Decompositions and Packings of Digraphs with Orientations of a 4-Cycle

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Abstract. We present necessary and sufficient conditions for the decomposition of the complete symmetric bipartite digraph into each of the orientations of a 4–cycle (in the cases for which such decompositions are not already known). We use these results to find optimal packings of the complete symmetric digraph with each of the orientations of a 4–cycle. Finally we give necessary and sufficient conditions for the existence of a decomposition of the complete symmetric digraph on v vertices with a hole of size w into each of the orientations of a 4–cycle.

1 Introduction

A *maximal packing* of a digraph D with isomorphic copies of a digraph d is a set $\{d_1, d_2, \ldots, d_n\}$ where $d_i \cong d$ and $V(d_i) \subset V(D)$ for all i, $A(d_i) \bigcap A(d_j) = \emptyset$ if $i \neq j$, $\bigcup_{i=1}^{n} d_i \subset D$, and

$$\left| A(D) \setminus \bigcup_{i=1}^{n} A(d_i) \right|$$

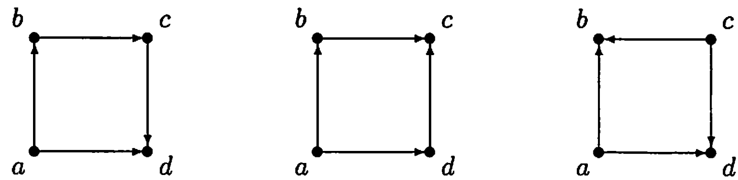
is minimal, where $V(D)$ is the vertex set of digraph D and $A(D)$ is the arc set of digraph D. A maximal packing of D with isomorphic copies of d such that $\bigcup_{i=1}^{n} d_i = D$ is an *isomorphic decomposition* of D into copies of d (or a "d–decomposition of D" for short). Packings and decompositions of (undirected) graphs are similarly defined. Decompositions of the complete graph on v vertices, K_v, into cycles have been extensively studied (see [5] for a survey). Packings of the complete graph with isomorphic copies of a graph g have been studied for g a 3-cycle [7], g a 4-cycle [9], $g = K_4$ [1], and g a 6-cycle [3, 4].

Let $K(v,w)$ denote the *complete graph on v vertices with a hole of size w*. Namely, $K(v,w)$ has vertex set $V(K(v,w)) = V_w \bigcup V_{v-w}$ where $|V_{v-w}| = v - w$ and $|V_w| = w$ and edge set

$$E(K(v,w)) = \{(a,b) \mid a \neq b, \{a,b\} \subset V_{v-w} \bigcup V_w \text{ and } \{a,b\} \not\subset V_w\}$$

(the complete symmetric digraph on v vertices with a hole of size w, $D(v,w)$, is similarly defined). A 3-cycle decomposition of $K(v,w)$ is a *Steiner triple system of order v with a hole of size w* and the existence of these designs is studied in [6].

There are four orientations of a 4-cycle:



X Y Z

and the 4−circuit. We denote these digraphs as $[a,b,c,d]_X$, $[a,b,c,d]_Y$, and $[a,b,c,d]_Z$, respectively, we denote the 4−circuit with arcs (a,b), (b,c), (c,d) and (d,a) as $[a,b,c,d]_C$ and we denote the complete symmetric digraph on v vertices as D_v. An X−decomposition of D_v exists if and only if $v \equiv 0$ or $1 \pmod 4$, $v \neq 5$, a Y−decomposition of D_v exists if and only if $v \equiv 0$ or $1 \pmod 4$, $v \notin \{4,5\}$, and a Z−decomposition of D_v exists if and only if $v \equiv 1 \pmod 4$ [2]. A C_4− decomposition of D_v exists if and only if $v \equiv 0$ or $1 \pmod 4$, $v \neq 4$ [8].

The purpose of this paper is to give necessary and sufficient conditions for the decomposition of the complete bipartite symmetric digraph, $D_{m,n}$, into each of the orientations of a 4−cycle (in the cases where such decompositions are not already known). We will then use these results to solve the problem of packing D_v with each of the orientations of a 4−cycle. Finally, we give necessary and sufficient conditions for the existence of a decomposition of $D(v,w)$ into each of the orientations of a 4−cycle.

2 Decompositions of $D_{m,n}$

The following result is due to Sotteau [10]:

Theorem 2.1 *A C_4–decomposition of $D_{m,n}$ exists if and only if $m, n \geq 2$ and $mn \equiv 0$ (mod 2).*

We now give necessary and sufficient conditions for the existence of a d–decomposition of $D_{m,n}$ where $d \in \{X, Y\}$. Throughout this section, we assume $D_{m,n}$ has partite vertex sets $\{0_0, 1_0, \ldots, (m-1)_0\}$ and $\{0_1, 1_1, \ldots, (n-1)_1\}$.

Theorem 2.2 *An X–decomposition of $D_{m,n}$ exists if and only if* <u>*either*</u> *$m \equiv n \equiv 0$ (mod 2)* <u>*or*</u> *$m \equiv 1$ (mod 2), $m \geq 3$, and $n \equiv 0$ (mod 4).*

Proof. Since $D_{m,n}$ contains $2mn$ arcs and X contains 4 arcs, it is clearly necessary that either $m \equiv 0$ (mod 2) or $n \equiv 0$ (mod 2).

Now suppose $m \equiv 2$ (mod 4), $n \equiv 1$ (mod 2), and let V_1 and V_2 be the partite vertex sets of $D_{m,n}$ where $|V_1| = m$ and $|V_2| = n$. If there is an X–decomposition of $D_{m,n}$, say $\{X_1, X_2, \ldots, X_x\}$, then $x = mn/2 \equiv 1$ (mod 2). For each $i \in \{1, 2, \ldots, x\}$, either $X_i = [a_0, b_1, c_0, d_1]_X$ or $X_i = [a_1, b_0, c_1, d_0]_X$ for some a, b, c, d. If $X_i = [a_0, b_1, c_0, d_1]_X$ then od(b_1)+od(d_1) $= 1$ (where od(b) is the out-degree of vertex b and id(b) is the in-degree of vertex b) and if $X_i = [a_1, b_0, c_1, d_0]_X$ then od(a_1)+od(c_1) $= 3$. In either case, in X_i the sum of the out-degrees of vertices in V_2 is odd. Since there are $x \equiv 1$ (mod 2) isomorphic copies of X in such a decomposition, it must be that the sum of out-degrees of all vertices in V_2 is odd. However, each vertex of V_2 has out-degree $m \equiv 2$ (mod 4), a contradiction. Therefore such a decomposition does not exist.

Now if $m \equiv n \equiv 0$ (mod 2), then the set

$$\{[(2i)_0, (1+2j)_1, (1+2i)_0, (2j)_1]_X, [(2i)_1, (1+2j)_0, (1+2i)_1, (2j)_0]_X \mid i \in \mathbf{Z}_{m/2}, j \in \mathbf{Z}_{n/2}\}$$

forms such a decomposition.

Finally, suppose $m \equiv 1$ (mod 2), $m \geq 3$, and $n \equiv 0$ (mod 4). Then the set

$$\{[(4i)_1, 0_0, (4i+1)_1, 1_0]_X, [1_0, (4i+1)_1, 2_0, (4i)_1]_X \mid i \in \mathbf{Z}_{n/4}\}$$
$$\bigcup\{[0_0, (4i+3)_1, 1_0, (4i+2)_1]_X, [(4i+2)_1, 2_0, (4i+1)_1, 0_0]_X \mid i \in \mathbf{Z}_{n/4}\}$$
$$\bigcup\{[2_0, (4i+2)_1, 1_0, (4i+3)_1]_X, [(4i+3)_1, 0_0, (4i)_1, 2_0]_X \mid i \in \mathbf{Z}_{n/4}\}$$

forms an X–decomposition of $D_{3,n}$. Since $D_{m,n}$ can clearly be decomposed into a copy of $D_{3,n}$ and a copy of $D_{m-3,n}$, this result along with the fact that an X–decomposition of $D_{m-3,n}$ exists as shown above (since $m-3 \equiv n \equiv 0$ (mod 2)) yield the existence of an X–decomposition of $D_{m,n}$. ∎

Theorem 2.3 *A Y−decomposition of $D_{m,n}$ exists if and only if $m, n \geq 2$ and $mn \equiv 0$ (mod 2).*

Proof. As argued in Theorem 2.2, it is necessary that either m or n is even. The case $m \equiv n \equiv 0$ (mod 2) is presented in [2]. So suppose $m \equiv 1$ (mod 2), $m \geq 3$, and $n \equiv 0$ (mod 2). Then the set

$$\{[0_0, (2i)_1, 1_0, (2i+1)_1]_Y, [1_0, (2i)_1, 2_0, (2i+1)_1]_Y,$$
$$[2_0, (2i+1)_1, 0_0, (2i)_1]_Y \mid i \in \mathbf{Z}_{n/2}\}$$

forms a Y−decomposition of $D_{3,n}$. As in the final case of Theorem 2.2, this theorem follows. ∎

Finally, Z−decompositions of $D_{m,n}$ were dealt with in [11]:

Theorem 2.4 *A Z−decomposition of $D_{m,n}$ exists if and only if $m \equiv n \equiv 0$ (mod 2).*

3 Packing D_v

If $\{d_1, d_2, \ldots, d_n\}$ is a packing of D_v with copies of d, then following the terminology of Kennedy [3, 4], we define the digraph L with arc set $A(L) = A(D_v) \backslash \bigcup_{i=1}^{n} A(d_i)$ and vertex set induced by $A(L)$, as the *leave* of the packing. Therefore a maximal packing of D_v minimizes $|A(L)|$. In this section, we give necessary conditions on the structure of L for a maximal packing of D_v with copies of each of the orientations of a 4−cycle (we only consider $v \geq 4$). We then show these necessary conditions are sufficient by presenting a specific packing which minimizes $|A(L)|$. Throughout this section we assume D_v has vertex set $\mathbf{Z}_v$.

Theorem 3.1 *An optimal packing of D_v with copies of C_4 and leave L satisfies:*

1. *$L = \emptyset$ if $v \equiv 0$ or 1 (mod 4), $v \neq 4$,*
2. *$|A(L)| = 4$ if $v = 4$,*
3. *$L = D_2$ if $v \equiv 2$ or 3 (mod 4).*

Proof. If $v \equiv 0$ or 1 (mod 4), $v \neq 4$, then there exists a C_4−decomposition of D_v [8] and the result follows.

Now $|A(D_4)| = 12$ and a C_4–decomposition of D_4 does not exist, so a packing of D_4 with leave L where $|A(L)| = 4$ would be optimal. Consider the packing $\{[0,1,2,3]_C, [0,3,2,1]_C\}$ and leave $L = \{(1,3),(3,1),(0,2),(2,0)\}$.

If $v = 6$, then we have the packing of D_6 of $\{[0,1,4,2]_C, [1,2,5,3]_C, [0,2,3,4]_C, [1,3,0,5]_C, [2,1,0,3]_C, [5,0,4,3]_C, [2,4,1,5]_C\}$ with leave $L = \{(4,5), (5,4)\}$.

If $v \equiv 2$ or $3 \pmod 4$, $v \geq 7$, then $|A(D_v)| \equiv 2 \pmod 4$. Each vertex of D_v has in-degree equal to out-degree and each vertex x of C_4 satisfies $\mathrm{id}(x) = \mathrm{od}(x) = 1$. Therefore the leave of a packing must have each vertex with in-degree equal to out-degree. So a packing of D_v with leave D_2 would be optimal. D_v can clearly be decomposed into a copy of D_{v-2}, a copy of $D_{v-2,2}$, and a copy of D_2. Since D_{v-2} can be decomposed into copies of C_4 [8], and $D_{v-2,2}$ can be decomposed into copies of C_4 by Theorem 2.1, then D_v can be packed with copies of C_4 with a leave of $L = D_2$. ∎

Theorem 3.2 *An optimal packing of D_v with copies of X and leave L satisfies:*

1. $L = \emptyset$ *if* $v \equiv 0$ *or* $1 \pmod 4$, $v \neq 5$,
2. $|A(L)| = 4$ *if* $v = 5$,
3. $L = D_2$ *if* $v \equiv 2$ *or* $3 \pmod 4$.

Proof. If $v \equiv 0$ or $1 \pmod 4$, $v \neq 5$, then there exists an X–decomposition of D_v [2] and the result follows.

Now $|A(D_5)| = 20$ and an X–decomposition of D_5 does not exist, so a packing of D_5 with leave L where $|A(L)| = 4$ would be optimal. Consider the packing $\{[0,3,2,4]_X, [1,4,2,3]_X, [3,0,2,1]_X, [4,1,2,0]_X\}$ and leave $L = \{(3,4),(4,3),(0,1),(1,0)\}$.

If $v \equiv 2$ or $3 \pmod 4$, then $|A(D_v)| \equiv 2 \pmod 4$. Each vertex of D_v has total degree $2(v-1)$ and each vertex of X has total degree 2. Therefore the leave of a packing must have each vertex with even total degree. So a packing of D_v with leae D_2 would be optimal.

If $v \equiv 2 \pmod 4$, then D_v can clearly be decomposed into a copy of D_{v-2}, a copy of $D_{v-2,2}$, and a copy of D_2. Since D_{v-2} can be decomposed into copies of X [2], and $D_{v-2,2}$ can be decomposed into copies of X by Theorem 2.2, it follows that D_v can be packed with copies of X with a leave of $L = D_2$.

If $v = 7$, then we have the packing of D_7 of $\{[0,3,5,4]_X, [1,4,5,0]_X, [2,0,5,1]_X, [3,1,5,2]_X, [4,2,5,3]_X, [4,6,3,0]_X, [0,6,4,1]_X, [1,6,0,2]_X, [2,6,1,3]_X, [3,6,2,4]_X\}$ with leave $L = \{(5,6), (6,5)\}$.

If $v \equiv 3 \pmod 4$, $v \geq 11$, then D_v can clearly be decomposed into a copy of D_{v-7}, a copy of $D_{v-7,7}$, and a copy of D_7. Since D_{v-7} can be decomposed into copies of X [2], $D_{v-7,7}$ can be decomposed into copies of X by Theorem 2.2, and D_7 can be packed with copies of X with a leave of D_2 as seen above, it follows that D_v can be packed with copies of X with a leave of $L = D_2$. ∎

Theorem 3.3 *An optimal packing of D_v with copies of Y and leave L satisfies:*

1. $L = \emptyset$ *if* $v \equiv 0$ *or* $1 \pmod 4$, $v \notin \{4, 5\}$,
2. $|A(L)| = 4$ *if* $v \in \{4, 5\}$,
3. $L = D_2$ *if* $v \equiv 2$ *or* $3 \pmod 4$.

Proof. If $v \equiv 0$ or $1 \pmod 4$, $v \notin \{4, 5\}$, then there exists a Y−decomposition of D_v [2] and the result follows.

As in Theorems 3.2 and 3.3, packings of D_v where $v \in \{4, 5\}$ with leave L satisfying $|A(L)| = 4$ would be optimal. Consider the packing of D_4 of $\{[1,0,3,2]_Y, [3,0,1,2]_Y\}$ and leave $L = \{(0,2),(2,0),(1,3),(3,1)\}$. Consider the packing of D_5 of $\{[0,1,4,2]_Y, [2,0,3,1]_Y, [3,4,0,1]_Y, [4,3,2,1]_Y\}$ and leave $L = \{(0,4),(4,2),(2,3),(3,0)\}$.

If $v \equiv 2$ or $3 \pmod 4$, then the argument of Theorem 3.2 shows that a packing of D_v with leave D_2 would be optimal.

If $v = 6$, then we have the packing of D_6 of $\{[3,2,0,1]_Y, [0,3,4,1]_Y, [5,0,2,1]_Y, [2,3,0,4]_Y, [2,1,3,5]_Y, [0,5,2,4]_Y, [4,3,5,1]_Y\}$ with leave $L = \{(4,5), (5,4)\}$.

If $v = 7$, then we have the packing of D_7 of $\{[4,5,3,0]_Y, [0,5,4,1]_Y, [1,5,0,2]_Y, [2,5,1,3]_Y, [3,5,2,4]_Y, [3,6,4,0]_Y, [4,6,0,1]_Y, [0,6,1,2]_Y, [1,6,2,3]_Y, [2,6,3,4]_Y\}$ with leave $L = \{(5,6), (6,5)\}$.

If $v \equiv 2$ or $3 \pmod 4$ and $v \geq 10$, then D_v can clearly be decomposed into a copy of D_{v-2}, a copy of $D_{v-2,2}$, and a copy of D_2. Since D_{v-2} can be decomposed into copies of Y [2], and $D_{v-2,2}$ can be decomposed into copies of Y by Theorem 2.3, it follows that D_v can be packed with copies of Y with a leave of $L = D_2$. ∎

Theorem 3.4 *An optimal packing of D_v with copies of Z and leave L satisfies:*

1. $L = \emptyset$ *if* $v \equiv 1 \pmod 4$,
2. $|A(L)| = v$ *and the arcs of L are arranged in a collection of disjoint circuits if* $v \equiv 0$ *or* $2 \pmod 4$,

3. $|A(L)| = 6$ *and the arcs of* L *are arranged in such a way that each vertex of the leave has in-degree = out-degree* $\equiv 0 \pmod 2$ *if* $v \equiv 3 \pmod 4$.

Proof. If $v \equiv 1 \pmod 4$, then there exists a Z−decomposition of D_v [2] and the result follows.

If $v \equiv 0$ or $2 \pmod 4$, then each vertex x of D_v satisfies $\mathrm{id}(x) = \mathrm{od}(x) \equiv 1 \pmod 2$. Since each vertex of Z has in-degree and out-degree even, the leave of an optimal packing must have each vertex with both in-degree and out-degree equal to 1. Therefore the leave of an optimal packing will consist of v arcs arranged in disjoint circuits. We show these necessary conditions are sufficient in the following 3 cases:

Case 1. Suppose $v = 4t$ where $t \equiv 0 \pmod 2$. Then the set

$$\{[i, 4t-1-2j+i, 1+i, 2+i+2j]_Z \mid i \in \mathbf{Z}_{4t}, j \in \mathbf{Z}_{(t-4)/2+1}\} \cup$$
$$\{[i, t+3+i+2j, 1+i, 3t-2+i-2j]_Z \mid i \in \mathbf{Z}_{4t}, j \in \mathbf{Z}_{(t-4)/2+1}\} \cup$$
$$\{[i, 3t+1+i, i+2, t+1+i]_Z \mid i \in \mathbf{Z}_{4t}\} \cup$$
$$\{[i, 3t+i, 2t+i, t+i]_Z \mid i \in \mathbf{Z}_{2t}\}$$

forms a packing with leave $L = \{(i, 2t+i), (2t+i, i) \mid i \in \mathbf{Z}_{2t}\}$ (a collection of $v/2$ disjoint 2−circuits).

Case 2. Suppose $v = 4t$ where $t \equiv 1 \pmod 2$. Then the set

$$\{[i, 4t-1-2j+i, 1+i, 2+i+2j]_Z \mid i \in \mathbf{Z}_{4t}, j \in \mathbf{Z}_{(t-3)/2+1}\} \cup$$
$$\{[i, t+2+i+2j, 1+i, 3t-1+i-2j]_Z \mid i \in \mathbf{Z}_{4t}, j \in \mathbf{Z}_{(t-3)/2+1}\} \cup$$
$$\{[i, 3t+i, 2t+i, t+i]_Z \mid i \in \mathbf{Z}_{2t}\}$$

forms a packing with leave $L = \{(i, 2t+i), (2t+i, i) \mid i \in \mathbf{Z}_{2t}\}$ (a collection of $v/2$ disjoint 2−circuits).

Case 3. Suppose $v = 4t+2$. Then the set

$$\{[i, 4t+1+i-2j, 1+i, 2+i+2j]_Z \mid i \in \mathbf{Z}_{4t+2}, j \in \mathbf{Z}_t\}$$

forms a packing with leave $L = \{(i, 2t+1+i), (2t+1+i, i) \mid i \in \mathbf{Z}_{2t}\}$ (a collection of $v/2$ disjoint 2−circuits).

If $v \equiv 3 \pmod 4$, then each vertex x of D_v satisfies $\mathrm{id}(x) = \mathrm{od}(x) \equiv 0 \pmod 2$. Since each vertex of Z has in-degree and out-degree even, the leave of an optimal packing must have each vertex with both in-degree and out-degree even. Since $|A(D_v)| \equiv 2 \pmod 4$, then for any packing, the leave L must satisfy $|A(L)| \equiv 2 \pmod 4$. We cannot have all vertices of both in-degree and out-degree even with only 2 arcs. This condition is possible with 6 arcs and a packing with leave $L = D_3$ would be optimal. D_v can clearly be decomposed into a copy of D_{v-2}, a copy of $D_{v-3,2}$, and a copy of D_3. Since D_{v-2} can be decomposed into copies of Z [2], and $D_{v-3,2}$ can be decomposed into copies of Z by Theorem 2.4, it follows that D_v can be packed with copies of Z with a leave of $L = D_3$. ∎

Theorems 3.1-3.4 give necessary and sufficient conditions for an optimal packing of D_v with each of the orientations of a 4−cycle.

4 Decompositions of D_v with holes

In this section we give necessary and sufficient conditions for the existence of a decomposition of $D(v, w)$ into each of the orientations of a 4−cycle. We call such designs *decompositions of D_v with a hole of size w*. We only consider $v - w > 1$ since if $v - w = 1$, $D(v, w) = D_v$.

Theorem 4.1 *A C_4−decomposition of D_v with a hole of size w exists if and only if $\{v(\text{mod } 4), w(\text{mod } 4)\} \subset \{0,1\}$ or $\{v(\text{mod } 4), w(\text{mod } 4)\} \subset \{2,3\}$ and $v - w > 3$.*

Proof. A C_4−decomposition of $D(v, w)$ where $v - w = 3$ induces a decomposition of D_3 into isomorphic copies of

Clearly such a decomposition does not exist. Therefore, $v - w > 3$ is necessary.

An obvious necessary condition is that $4 \mid |A(D(v, w))|$, which is equivalent to $\{v(\text{mod } 4), w \text{ (mod } 4)\} \subset \{0,1\}$ or $\{v(\text{mod } 4), w \text{ (mod } 4)\} \subset \{2,3\}$.

If either $v \equiv 1 \pmod 4$ and $w \equiv 0 \pmod 4$ or $v \equiv 3 \pmod 4$ and $w \equiv 2 \pmod 4$, then we can clearly decompose $D(v, w)$ into a copy of D_{v-w} and a copy of $D_{v-w,w}$. Since, in this case, D_{v-w} can be decomposed into copies of C_4 [7] and $D_{v-w,w}$ can be decomposed into copies of C_4 by Theorem 2.1, it follows that $D(v, w)$ can be decomposed into copies of C_4.

If $v = 6$ and $w = 2$, then a decomposition of $D(v, w)$ into copies of C_4 is equivalent to a packing of D_v with a leave of D_2. Such a structure is given in Theorem 3.1.

For each of the remaining cases, we can clearly decompose $D(v, w)$ into a copy of D_{v-w+1} and a copy of $D_{v-w,w-1}$. In these cases, $v - w + 1 \equiv 0$ or $1 \pmod 4$, and so D_{v-w+1} can be decomposed into copies of C_4 [7], and since either $v-w$ or $w-1$ is even, $D_{v-w,w-1}$ can be decomposed into copies of C_4 by Theorem 2.1. It follows that $D(v,w)$ can be decomposed into copies of C_4. ∎

In the remainder of this section, we assume the vertex set of $D(v,w)$ is $V_w \bigcup V_{v-w}$ where these sets are as described in Section 1 and $V_w = \{0_0, 1_0, \ldots, (w-1)_0\}$ and $V_{v-w} = \{0_1, 1_1, \ldots, (v-w-1)_1\}$.

Theorem 4.2 *An $X-$decomposition of D_v with a hole of size w exists if and only if $\{v(mod\ 4),\ w(mod\ 4)\} \subset \{0,1\}$ or $\{v(mod\ 4),\ w(mod\ 4)\} \subset \{2,3\}$, and $v-w \neq 3$ in the case of $v \equiv 2(mod\ 4)$ and $w \equiv 3(mod\ 4)$.*

Proof. First, suppose there exists an $X-$decomposition of $D(v,w)$ with $v \equiv 2 \pmod 4$, $w \equiv 3 \pmod 4$, and $v-w=3$. Let B be a set of isomorphic copies of X in such a decomposition. Then $|B|$ is even. Let

$$\begin{aligned}
x_1 &= |\{[a_1,b_1,c_1,d_0]_X \mid [a_1,b_1,c_1,d_0]_X \in B \text{ for some } a,b,c,d\}|\\
x_2 &= |\{[a_1,b_0,c_1,d_1]_X \mid [a_1,b_0,c_1,d_1]_X \in B \text{ for some } a,b,c,d\}|\\
x_3 &= |\{[a_1,b_1,c_0,d_1]_X \mid [a_1,b_1,c_0,d_1]_X \in B \text{ for some } a,b,c,d\}|\\
x_4 &= |\{[a_0,b_1,c_1,d_1]_X \mid [a_0,b_1,c_1,d_1]_X \in B \text{ for some } a,b,c,d\}|\\
x_5 &= |\{[a_1,b_0,c_1,d_0]_X \mid [a_1,b_0,c_1,d_0]_X \in B \text{ for some } a,b,c,d\}|\\
x_6 &= |\{[a_0,b_1,c_0,d_1]_X \mid [a_0,b_1,c_0,d_1]_X \in B \text{ for some } a,b,c,d\}|.
\end{aligned}$$

Then $|B| = x_1 + x_2 + \cdots + x_6$. Since the in-degree equals out-degree for each vertex in $\{0_0, 1_0, \ldots, (w-1)_0\}$, it follows that $x_1 + x_5 = x_4 + x_6$. Such a decomposition induces a decomposition of a copy of D_3 with vertex set $\{0_1, 1_1, 2_1\}$ into orientations of a 2−path and therefore $x_1+x_2+x_3+x_4 = 3$. Since there are only 4 such decompositions of D_3 (up to isomorphism), we deduce that either

(i) $x_2 = 3$,
(ii) $x_3 = 3$,
(iii) $x_2 = 1$ and $x_1 + x_4 = 2$, or
(iv) $x_3 = 1$ and $x_1 + x_4 = 2$.

In each case, we reach a contradiction as follows:

(i) If $x_2 = 3$, then $x_1 = x_3 = x_4 = 0$ and therefore $x_5 = x_6$. But then $|B| = 3 + 2x_5$, contradicting the fact that $|B|$ is even.

(ii) If $x_3 = 3$, then a similar argument to that given in (i) leads to a contradiction.

(iii) If $x_1 = 2$ and $x_2 = 1$, then $x_3 = x_4 = 0$ and $2 + x_5 = x_6$. But then $|B| = 5 + 2x_5$, a contradiction. Similarly if $x_2 = 1$ and $x_4 = 2$, we get $|B| = 5 + 2x_6$, a contradiction. Now if $x_1 = x_2 = x_4 = 1$ then $x_3 = 0$ and $x_5 = x_6$. But then $|B| = 3 + 2x_5$, a contradiction.

(iv) If $x_3 = 1$ and $x_1 + x_4 = 2$ then a similar argument to that given in (iii) leads to a contradiction.

If $v \equiv 0 \pmod 4$ and $w \equiv 1 \pmod 4$, then we can clearly decompose $D(v,w)$ into a copy of D_{v-w+1} and a copy of $D_{v-w,w-1}$. Since D_{v-w+1} can be decomposed into copies of X [2] and $D_{v-w,w-1}$ can be decomposed into copies of X by Theorem 2.2, it follows that $D(v,w)$ can be decomposed into copies of X.

If $v \equiv 3 \pmod 4$ and $w \equiv 2 \pmod 4$, then we can decompose $D(v,w)$ into a copy of $D_{5,w-2}$, a copy of $D(7,2)$, a copy of $D_{v-w-5,2}$, a copy of $D_{v-w-5,w-2}$, a copy of $D_{v-w-5,5}$, and a copy of D_{v-w-5}. Since D_{v-w-5} can be decomposed into copies of X [2], the bipartite digraphs can be decomposed into copies of X by Theorem 2.2, and a decomposition of $D(7,2)$ is given in Theorem 3.2, it follows that $D(v,w)$ can be decomposed into copies of X.

If $v = 10$ and $w = 3$, then the following is an X−decomposition of $D(10,3)$:

$$\{[3_1,0_1,1_1,2_1]_X,[4_1,0_1,2_1,0_0]_X,[1_0,0_1,0_0,2_1]_X,[2_1,0_1,3_1,1_0]_X,$$
$$[0_0,0_1,4_1,3_1]_X,[4_1,1_1,0_1,1_0]_X,[0_0,1_1,3_1,4_1]_X,[1_0,4_1,2_1,1_1]_X,$$
$$[1_1,1_0,3_1,0_0]_X,[2_1,3_1,1_1,4_1]_X,[5_1,0_1,2_0,1_1]_X,[5_1,2_1,2_0,3_1]_X,$$
$$[5_1,4_1,2_0,6_1]_X,[6_1,1_1,2_0,0_1]_X,[6_1,3_1,2_0,2_1]_X,[6_1,5_1,2_0,4_1]_X,$$
$$[0_1,6_1,2_0,5_1]_X,[1_1,5_1,0_0,6_1]_X,[2_1,6_1,0_0,5_1]_X,[3_1,5_1,1_0,6_1]_X,$$
$$[4_1,6_1,1_0,5_1]_X\}.$$

If $v \equiv 2 \pmod 4$, $w \equiv 3 \pmod 4$ $v-w \neq 3$, and $(v,w) \neq (10,3)$, then $D(v,w)$ can be decomposed into a copy of $D(10,3)$, a copy of $D_{7,w-3}$, a copy of $D_{v-w-7,3}$, a copy of $D_{v-w-7,w-3}$, a copy of $D_{v-w-7,7}$, and a copy of D_{v-w-7}. Since D_{v-w-7} can be decomposed into copies of X [2], the bipartite digraphs can be decomposed into copies of X by Theorem 2.2, and an X−decomposition of $D(10,3)$ is given above, it follows that $D(v,w)$ can be decomposed into copies of X.

If $v = 9$ and $w = 4$, then the following is an X−decomposition of $D(9,4)$:

$$\{[0_1,1_1,3_1,2_1]_X,[2_1,1_1,0_1,3_1]_X,[3_1,1_1,2_1,0_1]_X,[4_1,0_1,2_0,1_1]_X,$$
$$[4_1,2_1,3_0,3_1]_X,[3_1,3_0,2_1,4_1]_X,[4_1,0_0,1_1,1_0]_X,[4_1,2_0,0_1,3_0]_X,$$
$$[1_0,3_1,0_0,4_1]_X,[3_0,1_1,2_0,4_1]_X,[2_0,3_1,1_0,2_1]_X,[2_1,0_0,3_1,2_0]_X,$$
$$[0_1,1_0,1_1,0_0]_X,[0_0,2_1,1_0,0_1]_X,[1_1,3_0,0_1,4_1]_X\}.$$

If $v \equiv 1 \pmod 4$, $w \equiv 0 \pmod 4$, and $v - w = 5$ then $D(v,w)$ can be decomposed into a copy of $D(9,4)$ and a copy of $D_{5,w-4}$. Since $D_{5,w-4}$ can be decomposed into copies of X by Theorem 2.2 and an X−decomposition of $D(9,4)$ is given above, it follows that $D(v,w)$ can be decomposed into copies of X. If $v-w \geq 9$, then $D(v,w)$ can be decomposed into a copy of D_{v-w} and a copy of $D_{v-w,w}$. Since D_{v-w} can be decomposed into copies of X [2], and $D_{v-w,w}$ can be decomposed into copies of X by Theorem 2.2 it follows that $D(v,w)$ can be decomposed into copies of X.

In the remaining cases, $v-w \equiv 0 \pmod 4$. $D(v,w)$ can be decomposed into a copy of D_{v-w} and a copy of $D_{v-w,w}$. Since D_{v-w} can be decomposed into copies of X [2] and $D_{v-w,w}$ can be decomposed into copies of X by Theorem 2.2, it follows that $D(v,w)$ can be decomposed into copies of X.

∎

Theorem 4.3 *A $Y-$decomposition of D_v with a hole of size w exists if and only if $\{v(\text{mod } 4),\ w(\text{mod } 4)\} \subset \{0,1\}$ or $\{v(\text{mod } 4),\ w(\text{mod } 4)\} \subset \{2,3\}$, and $v-w \neq 3$.*

Proof. First, suppose there exists an $Y-$decomposition of $D(v,w)$ with $\{v(\text{mod } 4),\ w(\text{mod } 4)\} \subset \{0,1\}$ or $\{v(\text{mod } 4),\ w(\text{mod } 4)\} \subset \{2,3\}$, and $v-w=3$. Let B be a set of isomorphic copies of Y in such a decomposition. Let

$$\begin{aligned}
y_1 &= |\{[a_1,b_1,c_0,d_1]_Y \mid [a_1,b_1,c_0,d_1]_Y \in B \text{ for some } a,b,c,d\}| \\
y_2 &= |\{[a_0,b_1,c_1,d_1]_Y \mid [a_0,b_1,c_1,d_1]_Y \in B \text{ for some } a,b,c,d\}| \\
y_3 &= |\{[a_0,b_1,c_0,d_1]_Y \mid [a_0,b_1,c_0,d_1]_Y \in B \text{ for some } a,b,c,d\}| \\
y_4 &= |\{[a_1,b_1,c_1,d_0]_Y \mid [a_1,b_1,c_1,d_0]_Y \in B \text{ for some } a,b,c,d\}| \\
y_5 &= |\{[a_1,b_0,c_1,d_0]_Y \mid [a_1,b_0,c_1,d_0]_Y \in B \text{ for some } a,b,c,d\}|.
\end{aligned}$$

Since each vertex of $\{0_0,1_0,\ldots,(w-1)_0\}$ has in-degree equal to out-degree, $y_1=y_2$. Such a decomposition induces a decomposition of a copy of D_3 with vertex set $\{0_1,1_1,2_1\}$ into orientations of a 2−path. Therefore $y_1+y_2+y_4=3$. However, if <u>either</u> $y_1=y_2=y_4$ <u>or</u> $y_1=y_2=0$ and $y_4=3$, then it is easily seen that the necessary induced decomposition of D_3 does not exist.

We now consider several cases to establish sufficiency.

Case 1. Suppose that <u>either</u> $v \equiv 0 \pmod 4$ and $w \equiv 1 \pmod 4$ <u>or</u> $v \equiv 2 \pmod 4$ and $w \equiv 3 \pmod 4$ and in either case, $v-w>3$. We can clearly decompose $D(v,w)$ into a copy of D_{v-w+1} and a copy of $D_{v-w,w-1}$. Since D_{v-w+1} can be decomposed into copies of Y [2] and $D_{v-w,w-1}$ can be decomposed into copies of Y by Theorem 2.3, it follows that $D(v,w)$ can be decomposed into copies of Y.

Case 2. If $v=8$ and $w=4$, then the following is a $Y-$decomposition of $D(8,4)$:

$$\{[0_1,1_1,0_0,2_1]_Y,[1_1,2_1,3_1,1_0]_Y,[2_1,1_0,0_1,3_0]_Y,[0_1,0_0,1_1,1_0]_Y,$$

$[1_1, 3_1, 0_1, 2_0]_Y, [0_1, 2_0, 1_1, 3_0]_Y, [1_1, 0_1, 3_1, 3_0]_Y, [3_1, 2_1, 0_1, 0_0]_Y,$
$[3_1, 3_0, 2_1, 1_0]_Y, [2_0, 2_1, 1_1, 3_1]_Y, [0_0, 2_1, 2_0, 3_1]_Y\}.$

Suppose that $v \equiv 0 \pmod 4$, $v > 8$, and $w \equiv 0 \pmod 4$. If, in addition, $v - w = 4$, then $D(v, w)$ can be decomposed into a copy of $D(8, 4)$ and a copy of $D_{4,w-4}$. Since a decomposition of $D(8, 4)$ exists as given here and $D_{4,w-4}$ can be decomposed into copies of Y by Theorem 2.3, the result follows in this special case. Now if $v - w > 4$, then $D(v, w)$ can clearly be decomposed into a copy of D_{v-w} and a copy of $D_{v-w,w}$. Since D_{v-w} can be decomposed into copies of Y [2] and $D_{v-w,w}$ can be decomposed into copies of Y by Theorem 2.3, it follows that $D(v, w)$ can be decomposed into copies of Y.

Case 3. If $v = 9$ and $w = 4$, then the following is a Y–decomposition of $D(9, 4)$:

$\{[1_0, 0_1, 0_0, 1_1]_Y, [0_0, 2_1, 1_0, 4_1]_Y, [1_0, 2_1, 0_0, 3_1]_Y, [0_0, 1_1, 1_0, 0_1]_Y,$
$[4_1, 0_1, 3_1, 0_0]_Y, [1_1, 2_1, 4_1, 3_1]_Y, [3_1, 1_1, 4_1, 1_0]_Y, [3_1, 2_1, 1_1, 0_1]_Y,$
$[4_1, 1_1, 0_1, 2_1]_Y, [0_1, 2_1, 3_1, 4_1]_Y, [1_1, 2_0, 0_1, 3_0]_Y, [2_1, 2_0, 1_1, 3_0]_Y,$
$[3_1, 2_0, 2_1, 3_0]_Y, [4_1, 2_0, 3_1, 3_0]_Y, [0_1, 2_0, 4_1, 3_0]_Y\}.$

Suppose that $v \equiv 1 \pmod 4$, $v > 9$ and $w \equiv 0 \pmod 4$. If, in addition, $v - w = 5$, then $D(v, w)$ can be decomposed into a copy of $D(9, 4)$ and a copy of $D_{5,w-4}$. Since a decomposition of $D(9, 4)$ exists as given here and $D_{5,w-4}$ can be decomposed into copies of Y by Theorem 2.3, the result follows in this special case. Now if $v - w > 5$, then $D(v, w)$ can clearly be decomposed into a copy of D_{v-w} and a copy of $D_{v-w,w}$ and the result follows as in Case 2.

Case 4. If $v = 6$ and $w = 2$, then a decomposition of $D(6, 2)$ is given in Theorem 3.3. Suppose that $v \equiv 2 \pmod 4$, $v > 6$, and $w \equiv 2 \pmod 4$. If, in addition, $v - w = 4$, then $D(v, w)$ can be decomposed into a copy of $D(6, 2)$ and a copy of $D_{4,w-2}$. Since a decomposition of $D(6, 2)$ exists and $D_{4,w-2}$ can be decomposed into copies of Y by Theorem 2.3, the result follows in this special case. Now if $v - w > 4$, then $D(v, w)$ can clearly be decomposed into a copy of D_{v-w} and a copy of $D_{v-w,w}$ and the result follows as in Case 2.

Case 5. If $v = 9$ and $w = 5$, then we observe that $D(9, 5)$ can be decomposed into a copy of $D(6, 2)$ and a copy of $D_{4,3}$. Since $D(6, 2)$ can be decomposed into copies of Y by case 4, and $D_{4,3}$ can be decomposed into copies of Y by Theorem 2.3, it follows that $D(9, 5)$ can be decomposed into copies of Y. Suppose that $v \equiv 1 \pmod 4$, $v > 9$, and $w \equiv 1$

(mod 4). If, in addition, $v-w=4$, then $D(v,w)$ can be decomposed into a copy of $D(9,5)$ and a copy of $D_{4,w-5}$. Since a decomposition of $D(9,5)$ exists as given here and $D_{4,w-5}$ can be decomposed into copies of Y by Theorem 2.3, the result follows in this special case. Now if $v-w>4$, then $D(v,w)$ can clearly be decomposed into a copy of D_{v-w} and a copy of $D_{v-w,w}$ and the result follows as in Case 2.

Case 6. If $v=7$ and $w=2$, then a decomposition of $D(7,2)$ is given in Theorem 3.3. Suppose that $v \equiv 3 \pmod 4$, $v>7$, and $w \equiv 2 \pmod 4$. If, in addition, $v-w=5$, then $D(v,w)$ can be decomposed into a copy of $D(7,2)$ and a copy of $D_{5,w-2}$. Since a decomposition of $D(7,2)$ exists and $D_{5,w-2}$ can be decomposed into copies of Y by Theorem 2.3, the result follows in this special case. Now if $v-w>5$, then $D(v,w)$ can clearly be decomposed into a copy of D_{v-w} and a copy of $D_{v-w,w}$ and the result follows as in Case 2.

Case 7. If $v=7$ and $w=3$, then the following is a Y-decomposition of $D(7,3)$:

$$\{[3_1,2_1,0_1,1_1]_Y,[0_1,3_1,0_0,1_1]_Y,[1_0,0_1,2_1,1_1]_Y,$$
$$[2_1,3_1,0_1,0_0]_Y,[2_1,1_1,3_1,1_0]_Y,[0_0,1_1,2_0,2_1]_Y,$$
$$[2_0,0_1,1_0,1_1]_Y,[3_1,1_0,2_1,2_0]_Y,[0_1,0_0,3_1,2_0]_Y\}.$$

Suppose that $v \equiv 3 \pmod 4$, $v>7$, and $w \equiv 3 \pmod 4$. If, in addition, $v-w=4$, then $D(v,w)$ can be decomposed into a copy of $D(7,3)$ and a copy of $D_{4,w-3}$. Since a decomposition of $D(7,3)$ exists and $D_{4,w-3}$ can be decomposed into copies of Y by Theorem 2.3, the result follows in this special case. Now if $v-w>4$, then $D(v,w)$ can clearly be decomposed into a copy of D_{v-w} and a copy of $D_{v-w,w}$ and the result follows as in Case 2. ∎

Theorem 4.4 *A Z-decomposition of D_v with a hole of size w exists if and only if* <u>*either*</u> $v \equiv w \equiv 1 \pmod 4$, $v>1, w>1$ <u>*or*</u> $v \equiv w \equiv 3 \pmod 4$.

Proof. As in Theorem 4.1, it is necessary that $4 \mid |A(D(v,w))|$. The in-degree of each vertex of Z is even. Since $D(v,w)$ contains $v-w$ vertices of in-degree w, it is necessary that w is odd. Also, $D(v,w)$ contains w vertices of in-degree $v-w$ and so it is necessary that v is odd. This establishes the necessary conditions.

$D(v,w)$ can clearly be decomposed into a copy of D_{v-w+1} and a copy of $D_{v-w,w-1}$. Since $v-w+1 \equiv 1 \pmod 4$, D_{v-w+1} can be decomposed into copies of Z [2]. Since both $v-w$ and $w-1$ are even, $D_{v-w,w-1}$ can be decomposed into copies of Z by Theorem 2.4. It follows that $D(v,w)$ can be decomposed into copies of Z. ∎

Theorems 4.1-4.4 give necessary and sufficient conditions for a decomposition of D_v with a hole of size w for each of the orientations of a 4−cycle.

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