for all $1 \leq i \leq n$, then $J$ is a cyclic group of order $d$ with a generator $\left[\frac{-b_1}{d}, \frac{a_1}{d}\right]$. In fact, it is sufficient to consider $r = 1$.

Trivially $a_1 \frac{-b_1}{d} + b_1 \frac{a_1}{d} = 0$ and $a_2 \frac{-b_1}{d} + b_2 \frac{a_1}{d} = \frac{d_{11}}{d}$ are integers. For any $3 \leq i \leq n$, let $\ell_{i1}$, $\ell_{i2}$ and $\ell_i$ be integers so that $\gcd(\ell_{i1}, \ell_{i2}) = 1$ and $\ell_{i1}(a_1, b_1) + \ell_{i2}(a_2, b_2) = \ell_i(a_i, b_i)$. Then, $d_{1i} = \frac{\ell_{i2} d_{12}}{\ell_i}$ and $d_{2i} = \frac{\ell_{i1} d_{12}}{\ell_i}$ for all $3 \leq i \leq n$. Since $\gcd(\ell_{i1}, \ell_{i2}) = 1$ and $d$ divides both $d_{1i}$ and $d_{2i}$, we have that for $3 \leq i \leq n$, $\ell_i$ divides $d_{12}$ and $d$ divides $\frac{d_{12}}{\ell_i}$. This implies that $\ell_i$ divides $\frac{d_{12}}{d}$ for all $3 \leq i \leq n$. Now, for any $3 \leq i \leq n$, $a_i \frac{-b_1}{d} + b_i \frac{a_1}{d} = \frac{d_{1i}}{d} = \frac{\ell_{i2} d_{12}}{\ell_i d}$ is an integer. This completes the proof.

# References


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