The existence of group divisible designs with first and second associates, having block size 3

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Abstract

We completely settle the existence problem for group divisible designs with first and second associates in which the block size is 3, and with m groups each of size n, where n, $m \geq 3$.

1 Introduction

According to Raghavarao [15] partially balanced designs with two association classes were classified in 1952 by Bose and Shimamoto into five types:

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group divisible designs, simple, triangular, latin square type and cyclic. We will concentrate here on group divisible designs.

A *group divisible design* $GDD(n, m; k; \lambda_1, \lambda_2)$ is an ordered triple (V, G, B) where V is a set of *varieties* or *symbols*, G is a partition of V into m sets of size n, each set being called a *group*, and B is collection of subsets of V, called *blocks* each of size k, such that

(1) each pair of varieties that occur together in the same group, occur together in exactly λ_1 blocks, and

(2) each pair of varieties that occur together in no group, occur together in exactly λ_2 blocks.

Elements occurring together in the same group are called *first associates*, and elements occurring in different groups we called *second associates.* We say that the GDD is *defined on the set* V.

In 1952 Bose and Connor further subdivided group divisible designs into three classes:

Singular if $r - \lambda_1 = 0$

Semiregular if $r - \lambda_1 > 0$ and $rk - v\lambda_2 = 0$ and

Regular if $r - \lambda_1 > 0$ and $k - v\lambda_2 > 0$.

For a wealth of information on GDDs see Raghavarao [15]. Clatworthy [5] gives tables for all three classes of GDDs. Hanani [13] considered group divisible designs with $\lambda_1 = 0$. He proved that the elementary necessary conditions are sufficient for the existence of such GDDs with block size 3 (see Theorem 1.2). Brouwer, Schrijver and Hanani [3] proved that the necessary conditions are sufficient for the existence of GDDs with $k = 4$ and $\lambda_1 = 0$.

In this paper, we completely solve the problem of constructing GDDs with $k = 3$ and n, $m \geq 3$.

To simplify the notation, let a $GDD(n,m;3;\lambda_1,\lambda_2)$ be denoted by a $GDD(n,m)$ of *index* (λ_1,λ_2), and let a block of size 3 be called a *triple*. GDDs with $m=1$ (so λ_2 is irrelevant) and $k=3$ are the well known triple systems, so we denote a $GDD(n,1;3;\lambda_1,\lambda_2)$ by a $TS(n)$ of index λ_1. Since a $TS(n)$ has only one group, $V=G$ so it can be simply represented by (V,B). We will use the following famous result.

Theorem 1.1 ([12]) *Let $n\geq 3$. There exists a $TS(n)$ of index λ iff*

(a) 2 divides $\lambda(n-1)$, and

(b) 3 divides $\lambda n(n-1)$.

So the existence of a $GDD(n,m)$ of order (λ_1,λ_2) has been completely settled if $m=1$ (and so has been settled if $n=1$, and if $\lambda_1=0$). It has also been settled if $\lambda_1=0$ with the following result.

Theorem 1.2 ([13]) *There exists a $GDD(n,m)$ of index $(0,\lambda_2)$ iff*

(a) 2 divides $\lambda_2(m-1)n$,

(b) 3 divides $\lambda_2 m(m-1)n^2$, and

(c) $m\geq 3$.

In this paper we settle the existence problem for a $GDD(n,m)$ of index (λ_1,λ_2) with $n\geq 3$, $m\geq 3$, $\lambda_1\geq 1$ and $\lambda_2\geq 1$ (see Theorem 3.2). The case where $n=2$ or $m=2$ is quite complicated and of less statistical interest, so is solved in another paper [11].

We can immediately obtain some necessary conditions.

Lemma 1.3 *If there exists a $GDD(n,m)$ of index (λ_1,λ_2), then*

(1) 2 divides $\lambda_1(n-1)+\lambda_2(m-1)n$, and

(2) 3 divides $\lambda_1 mn(n-1)+\lambda_2 m(m-1)n^2$.

Proof: (1) Each block containing a particular symbol s in V has 2 pairs of symbols that contain s, so the total number of pairs of symbols containing s must be divisible by 2; the number of blocks $r = (\lambda_1(n-1)+\lambda_2(m-1)n)/2$ containing s is called the replication number of the GDD.

(2) Each block contains 3 pairs, so the total number of pairs must be divisible by 3. So $b = (\lambda_1 mn(n-1)/2 + \lambda_2 m(m-1)n^2/2)/3$ is the number of blocks in the GDD. □

It turns out that these necessary conditions are sufficient, as is stated in Theorem 3.2. Proving this is the main result in this paper. Table 1 expands on Theorem 3.2, explicitly listing the values of n and m for each value of λ_1 and λ_2 for which there exists a $GDD(n,m)$ of index (λ_1, λ_2).

Throughout the rest of this paper we assume that $\lambda_1 \geq 1$, $\lambda_2 \geq 1$, $n \geq 3$, $m \geq 3$ and $k = 3$. Let $\mathbb{Z}_x = \{0, 1, \ldots, x-1\}$.

2 Preliminary Results

The following lemmas will be extremely useful in fulfilling our task.

Lemma 2.1 *Let $m \geq 3$. If there exists a $TS(n)$ of index λ_1, and if conditions (1) and (2) of Lemma 1.3 hold, then there exists a $GDD(n,m)$ of index (λ_1, λ_2).*

Proof: Since there exists a $TS(n)$ of index λ_1, conditions (a) and (b) of Theorem 1.1 hold. Since we also know that conditions (1) and (2) of Lemma 1.3 hold, we have that conditions (a), (b) and (c) of Theorem 1.2 hold. Therefore there exists a $GDD(n,m)$ of index $(0, \lambda_2)$. So putting this together with m copies of the $TS(n)$ of index λ_1 (one copy is defined on each group), produces a $GDD(n,m)$ of index (λ_1, λ_2). □

Lemma 2.2 *Let $m \geq 3$ and $\lambda_1 \leq \lambda_2$. If there exists a $TS(nm)$ of index λ_1, and if conditions (1) and (2) of Lemma 1.3 hold, then there exists a*

$GDD(n, m)$ *of index* (λ_1, λ_2).

Proof: From Theorem 1.1 we have that 2 divides $\lambda_1(mn-1)$ and that 3 divides $\lambda_1 mn(mn-1)$. Adding conditions (1) and (2) of Lemma 1.3 means that conditions (a), (b) and (c) for the existence of a $GDD(n, m)$ of index $(0, \lambda_2 - \lambda_1)$ are satisfied. This GDD together with the postulated $TS(mn)$ of index λ_1, produce a $GDD(n, m)$ of index (λ_1, λ_2). □

Lemma 2.3 *If* $\lambda < \lambda_1$, *if there exists a* $TS(n)$ *of index* λ, *and if there exists a* $GDD(n, m)$ *of index* $(\lambda_1 - \lambda, \lambda_2)$, *then there exists a* $GDD(n, m)$ *of index* (λ_1, λ_2).

Proof: Add m copies of the $TS(n)$ of index λ, one defined on each of the groups of the $GDD(n, m)$, to the $GDD(n, m)$ of index $(\lambda_1 - \lambda, \lambda_2)$. □

Lemma 2.4 *If* $\lambda < \min\{\lambda_1, \lambda_2\}$, *if there exists a* $TS(mn)$ *of index* λ, *and if there exists a* $GDD(n, m)$ *of index* $(\lambda_1 - \lambda, \lambda_2 - \lambda)$, *then there exists a* $GDD(n, m)$ *of index* (λ_1, λ_2).

Proof: Add the $TS(mn)$ to the $GDD(n, m)$ of index $(\lambda_1 - \lambda, \lambda_2 - \lambda)$ to produce a $GDD(n, m)$ of index (λ_1, λ_2). □

The following GDDs will be of use in Section 2, as well as being important in Section 3.

Lemma 2.5 *If* $m \equiv 0$ *(mod 3) then there exists a* $GDD(2, m)$ *of index* $(2, 1)$.

Proof: This is also known as a minimum covering of a $TS(2m)$ of index 1, and is constructed in [10]. □

A $TS(n)$ (V, B) or a $GDD(n, m)$ (V, G, B) is said to be *resolvable* if B can be partitioned into sets of $|V|/3$ edge-disjoint triples; each set in this partition of B is known as a *parallel class*. The following is well known [2, 4, 12, 17].

Lemma 2.6 *For all $m \equiv 3 \pmod 6$ there exists a resolvable $TS(m)$ of index 1 (also known as a Kirkman triple system). For all $m \equiv 0 \pmod 6$ with $m > 12$, there exists a resolvable $GDD(2, m/2)$ of index $(0,1)$ (also known as a nearly Kirkman triple system).*

At various stages it will be helpful to consider a graph theoretic description of a $TS(v)$, or some related structure. It is easy to see (and well known) that a $TS(v)$ of index 1 (that is, a Steiner triple system) is equivalent to an edge-disjoint decomposition of K_v, the complete graph on v vertices, into copies of K_3. A 2-factor of K_v is a subgraph of K_v in which each vertex has degree 2. The following result shows that if $m \equiv 5 \pmod 6$, so Theorem 1.1 shows that no Steiner triple system exists, we can find a useful related structure.

Lemma 2.7 *For all $m \equiv 3$ or $5 \pmod 6$, $m \geq 5$, there exists an edge disjoint decomposition of K_m into two 2-factors and $m(m-5)/6$ copies of K_3, and in the case $m \equiv 3 \pmod 6$, each component of each 2-factor is a K_3.*

Proof: If $m \equiv 3 \pmod 6$ then by Lemma 2.6 there exists a resolvable $TS(m)$ of index 1. Use two parallel classes to form the two 2-factors and the result follows.

If $m \equiv 5 \pmod 6$ then let $\mathbb{Z}_m$ be the vertex set and let $m = 6x + 5$. We use the result of Davies [8] and Skolem [19] (see also Simpson [18] for a generalization) who showed that the integers in $\{x+1, x+2, \ldots, 3x+1\} \setminus \{s\}$, where $s \in \{3x, 3x+1\}$, can be partitioned into pairs (a_i, b_i) with $b_i > a_i$ such that if $b_i - a_i = c_i$ then $\{c_i | 1 \leq i \leq x\} = \{1, 2, \ldots, x\}$. The two 2-factors are formed by the edges $\{j, j+s\}$ and $\{j, j+3x+2\}$ for each $j \in \mathbb{Z}_m$. The remaining edges can be partitioned by the following triples: $\{j, a_i + j, b_i + j\}$ for each $j \in \mathbb{Z}_m$ and for $1 \leq i \leq x$. □

The following turns out to be one of the more difficult cases in our quest to prove Theorem 3.2. Since it will be used in Sections 2 (where we assume $n > 2$) and 3, we will present it here.

Lemma 2.8 *Let $m \equiv 0$ or 2 (mod 3) with $m > 3$. There exists a $GDD(2, m)$ of index* $(4, 1)$.

Proof: Suppose that we have an edge-disjoint decomposition of K_m with vertex set $\mathbb{Z}_m$ into a graph H together with a set T of copies of K_3. Suppose further that the edges of H can be directed so that each vertex has out-degree 2. Then we can construct a $GDD(2, m)$ of index $(4, 1)$ with $(V, G, B) = (\mathbb{Z}_m \times \mathbb{Z}_2, \{i \times \mathbb{Z}_2 \mid i \in \mathbb{Z}_m\}, B)$ by defining B as follows.

(a) For each directed edge (a, b) in H, let $\{\{(a, 0), (a, 1), (b, 0)\}, \{(a, 0), (a, 1), (b, 1)\}\} \subseteq B$, and

(b) for each copy of K_3 on the vertex set $\{a, b, c\}$ in T, let $\{(a, 0), (b, 0), (c, 0)\}$, $\{(a, 1), (b, 1), (c, 0)\}$, $\{(a, 1), (b, 0), (c, 1)\}$, $\{(a, 0), (b, 1), (c, 1)\}\} \subseteq B$ (this is just a $GDD(2, 3)$ of index $(0, 1)$).

Since each vertex $a \in \mathbb{Z}_m$ has out degree 2, the pair $\{(a, 0), (a, 1)\}$ occurs in $\lambda_1 = 4$ triples defined in (a). Since each edge $\{a, b\}$ is in H or in exactly one copy of K_3, the pair $\{(a, i), (b, j)\}$ (with $a \neq b$, $\{a, b\} \subseteq \mathbb{Z}_m$, $\{i, j\} \subseteq \mathbb{Z}_2$) occurs in exactly one triple defined in (a) or (b) respectively.

So it remains to find the edge-disjoint decomposition H and T of K_m for all $m \equiv 0$ or 2 (mod 3) with $m > 3$.

If $m \equiv 3$ or 5 (mod 6) then by Lemma 2.7 there exists an edge-disjoint decomposition of K_m into two 2-factors and copies of K_3. By orienting each cycle in each component of each 2-factor to form a directed cycle, we form the required graph H in which each vertex has out-degree 2.

If $m \equiv 0$ (mod 6) then for all $m \neq 12$, partition the vertices into $m/6$ sets of size 6. For each such vertex set, say $\{a_0, a_1, a_2, a_3, a_4, a_5\}$, let H

contain the directed edges (a_i, a_{i+1}) for $i \in \mathbb{Z}_6$, (a_i, a_{i+2}) for $i \in \{0, 2, 4\}$ and (a_i, a_{i+3}) for $i \in \{1, 3, 5\}$, so each vertex has outdegree 2. The only edges not covered in this copy of K_6 form a copy of K_3 on the vertex set $\{a_1, a_3, a_5\}$. Finally, the edges joining different copies of K_6 can be partitioned into copies of K_3, since this is equivalent to a $GDD(6, m/6)$ of index $(0, 1)$, which exists by Theorem 1.2 since $m/6 \neq 2$. For $m = 12$, let K_m have vertex set $\mathbb{Z}_m$, let H contain the directed edges $(i, i+4)$ for $i \in \{0, 1, 4, 5, 8, 9\}$, $(i, i+6)$ for $i \in \{2, 3, 6, 7, 10, 11\}$, and $(i, i+5)$ for $i \in \mathbb{Z}_{12}$; so every vertex has outdegree 2. The remaining edges can be partitioned into copies of K_3: $\{i, i+1, i+3\}$ for $i \in \mathbb{Z}_{12}$, and $\{i, i+4, i+8\}$ for $i \in \{2, 3\}$.

If $m = 8$ then let H contain the directed edges $(7, 0)$, $(7, 1)$, $(2, 7)$, $(3, 7)$, $(4, 7)$, $(5, 7)$, $(6, 7)$, $(0, 6)$, $(1, 6)$, $(2, 6)$, $(3, 6)$, $(4, 6)$, $(6, 5)$, $(0, 3)$, $(1, 4)$, and $(5, 2)$. The remaining edges occur in the triples $\{3, 4, 5\}$, $\{1, 2, 3\}$, $\{0, 2, 4\}$ and $\{0, 1, 5\}$.

If $m \equiv 2 \pmod 6$ and $m > 8$, we will use a $TS(m-1)$ $(\mathbb{Z}_{m-1}, B)$ that has a set P of $(m-2)/3$ vertex-disjoint triples. If $m = 14$ then let $B = \{\{i, i+1, i+4\}, \{i, i+2, i+8\} \mid i \in \mathbb{Z}_{13}\}$, and let $P = \{\{0, 1, 4\}, \{3, 5, 11\}, \{6, 7, 10\}, \{8, 9, 12\}\}$. If $m > 14$ then B can be obtained from the resolvable $GDD(2, m/2 - 1)(\mathbb{Z}_{m-2}, B')$ of index $(0, 1)$ in Lemma 2.6 by adding a new symbol $m-2$ to each group; then choose P to be any parallel class in the resolvable GDD. In any case, let p be in no triple in P, and let $\{p, a, b\}$ be a triple in B. To form H, orient the edges in P to form directed cycles and add to them the directed edges $(p, a), (a, b), (b, p), (m-1, a), (m-1, b)$ and $(i, m-1)$ for each $i \in \mathbb{Z}_{m-1} \setminus \{a, b\}$. Then $T = B \setminus P \cup \{\{p, a, b\}\}$. □

Since $m = 3$ is an exception to Lemma 1.11, it will be necessary to obtain some GDDs directly. Before we construct these GDDs, we need some notation.

A *quasigroup* $(L, \circ)$ (or simply L) of order v on the symbols $\{1, \ldots, v\}$ is a $v \times v$ array in which each cell contains exactly one symbol, and each symbol occurs exactly once in each row of L, and exactly once in each column of L. The symbol in cell (i, j) of L is denoted by $i \circ j$. L is *idempotent* if $i \circ i = i$ for $1 \leq i \leq v$. L is *symmetric* if $i \circ j = j \circ i$ for all $1 \leq i, j \leq v$.

Lemma 2.9 *For all $v \geq 1$ there exists a quasigroup of order v. For all $v \neq 2$ there exists an idempotent quasigroup of order v. For all odd v there exists a symmetric idempotent quasigroup of order v. For all $v \geq 8$ there exists a quasigroup of order v that contains a subquasigroup of order 4.*

Proof: The last result follows from the embeddings of latin squares obtained by Evans [9]. The other results are easy to obtain and are well known. □

Lemma 2.10 ([20]) *Let $n \equiv 2 \pmod 6$ and $n > 2$. Let F be a pair of independent edges in K_n. Then there exists an edge-disjoint decomposition of $4K_n$ into copies of K_3 together with one 4-cycle, and into copies of K_3 together with one copy of $2F$.*

Lemma 2.11 *There exists a $GDD(n, 3)$ of index $(4, 1)$ for $n \equiv 2 \pmod 6$ and $n > 2$.*

Proof: For each $i \in \mathbb{Z}_2$, let $(\mathbb{Z}_n \times \{i\}, B_i)$ be an edge disjoint decomposition of $4K_n$ into copies of K_3 together with the 4-cycle $((0, i), (1, i), (2, i), (3, i))$ (see Lemma 2.10). Let $(\mathbb{Z}_n \times \{2\}, B_2)$ be an edge-disjoint decomposition of K_n into copies of K_3 and two copies of the edges $\{(0, 2), (2, 2)\}$ and $\{(1, 2), (3, 2)\}$ (see Lemma 2.10). Let $(\mathbb{Z}_n, \circ)$ be a quasigroup that contains a subquasigroup $(\mathbb{Z}_4, \circ)$ (see Lemma 2.9) of order 4. Let $B' = \{\{(i, 0), (j, 1), (i \circ j, 2)\} \mid i, j \in \mathbb{Z}_n, \{i, j\} \not\subseteq \mathbb{Z}_4\}$, and $B'' = \{\{(j, 0), (j, 1), (j, 2)\}$ $\{(j, 0), (j+1, 1), (j+2, 2)\}, \{(j, 0), (j+1, 0), (j+3, 1)\}, \{(j, 1), (j+1, 1), (j+3, 2)\}$ $\{(j, 2), (j+2, 2), (j+1, 0)\} \mid 0 \leq j \leq 3\}$. Finally, let $B = B_0 \cup B_1 \cup B_2 \cup B' \cup B''$. Then $(\mathbb{Z}_n \times \mathbb{Z}_3, B)$ is a $GDD(n, 3)$ of index $(4, 1)$. □

3 Constructing GDDs: $n \geq 3$ and $m \geq 3$

Throughout this section, we assume that $n \geq 3$ and $m \geq 3$. Clearly we still have many cases to consider! We will approach this by considering each of the possible congruence classes of λ_1 (modulo 6). In fact, the following observation allows us to simply consider the cases where $\lambda_1, \lambda_2 \in \{1, 2, 3, 4, 5, 6\}$.

Lemma 3.1 *Let n, $m \geq 3$. If conditions (1) and (2) of Lemma 1.3 are sufficient for the existence of a $GDD(n, m)$ of index (λ_1', λ_2') when $1 \leq \lambda_1'$, $\lambda_2' \leq 6$, then they are sufficient for all λ_1, $\lambda_2 \geq 1$.*

Proof: Let n, m, λ_1 and λ_2 satisfying conditions (1) and (2) of Lemma 1.3. Let $\lambda_1 = 6x + \lambda_1'$ and $\lambda_2 = 6y + \lambda_2'$, where $1 \leq \lambda_1', \lambda_2' \leq 6$. Then n, m, λ_1' and λ_2' also satisfy conditions (1) and (2) of Lemma 1.3, so by assumption there exists a $GDD(n, m)$ $(\mathbb{Z}_n \times \mathbb{Z}_m, \{\mathbb{Z}_n \times \{i\} \mid i \in \mathbb{Z}_m\}, B')$ of index (λ_1', λ_2'). By Theorem 1.1, for each $i \in \mathbb{Z}_m$ there exists a $TS(n)$ $(\mathbb{Z}_n \times \{i\}, B_i)$ of index $6x$. By Theorem 1.2 there exists a $GDD(n, m)$ $(\mathbb{Z}_n \times \mathbb{Z}_m, \{\mathbb{Z}_n \times \{i\} \mid i \in \mathbb{Z}_m\}, B'')$ of index $(0, 6y)$. Let $B = B' \cup B'' \cup (\cup_{i \in \mathbb{Z}_m} B_i)$. Then clearly $(\mathbb{Z}_n \times \mathbb{Z}_m, \{\mathbb{Z}_n \times \{i\} \mid i \in \mathbb{Z}_m\}, B)$ is a $GDD(n, m)$ of index (λ_1, λ_2).

Theorem 3.2 *Let n, $m \geq 3$ and $\lambda_1, \lambda_2 \geq 1$. There exists a $GDD(n, m)$ of index (λ_1, λ_2) iff*

(1) 2 divides $\lambda_1(n-1) + \lambda_2(m-1)n$, and

(2) 3 divides $\lambda_1 mn(n-1) + \lambda_2 m(m-1)n^2$.

λ_2 λ_1	0	1	2	3	4	5
0	(any;any)	(0;any) (1,5;1,3) (2,4;0,1,3,4) (3;odd)	(0,3;any) (1,2,4,5;0,1,3,4)	(even;any) (odd;odd)	(0,3;any) (1,2,4,5;0,1,3,4)	(0;any) (1,5;1,3) (2,4;0,1,3,4) (3;odd)
1	(1,3;any) (5;0,3)	(1;1,3) (3;odd) (5;3,5)	(1;0,1,3,4) (3;any) (5;0,3)	(1;odd) (3;odd) (5;3)	(1;0,1,3,4) (3;any) (5;0,2,3,5)	(1;1,3) (3;odd) (5;3)
2	(0,3;any) (1,4;any) (2,5;0,3)	(0;any) (1;1,3) (2;0,3) (3;odd) (4;0,1,3,4) (5;3)	(0,3;any) (1,4;0,1,3,4) (2,5;0,2,3,5)	(0,4;any) (1,3;odd) (2;0,3) (5;3)	(0,3;any) (1,4;0,1,3,4) (2,5;0,3)	(0;any) (1;1,3) (2;0,2,3,5) (3;odd) (4;0,1,3,4) (5;3,5)
3	(1;any) (3;any) (5;any)	(1;1,3) (3;odd) (5;1,3)	(1,5;0,1,3,4) (3;any)	(odd;odd)	(1,5;0,1,3,4) (3;any)	(1,5;1,3) (3;odd)
4	(0,1,3,4;any) (2,5;0,3)	(0;any) (1;1,3) (2;0,2,3,5) (3;odd) (4;0,1,3,4) (5;3,5)	(0,3;any) (1,4;0,1,3,4) (2,5;0,3)	(0,4;any) (1,3;odd) (2;0,3) (5;3)	(0,3;any) (1,4;0,1,3,4) (2,5;0,2,3,5)	(0;any) (1;1,3) (2;0,3) (3;odd) (4;0,1,3,4) (5;3)
5	(1,3;any) (5;0,3)	(1;1,3) (3;odd) (5;3)	(1;0,1,3,4) (3;any) (5;0,2,3,5)	(1,3;odd) (5;3)	(1;0,1,3,4) (3;any) (5;0,3)	(1;1,3) (3;odd) (5;3,5)

Table 1

Each cell (λ_a, λ_b) in this table lists $(n_1, \ldots, n_i; m_1, \ldots, m_j)$, where $n \equiv n_x \pmod 6$, $m \equiv m_y \pmod 6$, λ_a and λ_b satisfy the necessary conditions (1) and (2) of Lemma 1.3 for the existence of a $GDD(n, m)$ of index (λ_a, λ_b).

□

Proof: The necessity is proved in Lemma 1.3, so now we assume that (1) and (2) hold. To consider the various possibilities we take the congruence classes of $\lambda_1 \pmod 6$ in turn.

Case 1: $\lambda_1 = 1$. If $n \equiv 1$ or $3 \pmod 6$ then by Theorem 1.1 there exists a $TS(n)$ of index $\lambda = 1$, so by Lemma 2.1 there exists a $GDD(n, m)$ of index (λ_1, λ_2).

If $n \equiv 5 \pmod 6$ and $m \equiv 3$ or $5 \pmod 6$ then by Theorem 1.1 there exists a $TS(nm)$ of index λ_1, so by Lemma 2.2 the result follows.

If $n \equiv 5 \pmod 6$ and $m = 6$ then λ_2 is even (see Table 1). Let $T = \{\{j, j+1, j+3\} \mid j \in \mathbb{Z}_6\} \cup \{\{j, j+2, j+4\} \mid j \in \mathbb{Z}_2\}$, reducing the sums modulo 6. Each pair in $\mathbb{Z}_6$ occurs in two triples in T except for the pairs in $P = \{\{i, i+1 \pmod 6\} \mid i \in \mathbb{Z}_6\}$ which each occurs once. For each $x \in \mathbb{Z}_n$, let $(\{x\} \times \mathbb{Z}_6, B_x)$ be a $TS(6)$ of index 2. Let $(\mathbb{Z}_n, \circ)$ be an idempotent symmetric quasigroup (see Lemma 2.9). Let $(\mathbb{Z}_n \times \mathbb{Z}_6, \{\mathbb{Z}_n \times \{i\} \mid i \in \mathbb{Z}_6\}, B')$ be a $GDD\ (n, 6)$ of index $(0, \lambda_2 - 2)$ (see Theorem 1.2). Form a $GDD(n, m)$ of index $(1, \lambda_2)$ $(\mathbb{Z}_n \times \mathbb{Z}_6, \{\mathbb{Z}_n \times \{i\} \mid i \in \mathbb{Z}_6\}, B)$ as follows.

(i) For each $\{i, i+1\} \in P$, let $\{\{(x, i), (y, i), (x \circ y, i+1)\} \mid 0 \le x < y < n\} \in B$,

(ii) for each $\{a, b, c\} \in T$ with $a < b < c$, let $\{\{(x, a), (y, b), (x \circ y, c)\} \mid x, y \in \mathbb{Z}_n, x \neq y\}$,

(iii) for each $x \in \mathbb{Z}_n$ let $B_x \subseteq B$, and

(iv) let $B' \subseteq B$.

Then: each pair $\{(x, i), (y, i)\}$ is in one triple defined in (i); each pair $\{(x, i), (x, j)\}$ is in two triples defined in (iii) and in $\lambda_2 - 2$ triples defined in (iv); and each pair $\{(x, i), (y, j)\}$ with $x \neq y$ and $i \neq j$ is in $\lambda_2 - 2$ triples defined in (iv), is in two triples defined in (ii) if $\{i, j\} \notin P$, and is in one triple defined in (i) and one triple defined in (ii) if $\{i, j\} \in P$.

If $n \equiv 5 \pmod 6$ and $m \equiv 0 \pmod 6$ with $m > 6$ then λ_2 is even (see Table 1). There exists a $GDD(3, m/3)$ of index $(0,2)$ say $(\mathbb{Z}_m, G, B)$ by Theorem 1.2. Construct a $GDD(n,m)$ of index $(1,\lambda_2)$, say $(\mathbb{Z}_n \times \mathbb{Z}_m,$ $\{\mathbb{Z}_n \times \{i\} \mid i \in \mathbb{Z}_m\}$, $B^*)$ as follows.

(i) For each triple $A = \{a,b,c\}$ in G or B, let $(\mathbb{Z}_n \times A$, $\{\mathbb{Z}_n \times \{i\} \mid i \in A\}, B_A)$ be a $GDD(n,3)$ of index $(0,1)$ (see Theorem 1.2) and let $B_A \subseteq B^*$,

(ii) for each triple $A = \{a,b,c\}$ in G, let $(\mathbb{Z}_n \times A, B'_A)$ be a $TS(3n)$ of index 1 (see Theorem 1.1) and let $B'_A \subseteq B^*$, and

(iii) let $(\mathbb{Z}_n \times \mathbb{Z}_m$, $\{\mathbb{Z}_n \times \{i\} \mid i \in \mathbb{Z}_m\}, B')$ be a $GDD(n,m)$ of index $(0, \lambda_2 - 2)$ and let $B' \subseteq B^*$.

Then each pair $\{(x,i),(y,i)\}$ with $x \neq y$ occurs in exactly 1 triple defined in (ii). Also if $i \neq j$ then $\{(x,i),(y,j)\}$: occurs in $\lambda_2 - 2$ triples defined in (iii); occurs in 2 triples defined in (i) if i and j are in different groups in G; and occurs in 1 triple defined in (i) and 1 triple defined in (ii) if i and j are in the same group in G.

If $n \equiv 5 \pmod 6$ and $m \equiv 2 \pmod 6$ then $\lambda_2 = 4$ (see Table 1). Let $(\mathbb{Z}_m, G_1, B_1)$ be a $GDD(2, m/2)$ of index $(0,1)$ (see Theorem 1.2). By Lemma 2.7, there exists a decomposition of K_n defined on the vertex set $\mathbb{Z}_n$ into a set T of copies of K_3 together with two 2-factors; direct the edges in the two factors so that each component of each 2-factor becomes a directed cycle and call this directed graph H. Let $(\mathbb{Z}_m, \circ_m)$ and $(\mathbb{Z}_n, \circ_n)$ be idempotent quasigroups of order m and n respectively, and let $(\mathbb{Z}_n, \times)$ be a symmetric idempotent quasigroup of order n (see Lemma 2.9). Form a $GDD(n,m)$ of index $(1,4)$, say $(\mathbb{Z}_n \times \mathbb{Z}_m, \{\mathbb{Z}_n \times \{i\} \mid i \in \mathbb{Z}_m\}, B)$ as follows.

(i) For each $\{i,j\} \in G_1$, let $\{\{(x,i),(y,i),(x \times y, j)\}$, $\{(x,j),(y,j),(x \times y, i)\} \mid 0 \le x < y \le n-1\} \subseteq B$,

(ii) for each $\{i,j,k\} \in B_1$ with $i < j < k$, let each of the triples $\{(x,i),$ $(y,j),(x \circ_n y,k)\}$ with $x,\ y \in \mathbb{Z}_n$, $x \neq y$ be placed twice in B,

(iii) for each directed edge (x,y) in H, let $\{\{(x,i),(x,j),(y,i \circ_m j)\} \mid i,j \in \mathbb{Z}_m, i \neq j\} \subseteq B$, and

(iv) for each triple $\{x,y,z\} \in T$ with $x < y < z$, let each of the triples $\{(x,i),(y,j),(z,i \circ_m j)\}$ with $i,\ j \in \mathbb{Z}_m$, $i \neq j$ be placed twice in B.

Then each pair $\{(x,i),(y,i)\}$ occurs in one triple defined in (i), and each pair $\{(x,i),(x,j)\}$, occurs in 4 triples defined in (iii) (notice (iii) also includes the triple $\{(x,i),(x,j),(y,j \circ_m i)\}$). Also each pair $\{(x,i),(y,j)\}$ with $x \neq y$ and $i \neq j$ occurs in 2 triples in (i) if $\{i,j\} \in G_1$, 2 triples in (ii) if $\{i,j\} \notin G_1$, 2 triples in (iii) if (i,j) is a directed edge in H, and 2 triples in (iv) if (i,j) is not a directed edge in H.

Case 2: $\lambda_1 = 2$. If $n \equiv 0$ or $1 \pmod 3$ then by Theorem 1.1 there exists a $TS(n)$ of index $\lambda_1 = 2$, so by Lemma 2.1 the result follows.

If $n \equiv 2 \pmod 3$ and $\lambda_2 \in \{2,3,4,5,6\}$ then since $m \equiv 0$ or $2 \pmod 3$ (see Table 1) there exists by Theorem 1.1 a $TS(nm)$ of index $\lambda_1 = 2$. So the result follows by Lemma 2.2 since $\lambda_1 \leq \lambda_2$.

If $n \equiv 5 \pmod 6$ and $\lambda_2 = 1$, then $m \equiv 3 \pmod 6$ (see Table 1). By Lemma 2.7, there exists an edge disjoint decomposition of K_m, with vertex set $\mathbb{Z}_m$ into $m(m-5)/3$ copies of K_3 and two 2-factors F_1 and F_2, each component in each 2-factor being a K_3. Form a $GDD(n,m)$ of index $(2,1)$, $(\mathbb{Z}_n \times \mathbb{Z}_m, \{\mathbb{Z}_n \times \{i\} \mid i \in \mathbb{Z}_m\}, B)$ by defining B as follows.

(i) For each K_3 with vertex set A in F_1 or F_2, let $(\mathbb{Z}_n \times A, B_A)$ be a $TS(3n)$ of index 1 (see Theorem 1.1), and let $B_A \subseteq B$, and

(ii) for each remaining K_3 with vertex set $A = \{a_0,a_1,a_2\}$, let $(\mathbb{Z}_n \times A, \{\mathbb{Z}_n \times \{a_i\} \mid i \in \mathbb{Z}_3\}, B_A)$ be a $GDD(n,3)$ of index $(0,1)$ and let $B_A \subseteq B$.

Each pair $\{(x,a),(y,a)\}$ occurs in two triples defined in (i) because each two factor contains a triple containing a. Each pair $\{(x,a),(y,b)\}$ with $a \neq b$ occurs in one triple because the edge $\{a,b\}$ occurs in exactly one copy of K_3 (possibly in a 2-factor).

If $n \equiv 2 \pmod 6$ and $\lambda_2 = 1$ then $m \equiv 0 \pmod 3$ (see Table 1). Informally, there exists a set of triples that covers all the pairs once, except that one set of pairs forming a 1-factor on each group are covered twice (Lemma 2.5). There also exists a set of triples that covers each pair in a group once except for the edges forming a 1-factor of the group which are not covered at all (Theorem 1.2). Together we obtain the desired $GDD(n,m)$ of index $(2,1)$. More formally, by Lemma 2.5 there exists a $GDD(2,nm/2)$ of index $(2,1)$, say $(\mathbb{Z}_2 \times \mathbb{Z}_{n/2} \times \mathbb{Z}_m, \{\mathbb{Z}_2 \times \mathbb{Z}_i \times \mathbb{Z}_j \mid i \in \mathbb{Z}_{n/2}, j \in \mathbb{Z}_m\}, B)$. By Theorem 1.2, there exists a $GDD(2,n/2)$ of index $(0,1)$; for each $a \in \mathbb{Z}_m$ we define a copy of this $GDD(2,n/2)$ $(\mathbb{Z}_2 \times \mathbb{Z}_{n/2} \times \{a\}, \{\mathbb{Z}_2 \times \{i\} \times \{a\} \mid i \in \mathbb{Z}_{n/2}\}, B_a)$. Then we can define a $GDD(n,m)$ of index $(2,1)$ by $(\mathbb{Z}_2 \times \mathbb{Z}_{n/2} \times \mathbb{Z}_m, \{\mathbb{Z}_2 \times \mathbb{Z}_{n/2} \times \{a\} \mid a \in \mathbb{Z}_m\}, B \cup (\bigcup_{a \in \mathbb{Z}_m} B_a)\}$.

Case 3: $\lambda_1 = 3$. In every case n is odd. So by Theorem 1.1 there exists a $TS(n)$ of index 3, so the result follows from Lemma 2.1.

Case 4: $\lambda_1 = 4$. If $n \equiv 0$ or $1 \pmod 3$, then by Theorem 1.1 there exists a $TS(n)$ of index 4, so the result follows from Lemma 2.1.

If $n \equiv 2 \pmod 3$ and $\lambda_2 \in \{3,4,5,6\}$ then $nm \equiv 0$ or $1 \pmod 3$ (see Table 1) so there exists a $TS(nm)$ of index $\lambda = 2$. So the result follows from Lemma 2.4 and the case $\lambda_1 = 2$ considered earlier (see Table 1).

If $n \equiv 5 \pmod 6$ and $\lambda_2 \in \{1,2\}$ then there exists a $TS(n)$ of index $\lambda = 3$. So the result follows from Lemma 2.3 and the case $\lambda_1 = 1$ considered earlier (see Table 1).

If $n \equiv 2 \pmod 6$ and $\lambda_2 = 1$ then $m \equiv 0$ or $2 \pmod 3$ (see Table 1). So by Lemma 2.8 there exists a $GDD(2,m)$ of index $(4,1)$ except when $m = 3$.

Since $n \equiv 2 \pmod 6$, by Theorem 1.2 there exists a $GDD(2, n/2)$ of index $(0, 1)$, say $(\mathbb{Z}_n, G, B)$. Hence for $m > 3$ we can construct a $GDD(n, m)$ of index $(4, 1)$ $(\mathbb{Z}_n \times \mathbb{Z}_m, \{\mathbb{Z}_n \times \{i\} \mid i \in \mathbb{Z}_m\}, B^*)$, by defining B^* as follows.

(i) For each $\{a, b\} \in G$, let $(\{a, b\} \times \mathbb{Z}_m, \{\{a, b\} \times \{i\} \mid i \in \mathbb{Z}_m\}, B_{\{a,b\}})$ be a $GDD(2, m)$ of index $(4, 1)$ and let $B_{\{a,b\}} \subseteq B^*$,

(ii) for each $A = \{a_0, a_1, a_2\} \in B$, let $(A \times \mathbb{Z}_m, \{\{a_x\} \times \mathbb{Z}_m \mid x \in \mathbb{Z}_3\}, B_A)$ be a $GDD(m, 3)$ of index $(0, 1)$ (see Theorem 1.2) and let $B_A \subseteq B^*$, and

(iii) for each $i \in \mathbb{Z}_m$ and for each $\{a_0, a_1, a_2\} \in B$, let B^* contain 3 (more) copies of the triple $\{(a_0, i), (a_1, i), (a_2, i)\}$.

Each pair $\{(a, i), (b, i)\}$ occurs in 4 triples defined in (i) if $\{a, b\} \in G$, and occurs in 1 triple in (ii) and 3 triples in (iii) if $\{a, b\} \notin G$. Each pair $\{(a, i), (b, j)\}$ with $i \neq j$ occurs in 1 triple in (i) if $a = b$ or if $\{a, b\} \in G$, and occurs in 1 triple in (ii) if $a \neq b$ and $\{a, b\} \notin G$.

For $m = 3$, we have a $GDD(n, 3)$ of index $(4, 1)$ in Lemma 2.11.

If $n \equiv 2 \pmod 6$ and $\lambda_2 = 2$ then $m \equiv 0 \pmod 3$ (see Table 1), so by Theorem 1.2 there exists a $GDD(n, m)$ of index $(0, 1)$. Also, a $GDD(n, m)$ of index $(4, 1)$ was constructed earlier in this case, so together these two GDDs produce the required $GDD(n, m)$ of index $(4, 2)$.

Case 5: $\lambda_1 = 5$. If $n \equiv 1$ or $3 \pmod 6$ then there exists a $TS(n)$ of index $\lambda_1 = 5$, so the result follows from Lemma 2.1.

If $n \equiv 5 \pmod 6$ then there exists a $TS(n)$ of index 3 (see Theorem 1.1), and there exists a $GDD(n, m)$ of index $(\lambda_1 - 3, \lambda_2) = (2, 1)$ (see Table 1 and the case $\lambda_1 = 2$), so putting these together produces a $GDD(n, m)$ of index $(5, 1)$. □

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