

G -reconstruction of graphs

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Abstract

Let G be a group of permutations acting on an n -vertex set V , and X and Y be two simple graphs on V . We say that X and Y are G -isomorphic if Y belongs to the orbit of X under the action of G . One can naturally generalize the reconstruction problems so that when G is S_n , the symmetric group, we have the usual reconstruction problems. In this paper, we study G -edge reconstructibility of graphs. We prove some old and new results on edge reconstruction and reconstruction from end vertex deleted subgraphs.

1 Introduction

Unless specified, all the graphs in this paper are assumed to be undirected and without multiedges or loops, and to have n vertices and m edges. Distance between any two vertices u and v is denoted by $d(u, v)$, and maximum degree in a graph X is denoted by $\Delta(X)$ or simply Δ when there is no confusion. The automorphism group of a graph X is denoted by $\text{aut}X$.

The vertex deck of a graph X , denoted by $VD(X)$, is the collection of all its unlabelled vertex deleted subgraphs, and the graph X (or a property or a parameter of X) is said to be vertex reconstructible if X (or the property or the parameter) can be uniquely obtained from $VD(X)$. Similarly one also defines edge deck $ED(X)$ and edge reconstructibility. The collection of unlabelled subgraphs of X obtained by deleting degree-1-vertices, called end-vertex deck, is denoted by $VD_1(X)$, and correspondingly we have end-vertex reconstructibility. Vertex reconstruction conjecture (VRC) states that graphs with at least three vertices are vertex reconstructible. Edge reconstruction conjecture states that graphs with at least four edges are edge reconstructible. One can also pose the same conjectures in the language of hypomorphisms between labelled graphs as follows. Two graphs X and

Y are said to be vertex hypomorphic, denoted by $X \sim Y$, when there is a bijection f , called vertex hypomorphism, from $V(X)$ to $V(Y)$ such that $X - u \cong Y - f(u)$ for all $u \in V(X)$. VRC then states that $X \sim Y$ implies $X \cong Y$, provided $n \geq 3$. One similarly defines edge hypomorphism and can pose ERC. Reader is referred to [B] for survey of various reconstruction problems.

Let $V(X) = V(Y) = V$, and G be a group of permutations acting on V . The action of G defines the orbits of X and Y , and we say that X and Y are G -isomorphic, denoted by $X \stackrel{G}{\cong} Y$ if Y is in the orbit of X under the action of G . We can then naturally extend the above definitions to G -vertex (or edge) hypomorphism, (denoted by the symbol $\stackrel{G}{\sim}$), G -vertex (or edge) reconstructibility etc., and study the corresponding reconstruction problems.

Given a graph X and an edge set $E \subseteq E(X)$, an edge set F is called a replacing edge set of E if $X - E + F \sim X$ (or $X - E + F \stackrel{G}{\sim} X$) and $E \cap F = \emptyset$.

In this paper we demonstrate that edge or vertex reconstructibility of graphs can be proved under some circumstances by suitably choosing a group G and considering the problem as G -ERC or G -VRC. In Section 2, we state a generalization of the well known Nash-Williams' lemma. It is then applied to ERC in Section 2.1, and to vertex reconstruction of graphs from their end-vertex deleted subgraphs in Section 2.2.

This is an expanded version of [T3].

2 G -edge reconstruction

Let $V(X) = V(Y) = V$ and F be a spanning subgraph of X . For a group G , we denote $|\{g \in G | g(Y) \cap X = F\}|$ by $|Y \stackrel{G}{\rightarrow} X|_F$. The following lemma, which is a generalization of the Nash-Williams' lemma, is our tool in dealing with the reconstruction problems considered in the next two subsections. It can be proved along the same lines as Theorem 1.1 in [T1], and also follows from Theorem 2.1 in [ACKR].

Lemma 2.1 *If X and Y are G -edge hypomorphic but not G -isomorphic then for every spanning subgraph F of X , we have*

$$|X \stackrel{G}{\rightarrow} X|_F - |Y \stackrel{G}{\rightarrow} X|_F = (-1)^{m-|E(F)|} |G \cap \text{aut}X|$$

In the following, we demonstrate that many reconstruction problems can be naturally formulated as G -edge reconstruction problems, and Lemma 2.1 can be applied.

2.1 Edge reconstruction

The graphs considered in this section are 2-edge connected bipartite graphs or separable graphs with 3-connected pruned centre.

2-edge connected bipartite graphs

Proposition 2.2 *Let X be a 2-edge connected bipartite graph with s and t as the sizes of the two parts. Then X is edge reconstructible provided $m > st/2$ or $2^{m-1} > |\text{aut}K_{s,t}|$.*

Proof The recognition is trivial. Also, because of 2-edge connectivity, the vertex partitions are uniquely recognized in the subgraphs, so we assume the edge hypomorphism between X and a possible reconstruction Y to be a G -edge hypomorphism, where G is $\text{aut}K_{s,t}$. Now the claim is a simple corollary of Lemma 2.1.

That $m > st/2$ is sufficient for edge reconstructibility, was proved in [VY1].

Separable graphs with 3-edge connected pruned centre

For a graph X , define the *block-cutpoint tree* $T(X)$, whose vertex set has all the cutpoints and all the maximal two connected subgraphs (2-blocks) in it, and two vertices of $T(X)$ are joined by an edge iff one of them is a 2-block and other is a cutpoint on the same 2-block. The *pruned graph* $P(X)$ is the maximal subgraph without end-vertices. The center of $T(P(X))$ corresponds to a 2-block or a cut vertex of X , and is called the *pruned center* of X , denoted by $C(X)$.

Proposition 2.3 *Let X be a separable graph with end vertices, having a 3-connected pruned center $C(X)$. Suppose we colour the vertices of $C(X)$ blue and vertices outside $C(X)$ red. Let G' denote the automorphism group of the coloured graph $X - E(C(X))$, and G its subgroup induced by $V(C(X))$. Then X is edge reconstructible if $C(X)$ is G -edge reconstructible.*

Proof When an edge incident with an end vertex is deleted, (which is easily recognizable), we know the pruned graph and the pruned centre uniquely. This allows us to recognize the subset S of $ED(X)$ obtained by deleting an edge of $C(X)$ - given $X - e$, $e \in E(C(X))$ iff $T(P(X)) \cong T(P(X - e))$ and $|E(C(X))| = |E(C(X - e))| + 1$. Once S is recognized, we can assume that the pruned centres of X and Y , (where $Y \sim X$), have the same vertex set, and we are given only the graphs in S . Therefore, $C(X) \stackrel{G}{\cong} C(Y)$ is enough for an isomorphism between X and Y .

Following are some of the immediate consequences:

1. If $2^{|E(C(X))|} > |G|$ or $|E(C(X))| > |V(C(X))|/2$ then X is edge reconstructible. Note that we have $2^{|E(C(X))|}$ rather than $2^{|E(C(X))|-1}$ be-

cause we know $C(X)$ uniquely. Version of Lovász's result was proved earlier in [VY].

2. If $C(X)$ is claw-free or P_4 -free or chordal, then X is edge reconstructible. (A graph is chordal if no induced subgraph on four or more vertices is a cycle. A graph is claw-free if no induced subgraph is isomorphic to $K_{1,3}$. A graph is P_4 -free if no induced subgraph on four vertices is a path.)

We in fact have something stronger:

- (a) One can observe that all connected claw-free graphs other than paths are G -edge reconstructible for all groups G (irrespective of their connectivity). We do not give the details of the proof here, but refer the reader to [T2], where it is proved that a collection of connected claw-free graphs can be reconstructed from its shuffled edge deck. We only comment that all the steps in that proof are actually based on the fact that some edge set in a claw-free graph has no replacement unless it is a path. Paths are not G -reconstructible for some groups, (for example, a $2k$ -vertex path, for $k \geq 2$, is not A_{2k} -edge reconstructible, where A_{2k} is the alternating group).
- (b) In case of chordal graphs, we again follow the proofs in Section 4 of [T2], and claim that all 2-connected chordal graphs are G -edge reconstructible for all groups G . We also point out that, all trees except the thirteen trees listed in [CS] are G -edge reconstructible, since it is proved in [CS] that some edge sets in all the other trees have no replacing edge sets.
- (c) Connected P_4 -free graphs are G reconstructible for all groups G – the complement of any connected P_4 -free graph is disconnected, therefore, set of all the edges cannot be replaced.

Question Can one reduce ERC for graphs with 3-connected pruned centre to 3-connected graphs?

2.2 End vertex deleted subgraphs

Let Z be a graph with minimum degree at least 2, and X be a graph obtained by adding some more vertices of degree 1, making the new vertices adjacent to vertices in Z . We give simple proofs of some results on reconstruction from end vertex deleted subgraphs, some of which appeared in [L]. First we consider the case in which r_1 vertices of Z are made adjacent to r_1 new vertices of degree 1.

Proposition 2.4 *If $r_1 > |V(Z)|/2$ or $2^{r_1-1} > \text{aut}Z$ then X can be reconstructed from its end-vertex deleted subgraphs.*

Proof Since there are no end vertices in Z , vertices of Z are recognizable in every member of the deck. We identify all the end vertices of X , call the resulting vertex v and colour it blue, and rest of the vertices red. Call this graph Y . Since no two end vertices of X have a common neighbour, X is reconstructible from its end vertex deleted subgraphs, if Y is reconstructible from its subgraphs $Y - av$, where av is an edge between v and a red vertex a . Thus we are just edge reconstructing a graph which is a disjoint union of an r_1 -star and some isolated vertices, with the centre of the star coloured blue, with respect to the group $\text{aut}Z$. A direct application of Lemma 2.1 gives the result.

Now, we extend this idea to prove something stronger. Let Z be as above. Add end vertices in this graph to construct X as follows. Let $R_i \subseteq V(Z)$, $i = 1$ to k be disjoint sets, and $|R_i| = r_i \geq 0$. For $i = 1$ to k , we join each member of R_i to precisely i end-vertices. Set of remaining vertices of Z is denoted by R_0 . Following result is somewhat stronger than the results in [L].

Proposition 2.5 *If $r_j > r_{j-1}$ or $2^{r_j-1} > |\text{aut}Z|$ for some $j \leq k$, then X is end-vertex reconstructible.*

Proof Let $VD_1(Y) = VD_1(X)$. First we make some ‘recognition’ claims. As in Proposition 2.4, vertices of Z are recognized in every graph in the deck. Also, the $P(Y) \cong Z$. If Y is obtained as above by joining r'_i vertices to i end-vertices each, then $r'_i = r_i$. This is trivial to prove unless $r_1 = 2$, $r_i = 0$ for $i \geq 2$ and, $r'_1 = 0$, $r_2 = 1$ and $r'_i = 0$ for $i \geq 3$, in which case there are obvious counter examples. It is also trivial to recognize, for any given graph in the deck, the i for which a neighbour of $u \in R_i$ is deleted. Thus we have a natural partition of the given deck into decks D_i , $i = 1$ to k , where a member of D_i results from deleting an end-vertex adjacent to a vertex in R_i . Obviously, the multiplicity of any unlabelled graph in D_i is a multiple of i , so we can construct reduced decks D'_i by reducing all the multiplicities by a factor of i .

Construct a graph X_j , $j \geq 1$ from Z by colouring vertices of R_i with colour c_i , where $i \notin \{j-1, j\}$ and making vertices of R_j adjacent to one end-vertex each. If X_j is end-vertex reconstructible, then X is end-vertex reconstructible. Therefore, if $r_j > r_{j-1}$ or $2^{r_j-1} > |\text{aut}X_j|$ then X is end-vertex reconstructible, as in Proposition 2.4. Note that $|\text{aut}X_j| \leq |\text{aut}Z|$.

For some j , if $r_j > |V(Z)|/2$ then the graph is end-vertex reconstructible, which was proved in [La]. This is a corollary of our results in Propositions 2.4 and 2.5.

Remark Suppose that we are given the vertex deck of an arbitrary separable graph X with end-vertices. Given any end-vertex deleted graph $X - u$, it is easy to recognize the distance of u from the nearest vertex of $P(X)$. Thus we know the number of end vertices at any distance j from $P(X)$, and we can prove analogous results on the vertex reconstruction of X .

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