Connected (g, f)-factors and supereulerian digraphs

G. Gutin
Department of Maths and Stats
Brunel University
Uxbridge, Middlesex, UB8 3PH, U.K.
email: z.g.gutin@brunel.ac.uk

Dedicated to the memory of Paul Catlin

ABSTRACT. Given a digraph (an undirected graph, resp.) D and two positive integers $f(x), g(x)$ for every $x \in V(D)$, a subgraph H of D is called a (g, f)-factor if $g(x) \leq d^+_H(x) = d^-_H(x) \leq f(x) (g(x) \leq d_H(x) \leq f(x)$, resp.) for every $x \in V(D)$. If $f(x) = g(x) = 1$ for every x, then a connected (g, f)-factor is a hamiltonian cycle. The previous research related to the topic has been carried out either for (g, f)-factors (in general, disconnected) or for hamiltonian cycles separately, even though numerous similarities between them have been recently detected. Here we consider connected (g, f)-factors in digraphs and show that several results on hamiltonian digraphs, which are generalizations of tournaments, can be extended to connected (g, f)-factors. Applications of these results to supereulerian digraphs are also obtained.

1 Introduction and terminology

Given a digraph (an undirected graph, resp.) D and two positive integers $f(x), g(x)$ for every $x \in V(D)$, a subgraph H of D is called a *(g, f)-factor* if $g(x) \leq d^+_H(x) = d^-_H(x) \leq f(x) (g(x) \leq d_H(x) \leq f(x)$, resp.) for every $x \in V(D)$. If $f(x) = g(x) = 1$ for every x, then a connected (g, f)-factor is a hamiltonian cycle. The previous research related to the topic has been done for either (g, f)-factors (in general, disconnected) or hamiltonian cycles separately, even though numerous similarities between them have been recently detected [24] (see e.g. [10, 14, 15, 22, 23] where ideas from hamiltonian graph theory were used to discover new results in factor theory).

In this note we consider connected (g, f)-factors in digraphs and show that some results on hamiltonian digraphs, which are generalizations of tournaments, can be extended to connected (g, f)-factors using polynomial transformations from the hamiltonian cycle problem and the cycle covering given vertices problem. We investigate connected (g, f)-factors rather then general (g, f)-factors as the former rather than the latter are of interest in several applications. In a generalization of the traveling salesman problem, the vehicle routing problem [13], a route is a connected (g, f)-factor such that $f(x) = g(x) = 1$ for all vertices x but one z, called a depot, and $g(z) = 1, f(z) = k$, where k is the number of vehicles available.

We believe that many more characterizations and/or sufficient conditions for directed and undirected graphs to contain connected (g, f)-factors can be obtained.

A number of sufficient conditions for an undirected graph to be *supereulerian* (i.e. to contain a spanning eulerian subgraph) were obtained but no complete characterization is known (see e.g. [11, 12]). We show that one can verify whether a semicomplete multipartite digraph, locally in-semicomplete digraph or quasi-transitive digraph is supereulerian in polynomial time.

A *semicomplete* digraph is a digraph without non-adjacent vertices. Tournaments form a proper subclass of semicomplete digraphs. A digraph D is *locally in-semicomplete* (*locally out-semicomplete*, resp.) if the in-neighbours (out-neighbours, resp.) of every vertex in D induce a semicomplete digraph. A digraph which is both locally in-semicomplete and locally out-semicomplete is *locally semicomplete.*

A digraph D on p disjoint vertex classes (*partite sets*) is a *semicomplete p-partite* (or, *multipartite) digraph* if for any two vertices x and y in different partite sets at least one arc between x and y is in D and there are no arcs between vertices in a same partite set. Clearly, a semicomplete digraph with n vertices is a semicomplete n-partite digraph with only one vertex in every partite set. A digraph D is *quasi-transitive* if, for any triple x, y, z of distinct vertices of D such that (x, y) and (y, z) are arcs of D, there is at least one arc between x and z.

An *extension* of a digraph D is a new digraph H obtained from D by replacing every vertex $x \in V(D)$ with a set of independent vertices S_x such that, for every pair of distinct $x, y \in V(D)$, an arc (u, v), where $u \in S_x$, $y \in S_y$, is in H if and only if (x, y) is in D. An extension of a locally in-semicomplete digraph is called an *extended locally in-semicomplete digraph.* A class of digraphs Φ is called *extension-closed* if every extension of a digraph in Φ is a digraph in Φ. Clearly, the classes of extended locally in-semicomplete digraphs, semicomplete bipartite digraphs, semicomplete multipartite digraphs and quasi-transitive digraphs are extension-closed. Semicomplete digraphs and tournaments are not extension-closed.

Although study of locally semicomplete digraphs, locally in-semicomplete digraphs and quasi-transitive digraphs was initiated quite recently, numerous results on the topic have been already obtained (see e.g. [1, 5, 7, 8, 16, 18, 21]). Semicomplete multipartite digraphs have been investigated for longer time, many results on their path and cycle structure can be found in [19, 25, 27].

2 Results

In this section, n stands for the number of vertices in a digraph under consideration.

The proof of our characterization of semicomplete bipartite digraphs and extended locally in-semicomplete digraphs with connected (g, f)-factors, Theorem 2.2, is based on the following theorem proved in [17] and, independently, in [20] for semicomplete bipartite digraphs, and in [4] for extended locally in-semicomplete digraphs ([2], a short version of [4], does not contain the result for extended locally in-semicomplete digraphs but it has the same result for extended locally semicomplete digraphs with practically the same proof).

Theorem 2.1 *Let D be a semicomplete bipartite digraph or an extended locally in-semi-complete digraph. Then D is hamiltonian if and only if D is strongly connected and contains a (1,1)-factor.*

Theorem 2.2 and Corollary 2.3 are stated for semicomplete bipartite digraphs and extended locally in-semicomplete digraphs only, but they are certainly true for extended locally out-semicomplete digraphs as well since every extended locally out-semicomplete digraph can be obtained from an extended locally in-semicomplete digraph by reversing the arcs.

Theorem 2.2 *Let D be a semicomplete bipartite digraph or an extended locally in-semicomplete digraph. Then D has a connected (g, f)-factor if and only if D is strongly connected and contains a (g, f)-factor. One can check whether D has a connected (g, f)-factor in $O(n^3)$ time.*

Proof: Clearly, every digraph containing a connected (g, f)-factor is strongly connected ($g(x) \geq 1$) and has a (g, f)-factor.

Suppose that D is strongly connected and has a (g, f)-factor F. Let $V(D) = \{x_1, ..., x_n\}$ and let $r(x_i) = d_F^+(x_i)$ $(= d_F^-(x_i))$, $i = 1, ..., n$. Form an extension H of D by replacing every vertex x_i in D with set $\{x_i', x_i'', ..., x_i^{r(x_i)}\}$ of independent vertices. As the classes of semicomplete bipartite digraphs and extended locally in-semicomplete digraphs are extension-closed, H belongs to one of these classes.

Let F' be a component of F. Since F' is a eulerian digraph, it has an eulerian circuit $T(F') = y_1y_2...y_ky_1$. Let y_q $(1 \leq q \leq k)$ be the j'th appearance of x_i in $T(F')$. Then, substituting y_q with x_i^j, we obtain a cycle $C(F')$ in H. Clearly, $\cup\{C(F') : F'$ is a component of $F\}$ is a (1,1)-factor in H. As D is strongly connected so is H. Thus, by Theorem 2.1, H is hamiltonian. Clearly, a hamiltonian cycle in H corresponds to a connected (d_F^+, d_F^+)-factor, a connected (g, f)-factor, in D.

To check whether D contains a (g, f)-factor, construct a network N as follows: The vertex and arcs sets of N are $\{v^-, v^+ : v \in V(D)\}$ and $\{(u^+, w^-) : (u, w) \in E(D)\} \cup \{(v^-, v^+) : v \in V(D)\}$. Assign 0 and 1 as lower and upper bounds to every arc of the form (u^+, w^-), and $g(v)$ and $f(v)$ as lower and upper bounds to every arc of the form (v^-, v^+).

Clearly, D has a (g, f)-factor if and only if N admits a (feasible) flow, i.e. a circulation. It is well known that the problem of the existence of a (feasible) flow in a network M with p vertices can be transformed into the maximum flow problem in an auxiliary network and thus solved in $O(p^3)$ time [9]. It is well known that one can verify whether a digraph is strongly connected in $O(n^2)$ time. □

Clearly, a digraph H is supereulerian if and only if H has a connected $(1, d^*)$-factor, where $d^*(x) = \min\{d^+(x), d^-(x)\}$ for every $x \in V(H)$.

Corollary 2.3 *Let D be a semicomplete bipartite digraph or an extended locally in-semi-complete digraph. D is supereulerian if and only if D is strongly connected and contains a $(1, d^*)$-factor. One can check whether D is supereulerian in $O(n^3)$ time.*

The cycle covering given vertices problem (CCV) is the following: given a digraph D and a set of its vertices W, check whether D contains a cycle C such that $W \subseteq V(C)$.

Proposition 2.4 *The connected (g, f)-factor problem is polynomial reducible to CCV for any extension-closed class of digraphs.*

Proof: Let Φ be an extension-closed class of digraphs and $D \in \Phi$. Replacing every vertex x in D with $f(x)$ independent vertices we obtain a new digraph $H \in \Phi$. Let $W = \cup_{x \in V(D)} G_x$, where G_x consists of any $g(x)$ vertices in H corresponding to x (in D). Clearly, H has a cycle covering W if and only if D contains a connected (g, f)-factor. The above transformation is polynomial as $f(x) < n$. □

It was recently proved in [3] that CCV is polynomial time solvable for quasi-transitive digraphs. This result and Proposition 2.4 imply:

Corollary 2.5 *The connected (g, f)-factor problem is polynomial time solvable for quasi-transitive digraphs. One can check whether a quasi-transitive digraph is supereulerian in polynomial time.*

Notice that the complexity of CCV for quasi-transitive digraphs obtained in [3] is $O(n^5)$. Thus, by Proposition 2.4, the complexity of the connected (g, f)-factor problem for quasi-transitive digraphs is $O(n^{10})$. Using a simple modification of the approach to the hamiltonian cycle problem introduced in [18], one can give a direct proof of Corollary 2.5, which provides a better upper bound, $O(n^4)$, for the complexity of the connected (g, f)-factor problem for quasi-transitive digraphs.

A. Yeo [26] has informed us that he had a draft of a proof that CCV for semicomplete multipartite digraphs is polynomial time solvable. This result would extend Corollary 2.5 to semicomplete multipartite digraphs and generalize another recent result [6]: the hamiltonian cycle problem for semicomplete multipartite digraphs is polynomial time solvable.

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