Gracefulness of the union of cycles and paths

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ABSTRACT. Frucht and Salinas [1] conjectured that $C(k) \cup P(n)$ ($n \geq 3$) is graceful if and only if $k+n \geq 7$. We prove that $C(2k) \cup P(n)$ is graceful for $n \geq k+1$ ($k \geq 3$).

For smaller cases we prove that $C(2k) \cup P(n)$ is graceful for $k = 3, 4, 5, 6$; $n \geq 2$.

1 Introduction

Let G be a finite simple graph with n vertices and q edges ($q \geq n-1$). Then G is said to be *graceful* if there is an injection f (labelling)

$$f\colon V(G) \to \{0, 1, \ldots, q\}$$

such that the induced function

$$f^*\colon E(G) \to \{1, 2, \ldots, q\}$$

defined by

$$f^*(xy) = |f(x) - f(y)| \quad (\text{for all } xy \in E(G))$$

is an injection.

The images of f and f^* are called respectively *vertex* and *edge* labels.

Frucht and Salinas [1] conjectured that $C(k) \cup P(n)$ $(n \geq 3)$ is graceful if and only if $k+n \geq 7$: $C(k)$ and $P(n)$ denote respectively the cycle of length k and the path of length n. We prove (Corollary 7) that $C(2k) \cup P(n)$ is graceful for $n \geq k+1$, $k \geq 3$. For smaller cases we prove that $C(2k) \cup P(n)$ is graceful for $k = 3, 4, 5, 6$; $n \geq 2$.

Graceful labellings were first considered by Rosa [3] in 1966; a useful survey of results appears in Gallian [2].

2 Labelling a path with a constraint on the first vertex

Let $P(n)$ $(n \geq 0, n \neq 1)$ be the path with n vertices. If $n = 0$ then $P(n)$ is the empty path. Let $w_1, w_2, \ldots, w_n$ be the vertices of $P(n)$ $(n \geq 2)$. Write

$$n - 1 = 3s + \theta \quad (1 \leq \theta \leq 3).$$

Define a vertex labelling

$$h\colon V(P(n)) \to \{1, 2, \ldots, n\}$$

(setting $h(w_i) = h(i)$ and $\lfloor x \rfloor$ for the integer part of x) as follows:

(1) $h(1) = 2$.

(2) For t, $1 \leq t \leq n - \theta$, t odd,

$$h(t) = (t+1) - 3(\lfloor t/3 \rfloor - \lfloor t/6 \rfloor).$$

(3) For t, $1 \leq t \leq n - \theta$, t even,

$$h(t) = n + 2 + t/2 - 3\lfloor (t+2)/3 \rfloor.$$

(4) For t, $1 \leq t \leq \theta$,

$$h(n-\theta+t) = h(n-\theta) + \begin{cases} (-1)^{\lfloor(\theta-2)/2\rfloor + t} \lfloor (t+1)/2 \rfloor & (n-\theta \text{ even}) \\ (-1)^{\lfloor(\theta-1)/2\rfloor + t} \lfloor (t+1)/2 \rfloor & (n-\theta \text{ odd}) \end{cases}$$

□

We prove h is a graceful labelling:

Lemma 1. *h is a bijection.*

Proof: Suppose that

$$h(t) = h(t') \tag{1}$$

where $1 \leq t,\ t' \leq n$.

Case 1. $(1 \leq t, t' \leq n - \theta; t, t'$ both even).

From (1),

$$t - t' = 6(\lfloor (t+2)/3 \rfloor - \lfloor (t'+2)/3 \rfloor). \tag{2}$$

From (2), $t \equiv t' \pmod 6$ and hence $t + 2 \equiv t' + 2 \pmod 3$; again from (2), $t = t'$.

Case 2. $(1 \leq t, t' \leq n - \theta; t, t'$ both odd).

From (1),

$$t - t' = 3(\lfloor t/3 \rfloor - \lfloor t'/3 \rfloor) + (\lfloor t/6 \rfloor - \lfloor t'/6 \rfloor). \tag{3}$$

From (3), $t \equiv t' \pmod 3$ and so, since t and t' are both odd, $t \equiv t' \pmod 6$; again from (3), $t = t'$.

Case 3. $(1 \leq t, t' \leq n - \theta; t$ odd and t' even).

From (1),

$$2n \leq (t + t') + 3 \tag{4}$$
$$\leq 2n - 2\theta + 2. \tag{5}$$

Hence $\theta = 1$ and equality holds in (4) and (5): from (5), $\{t, t'\} = \{n-1, n-2\}$ and from (4), $t \equiv 1 \pmod 6$ and $t' \equiv 4 \pmod 6$. This is impossible.

Case 4. $\theta = |\{h(n - \theta + i) : i = 1, \ldots, \theta\}|$.

This follows immediately from the definition of $h(n - \theta + i)$.

Case 5. $(1 \leq t \leq n - \theta - 1, n - \theta + 1 \leq t' \leq n)$.

Set $t = 6k + \alpha$, $0 \leq \alpha \leq 5$ and $t' = n - \theta + i$, $1 \leq i \leq \theta$. Write

$$h(n - \theta + i) = (n + \theta)/2 + w(i, \theta)$$

where

$$w(i,\theta) = \begin{cases} (-1)^{\lfloor (\theta-2)/2 \rfloor + i} \lfloor (i+1)/2 \rfloor & (n - \theta \text{ even}) \\ (-1)^{\lfloor (\theta-1)/2 \rfloor + i} \lfloor (i+1)/2 \rfloor + (3 - 2\theta)/2 & (n - \theta \text{ odd}) \end{cases} \tag{6}$$

Set $w = w(i, \theta)$.

Firstly suppose that t is odd, which in turn implies α is odd. Then, from (1)

$$n + \theta + 2w = (t + 2) + \alpha - 6(\lfloor \alpha/3 \rfloor - \lfloor \alpha/6 \rfloor). \tag{7}$$

Set $t = n - \theta - \varepsilon$ $(\varepsilon \geq 1)$ and $\delta(\alpha) = \alpha - 6(\lfloor \alpha/3 \rfloor - \lfloor \alpha/6 \rfloor)$. Then $\delta(1) = 1$, $\delta(3) = -3$ and $\delta(5) = -1$. From (7)

$$2(\theta + w - 1) + \varepsilon = \delta(\alpha). \tag{8}$$

Now suppose that $n-\theta$ is even. From (6), $\theta+w \geq 1$. Hence, from (8), $1 \geq \delta(\alpha) \geq \varepsilon \geq 1$ and so $\alpha=\varepsilon=1$. Recall that, by definition, $n=3s+\theta+1$. Hence $t=n-\theta-\varepsilon=n-\theta-1 \equiv 0 \pmod 3$ which is impossible since $\alpha=1$. Consequently we may assume that $n-\theta$ is odd. From (6), $\theta+w=1/2$. Hence, from (8), $\delta(\alpha) \geq \varepsilon-1$. It follows that $\alpha=1$ and $\varepsilon=2$ (ε is even since both $n-\theta$ and t are odd). Therefore $t=n-\theta-2=3s-1$ which is impossible since $\alpha=1$.

Finally suppose that t is even, which in turn implies α is even. Then, from (1),

$$\theta+2w=n+4-t+2\alpha-6\lfloor(\alpha+2)/3\rfloor. \tag{9}$$

Set $t=n-\theta-\varepsilon$ ($\varepsilon \geq 1$) and $\delta(\alpha)=\alpha-3\lfloor(\alpha+2)/3\rfloor$. Then $\delta(0)=0$, $\delta(2)=-1$ and $\delta(4)=-2$. From (9),

$$2w-\varepsilon-4=2\delta(\alpha). \tag{10}$$

Suppose now that $n-\theta$ is even. Then, since t is even, so also is ε. From (6) and (10), $-4 \leq 2\delta(\alpha) \leq -2-\varepsilon \leq -4$. Hence $\varepsilon=2$ and $\alpha=4$. Therefore $t \equiv 1 \pmod 3$. But $t=n-\theta-\varepsilon=3s-1 \equiv -1 \pmod 3$ which is impossible.

Finally suppose that $n-\theta$ is odd. Then, from (6), $w \leq 1/2$ and hence, from (10), $-4 \leq 2\delta(\alpha) \leq -\varepsilon-3 \leq -4$. Hence $\varepsilon=1$ and $\alpha=4$. But $t=n-\theta-\varepsilon=3s \equiv 0 \pmod 3$. Since $\alpha=4$, $t \equiv 1 \pmod 3$. This is the final contradiction.

It is easy to show that h is a surjection and then the proof is complete. □

Theorem 2. *h is a graceful labelling of $P(n)$; $h(1)=2$.*

Proof: Set

$$\beta=\lfloor(n-\theta-2)/6\rfloor$$

and

$$t=6k+2, \quad 0 \leq k \leq \beta.$$

Suppose that $i \in \{0,1,2\}$, except when $n \in \{0,1,5\}$ and $k=\beta$, in which case $i=0$. Then

$$\begin{aligned} h(t+2i)-h(t+2i+1) &= n-t/2-i-3(\lfloor(6k+2i+4)/3\rfloor \\ &\quad -\lfloor(6k+2i+3)/3\rfloor+\lfloor(6k+2i+3)/6\rfloor) \\ &= n-t/2-i-3\delta(i) \end{aligned} \tag{1}$$

where $\delta(0)=k$ and $\delta(1)=\delta(2)=k+1$. Then, from (1),

$$h(t+2i)-h(t+2i+1)=n-t+\varepsilon(i) \tag{2}$$

where $\varepsilon(0) = 1$, $\varepsilon(1) = -3$ and $\varepsilon(2) = -4$.

Suppose that $i \in \{0,1,2\}$ except when $n \in \{0,1,5\}$ and $k = \beta$ in which case $i \in \{0,1\}$. Then

$$\begin{aligned} h(t+2i) - h(t+2i-1) &= n - t/2 - i + 2 - 3(\lfloor (6k+2i+4)/3 \rfloor \\ &\quad - \lfloor (6k+2i+1)/3 \rfloor + \lfloor (6k+2i+1)/6 \rfloor) \\ &= n - t/2 - i + 2 - 3(k+1) \\ &= n - t - i. \end{aligned} \tag{3}$$

By definition

$$|\{|h(n-\theta+i) - h(n-\theta+i+1)| : i = 0, \ldots, (\theta-1)\}| = \{1, \ldots, \theta\}. \tag{4}$$

The result follows from (2), (3) and (4). □

Corollary 3. *Write* $\overleftarrow{h}(t) = (n+1) - h(t)$ *for* $t = 1, 2, \ldots, n$. *Then* $\overleftarrow{h}$ *is a graceful labelling of* $P(n)$; $h(1) = n - 1$.

Proof: This is an immediate consequence of the theorem. □

We now label a disjoint union of a certain path P and cycle C where $|V(P \cup C)| = k$.

3 The main theorem: Theorem 6

Let k (≥ 7) and n $(\geq 0, \neq 1)$ be integers. Suppose that $k \equiv 2, 4, 5 \pmod 6$ and set $m = \lfloor k/6 \rfloor$. Define $\theta = \theta(k)$ and $w = w(k, n)$ as follows:

$$\theta = \begin{cases} 1 & (k \equiv 5 \pmod 6) \\ 0 & \text{otherwise} \end{cases} \qquad w = \begin{cases} 1 & (k \equiv 4 \pmod 6 \text{ and } n = 0) \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $k = 3\theta + 2w + 6m + 2$.

Set $C = C(4m + 2(\theta + w) + 2)$ and $P = P(2m + \theta)$. Suppose that

$$V(P) = \{u(i) \colon i = 1, 2, \ldots, 2m + \theta\}$$

and

$$V(C) = \{v(i) \colon i = 0, 1, \ldots, 4m + 2(\theta + w) + 1\}.$$

Set

$$\begin{aligned} X &= \{s \colon s = 0, 1, \ldots, 3m + 2\theta + w\} \\ &\quad \cup \{n + 3m + 2\theta + w + s \colon s = 1, 2, \ldots, 3m + \theta + 1 + w\} \end{aligned}$$

and define mappings

$$f \colon V(P) \to X, \quad g \colon V(C) \to X$$

as follows: write $f(u(i)) = f(i)$ and $g(v(i)) = g(i)$ then

(1) (i) $f(2i-1) = w - 2 + 3i, \quad (i = 1, 2, \ldots, m + \theta)$
(ii) $f(2i) = k + n - 3i \qquad (i = 1, 2, \ldots, m)$

(2) (i) $g(0) = 0$

(ii) $g(2i-1) = \begin{cases} k+n+2-3i & (i=1,2,\ldots,m+1) \\ n-w-3+3i & (i=m+2, m+3, \ldots, 2m+w+\theta+1) \end{cases}$

(iii) $g(2i) = \begin{cases} w-1+3i & (i=1,2,\ldots,m+\theta) \\ k-w+1-3i & (i=m+\theta+1, m+\theta+2, \ldots, 2m+w+\theta) \end{cases}$

□

Define

$$h^* : V(P) \cup V(C) \to X$$

to be the mapping which extends both f and g.

Lemma 4. *h^* is a bijection.*

Proof: We use

$$k = 3\theta + 2w + 6m + 2$$

and check separately the three cases: $\theta = w = 0$; $\theta = 1$, $w = 0$; $\theta = 0$, $w = 1$. □

Lemma 5. *The set of edge labels induced by h^* is*

$$\{n + i : i = 1, 2, \ldots, k - 1\}.$$

Proof: We use

$$k = 3\theta + 2w + 6m + 2 \tag{1}$$

and recall that when $w = 1$, $n = 0$.

Case 1. (path edge labels)

(i) For $i = 1, 2, \ldots, m$,

$$f(2i) - f(2i-1) = n + 3\theta + w + 6m + 4 - 6i$$

which gives label set

$$\{n + 3\theta + w - 2 + 6i : i = 1, 2, \ldots, m\}. \tag{2}$$

(ii) For $i = 1, 2, \ldots, m + \theta - 1$,

$$f(2i) - f(2i+1) = n + 3\theta + w + 6m + 1 - 6i$$

which gives label set

$$\{n - 3\theta + w + 1 + 6i : i = 1, 2, \ldots, m + \theta - 1\}. \tag{3}$$

Case 2. (cycle edge labels)

(i) For $i = 2, 3, \ldots, m+1$,

$$g(2i-1) - g(2i-2) = n + 3\theta + w - 10 + 6i$$

which gives label set

$$\{n + 3\theta + w - 4 + 6i \colon i = 1, 2, \ldots, m\}. \tag{4}$$

(ii) For $i = 1, 2, \ldots, m+\theta$,

$$g(2i-1) - g(2i) = n + 3\theta + w + 6m - 5 - 6i$$

which gives label set

$$\{n - 3\theta + w - 1 + 6i \colon i = 1, 2, \ldots, m+\theta\}. \tag{5}$$

(iii) For $i = m+2+\theta, m+3+\theta, \ldots, 2m+\theta+w+1$,

$$g(2i-1) - g(2i-2) = n - 3\theta - 2w - 6m - 9 + 6i$$

which gives label set

$$\{n + 3\theta - 2w - 3 + 6i \colon i = 1, 2, \ldots, m+w\}. \tag{6}$$

(iv) For $i = m+2, m+3, \ldots, 2m+\theta+1$,

$$g(2i-1) - g(2i) = n - 3\theta - 2w - 6m - 6 + 6i$$

which gives label set

$$\{n - 3\theta - 2w + 6i \colon i = 1, 2, \ldots, m+\theta\}. \tag{7}$$

(v)

$$g(1) - g(0) = n + 3\theta + 2w + 6m + 1 \tag{8}$$

$$g(2m + 2\theta + 1) - g(2m+2) = n + w + 1. \tag{9}$$

When $w = 1$,

$$g(4m+3) - g(0) = 3\theta + 2w + 6m \tag{10}$$

(recall that $n = 0$ when $w = 1$).

The Lemma now follows from (2) to (10). □

Notation. Set $H = C \cup P^*$ where

(i) $P^* = P(2m+n+\theta) = P(2m+\theta)P(n)$, i.e. P^* is the concatenation of the paths considered in Theorem 2 and Lemmas 4 and 5.

(ii) $C = C(4m+2(\theta+w)+2)$, i.e. the cycle considered in Lemmas 4 and 5.

Notice $|V(H)| = k+n$.

We now describe a graceful labelling h^{**} of H:

(i) h^{**} extends h^*.

(ii) Set $h^+ = h + 3m$ (see Theorem 2)

and

$$\overleftarrow{h}^{+} = \overleftarrow{h} + 3m + 2 \text{ (see Corollary 3).}$$

Then when $\theta = 0$, h^{**} extends h^+ and when $\theta = 1$, h^{**} extends $\overleftarrow{h}^{+}$.

Theorem 6. *h^{**} is a graceful labelling of H.*

Proof: The definitions of h^+ and $\overleftarrow{h}^{+}$ ensure that

$$|h^{**}(u(2m+\theta)) - h^{**}(w(1))| = n. \tag{1}$$

Since h and $\overleftarrow{h}$ are graceful labellings of $P(n)$ the set of edge labels, induced by h^{**}, on the edges of $P(n)$ is

$$\{i\colon i = 1, 2, \ldots, n-1\}. \tag{2}$$

Hence, from (1), (2) and Lemma 5, the set of edge labels is

$$\{i\colon i = 1, 2, \ldots, n+k-1\}. \tag{3}$$

Again, from the definition of h^+ and $\overleftarrow{h}^{+}$ and, from Lemma 4, the set of vertex labels induced by h^{**} is

$$\{i\colon i = 0, 1, \ldots, n+k-1\} \tag{4}$$

where if $w = 1$ then $n = 0$.

It follows from (3) and (4) that h^{**} is graceful. □

Corollary 7. *$C(2a) \cup P(b)$ is graceful for $b \geq a+1$ $(a \geq 3)$.*

Proof: This follows immediately from Theorem 6 on rearranging the parameters.

Theorem 8. *The graphs $C(2t) \cup P(n)$, $t \in \{3, 4, 5, 6\}$ $(n \geq 2)$ are graceful.*

Proof: We use Theorem 6.

When $t = 3$ take $m = 1$, $\theta = 0$, $w = 0$.

When $t = 4$ consider the cases: $m = 1$, $\theta = 1$, $w = 0$; $m = 1$, $w = 1$, $\theta = 0$.

When $t = 5$ take $m = 2$, $\theta = 0$, $w = 0$.

When $t = 6$ consider the cases: $m = 2$, $\theta = 1$, $w = 0$; $m = 2$, $\theta = 0$, $w = 1$.

Graceful labellings of $C(6) \cup P(3)$, $C(8) \cup P(4)$, $C(10) \cup P(i)$, $i \in \{2, 3, 5\}$ and $C(12) \cup P(i)$, $i \in \{2, 3, 6\}$ are given in Figure 1.

This completes the proof of the theorem. □

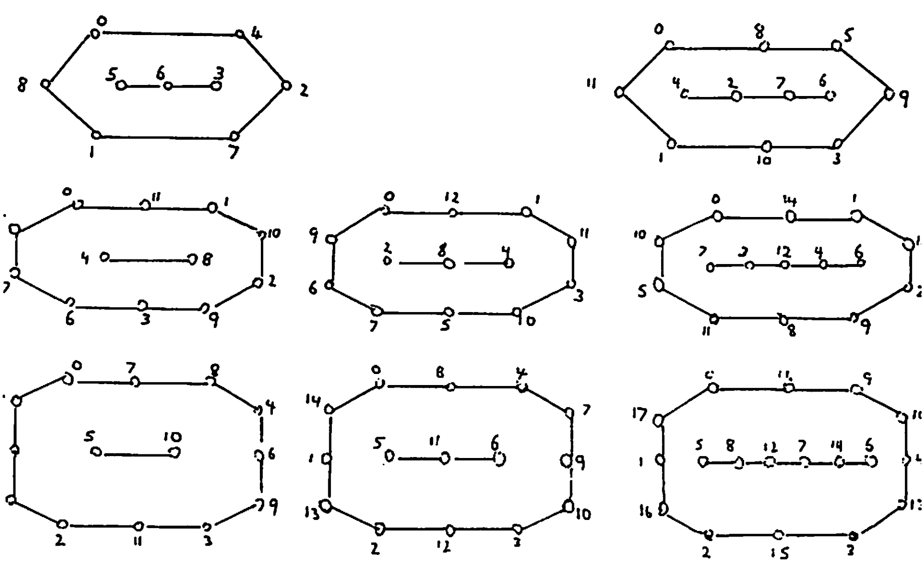

Figure 1

Final Comment

Using similar techniques we have shown that $C(m) \cup P(n)$ is graceful for $m = 5, 7, 9, 11$ and $n \geq 2$ but the details are as yet too complicated to be published here.

References

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