Isomorphisms of Cyclic Abelian Covers of Symmetric Digraphs

Hirobumi Mizuno
Department of Computer Science and Information Mathematics
University of Electro-Communications
1-5-1, Chofugaoka, Chofu
Tokyo 182
Japan

Iwao Sato
Oyama National College of Technology
Oyama
Tochigi 323
Japan

ABSTRACT. Let D be a connected symmetric digraph, A a finite abelian group with some specified property and $g \in A$. We present a characterization for two g-cyclic A-covers of D to be isomorphic with respect to a group Γ of automorphisms of D, for any g of odd order. Furthermore, we consider the number of Γ-isomorphism classes of g-cycic A-covers of D for an element g of odd order. We enumerate the number of isomorphism classes of g-cyclic $\mathbf{Z}_{p^n}$-covers of D with respect to the trivial group of automorphisms of D, for any prime p (> 2), where $\mathbf{Z}_{p^n}$ is the cyclic group of order p^n. Finally, we count Γ-isomorphism classes of cyclic $\mathbf{F}_p$-covers of D.

1 Introduction

Graphs and digraphs treated here are finite and simple.

Let D be a symmetric digraph and A a finite group. A function $\alpha\colon A(D) \to A$ is called *alternating* if $\alpha(y,x) = \alpha(x,y)^{-1}$ for each $(x,y) \in A(D)$. For $g \in A$, a *g-cyclic A-cover* $D_g(\alpha)$ of D is the digraph as follows:

$V(D_g(\alpha)) = V(D) \times A$ and $((u,h),(v,k)) \in A(D_g(\alpha))$ if and only if $(u,v) \in A(D)$ and $k^{-1}h\alpha(u,v) = g$.

The *natural projection* $\pi\colon D_g(\alpha) \to D$ is a function from $V(D_g(\alpha))$ onto $V(D)$ which erases the second coordinates. A digraph D' is called a *cyclic A-cover* of D if D' is a g-cyclic A-cover of D for some $g \in A$. In the case that A is abelian, then $D_g(\alpha)$ is called simply a *cyclic abelian cover.*

Let α and β be two alternating functions from $A(D)$ into A, and let Γ be a subgroup of the automorphism group $\mathrm{Aut}D$ of D, denoted $\Gamma \leq \mathrm{Aut}D$. Let $g, h \in A$. Then two cyclic A-covers $D_g(\alpha)$ and $D_h(\beta)$ are called *Γ-isomorphic*, denoted $D_g(\alpha) \cong_\Gamma D_h(\beta)$, if there exist an isomorphism $\Phi\colon D_g(\alpha) \to D_h(\beta)$ and a $\gamma \in \Gamma$ such that $\pi\Phi = \gamma\pi$, i.e., the diagram

$$\begin{array}{ccc} D_g(\alpha) & \xrightarrow{\Phi} & D_h(\beta) \\ \pi\downarrow & & \downarrow\pi \\ D & \xrightarrow{\gamma} & D \end{array}$$

commutes. Let $I = \{1\}$ be the trivial group of antomorphisms.

Cheng and Wells [1] discussed isomorphism classes of cyclic triple covers (1-cyclic $\mathbb{Z}_3$-covers) of a complete symmetric digraph. Furthermore, Mizuno and Sato [13] gave a formula for the characteristic polynomial of a cyclic A-cover of a symmetric digraph, for any finite group A. Mizuno and Sato [12] discussed the number of Γ-isomorphism classes of cyclic V-covers of a connected symmetric digraph for any finite dimensional vector space V over the finite field $GF(p)$ ($p > 2$).

A graph H is called a *covering* of a graph G with projection $\pi\colon H \to G$ if there is a surjection $\pi\colon V(H) \to V(G)$ such that $\pi\mid_{N(v')}\colon N(v') \to N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. The projection $\pi\colon H \to G$ is an *n-fold covering* of G if π is n-to-one. A covering $\pi\colon H \to G$ is said to be *regular* if there is a subgroup B of the automorphism group $\mathrm{Aut}H$ of H acting freely on H such that the quotient graph H/B is isomorphic to G.

Let G be a graph and A a finite group. Let $D(G)$ be the arc set of the symmetric digraph corresponding to G. Then a mapping $\alpha\colon D(G) \to A$ is called an *ordinary voltage assignment* if $\alpha(v,u) = \alpha(u,v)^{-1}$ for each $(u,v) \in D(G)$. The *(ordinary) derived graph* G^α derived from an ordinary voltage assignment α is defined as follows:

$V(G^\alpha) = V(G) \times A$, and $((u,h),(v,k)) \in D(G^\alpha)$ if and only if $(u,v) \in D(G)$ and $k = h\alpha(u,v)$.

The graph G^α is called an *A-covering* of G. The A-covering G^α is an $|A|$-fold regular covering of G. Every regular covering of G is an A-covering of G for some group A (see [3]). Furthermore the 1-cyclic A-cover $D_1(\alpha)$ of a symmetric digraph D can be considered as the A-covering $\tilde{D}^\alpha$ of the underlying graph $\tilde{D}$ of D.

A general theory of graph coverings is developed in [4]. $\mathbb{Z}_2$-coverings (double coverings) of graphs were dealed in [5] and [14]. Hofmeister [6] and, independently, Kwak and Lee [11] enumerated the I-isomorphism classes of n-fold coverings of a graph, for any $n \in N$. Dresbach [2] obtained a formula for the number of strong isomorphism classes of regular coverings of graphs with voltages in finite fields. The I-isomorphism classes of regular coverings of graphs with voltages in finite dimensional vector spaces over finite fields were enumerated by Hofmeister [7]. Hong, Kwak and Lee [9] gave the number of I-isomorphism classes of $\mathbf{Z}_n$-coverings, $\mathbf{Z}_p \oplus \mathbf{Z}_p$-coverings and D_n-coverings, n odd, of graphs, respectively.

In Section 2, we give a necessary and sufficient condition for two cyclic A-covers of a connected symmetric digraph D to be Γ-isomorphic for any finite abelian group A with the isomorphism extension property. Let the order of an element g of A be odd, and h be any element of the orbit containing g under the automorphism group Aut$A$. Then we show that the number of Γ-isomorphism classes of g-cyclic A-covers of D is equal to that of Γ-isomorphism classes of h-cyclic A-covers of D. In Section 3, we consider Γ-isomorphism classes of all h-cyclic A-covers of D which h belongs to the orbit on A containing a fixed element with odd order. In Section 4, we enumerate the number of I-isomorphism classes of g-cyclic $\mathbf{Z}_{p^n}$-covers of D for any prime p (> 2). In Section 5, we count Γ-isomorphism classes of cyclic $\mathbf{F}_p$-covers of D.

2 Isomorphisms of cyclic abelian covers

Let D be a symmetric digraph and A a finite group. The group Γ of automorphisms of D acts on the set $C(D)$ of alternating functions from $A(D)$ into A as follows.

$$\alpha^{\gamma}(x,y) = \alpha(\gamma(x),\gamma(y)) \text{ for all } (x,y) \in A(D),$$

where $\alpha \in C(D)$ and $\gamma \in \Gamma$. Any voltage $g \in A$ determines a permutation $\rho(g)$ of the symmetric group S_A on A which is given by $\rho(g)(h) = hg$, $h \in A$.

Mizuno and Sato [12] gave a characterization for two cyclic A-covers of D to be Γ-isomorphic.

Theorem 1. (12, Theorem 3.1) *Let D be a symmetric digraph, A a finite group, $g, h \in A$, $\alpha, \beta \in C(D)$ and $\Gamma \leq AutD$. Then the following are equivalent:*

1. $D_g(\alpha) \cong_{\Gamma} D_h(\beta)$

2. *There exist a family $(\pi_u)_{u \in V(D)} \in S_A^{V(D)}$ and $\gamma \in \Gamma$ such that*

$$\rho(\beta^{\gamma}(u,v)h^{-1}) = \pi_v \rho(\alpha(u,v)g^{-1})\pi_u^{-1} \text{ for each } (u,v) \in A(D),$$

where the multiplication of permutations is carried out from right to left.

From now on, assume that D is connected and A is abelian. Let G be the underlying graph, T a spanning tree of G and w a root of T. For any $\alpha \in C(D)$ and any walk W in G, the *net α-voltage* of W, denoted $\alpha(W)$, is the sum of the voltages of the edges of W. Then the *T-voltages* α_T of α is defined as follows:

$$\alpha_T(u,v) = \alpha(P_u) + \alpha(u,v) - \alpha(P_v) \text{ for each } (u,v) \in D(G) = A(D),$$

where P_u and P_v denote the unique walk from w to u and v in T, respectively. For a function $f: C(D) \to A$, the *net f-values* $f(W)$ of any walk W is defined as the net α-voltage of W.

Corollary 1. *Let D be a connected symmetric digraph, G its underlying graph, T a spanning tree of G. and $\alpha \in C(D)$. Furthermore, let A be a finite abelian group and $g \in A$. Then*

$$D_g(\alpha) \cong_I D_g(\alpha_T).$$

Moreover, there exists a function $s: V(D) \to A$ such that

$$\alpha_T(u,v) = -s(v) + \alpha(u,v) + s(u) \text{ for each } (u,v) \in D(G) = A(D).$$

Proof: Let $s(v) = \rho(\alpha(P_v))$ for $v \in V(D)$. Then by Theorem 1, the result follows. □

For a function $f: C(D) \to A$, let $A_f = A_f(v)$ denote the subgroup of A generated by all net f-values of the closed walk based at $v \in V(D)$. Let $ord(g)$ be the order of $g \in A$.

Theorem 2. *Let D be a connected symmetric digraph, A a finite abelian group, $g, h \in A$ and $\alpha, \beta \in C(D)$. Furthermore, let G be the underlying graph of D, T a spanning tree of G and $\Gamma \leq AutG$. Assume that the orders of g and h are equal and odd. Then the following are equivalent:*

1. $D_g(\alpha) \cong_\Gamma D_h(\beta)$.
2. *There exist $\gamma \in \Gamma$ and an isomorphism $\sigma: A_{\alpha_T - g}(w) \to A_{\beta_\gamma T - h}(\gamma(w))$ such that*

$$\beta^\gamma_{\gamma T}(u,v) - h = \sigma(\alpha_T(u,v) - g) \text{ for each } (u,v) \in A(D),$$

where $(\alpha_T - g)(u,v) = \alpha_T(u,v) - g$, $(u,v) \in A(D)$ and $w \in V(D)$.

Proof: At first, suppose that $D_g(\alpha) \cong_\Gamma D_h(\beta)$. By Corollary 1, we have $D_g(\alpha_T) \cong_\Gamma D_g(\beta_T)$. By Theorem 1, there exist a family $(\pi_u)_{u \in V(D)} \in S_A^{V(D)}$ and $\gamma \in \Gamma$ such that

$$\rho(\beta^\gamma_{\gamma T}(u,v) - h) = \pi_v \rho(\alpha_T(u,v) - g)\pi_u^{-1} \text{ for each } (u,v) \in A(D).$$

Let $(u,v) \in D(T)$. Then we have $\beta^{\gamma}_{\gamma T}(u,v) = \alpha_T(u,v) = 0$. Thus $\rho(-h) = \pi_v \rho(-g)\pi_u^{-1}$. Since $(u,v) \in D(T)$ we have $\rho(-h) = \pi_u \rho(-g)\pi_v^{-1}$. Therefore it follows that

$$\pi_v = \rho(h)\pi_u\rho(-g) \text{ and } \pi_u = \rho(2h)\pi_u\rho(-2g),$$

i.e.,

$$\pi_v(k+g) = \pi_u(k) + h \text{ and } \pi_u(k+2g) = \pi_u(k) + 2h \text{ for } k \in A.$$

Furthermore, we have

$$\pi_u(k+2mg) = \pi_u(k) + 2mh, \; m \in \mathbf{Z}.$$

Since $ord(g) = ord(h)$ is odd, there exists $t \in \mathbf{Z}$ such that $2tg = g$ and $2th = h$. Then we have

$$\begin{aligned} \pi_u^{-1}\pi_v(k+g) &= \pi_u^{-1}(\pi_u(k)+h) \\ &= \pi_u^{-1}(\pi_u(k)+2th) \\ &= \pi_u^{-1}(\pi_u(k+2tg) \\ &= k+2tg = k+g \text{ for any } k \in A, \end{aligned}$$

i.e.,

$$\pi_u^{-1}\pi_v = 1.$$

Thus $\pi_u = \pi_v$. Since D is connected, π is constant. Set $\pi_u = \zeta \in S_A$ for each $u \in V(D)$. Then we have $\rho(\beta^{\gamma}_{\gamma T}(u,v) - h) = \zeta\rho(\alpha(u,v) - g)\zeta^{-1}$ for each $(u,v) \in A(D)$. Hence there exists an isomorphism $\sigma\colon A_{\alpha T-g}(w) \to A_{\beta_\gamma T-h}(\gamma(w))$ such that

$$\beta^{\gamma}_{\gamma T}(u,v) - h = \sigma(\alpha_T(u,v) - g) \text{ for each } (u,v) \in A(D).$$

Conversely, assume that there exist $\gamma \in \Gamma$ and a group isomorphism $\sigma\colon A_{\alpha T-g}(w) \to A_{\beta_\gamma T-h}(\gamma(w))$ such that

$$\beta^{\gamma}_{\gamma T}(u,v) - h = \sigma(\alpha_T(u,v) - g) \text{ for each } (u,v) \in A(D).$$

Let $\{a_1 = 0, \ldots, a_m\}$ and $\{b_1 = 0, \ldots, b_m\}$ be the representatives of $A/A_{\alpha T-g}$ and $A/A_{\beta^{\gamma}_{\gamma T-h}}$, respectively. For any $c \in A$, there exist $c_\alpha \in A_{\alpha T-g}$ and $i(c) \in \{1, \ldots, m\}$ such that

$$c = c_\alpha + a_{i(c)}.$$

For each $v \in V(D)$, we define a mapping $\pi_v\colon A \to A$ by

$$\pi_v(c) = \sigma(c_\alpha) + b_{i(c)} \text{ for each } c \in A.$$

Then π_v is well-defined, and π_v is bijective.

Let $\pi = (\pi_v)_{v \in V(D)} \in S_A^{V(D)}$. Then π is constant and $\pi_v \mid A_{\alpha_{T-g}} = \sigma$ for each $v \in V(D)$. For any $c' \in A_{\alpha T-g}$, we have

$$\begin{aligned} c' + c &= (c' + c)_\alpha + a_{i(c'+c)} \\ &= c' + c_\alpha + a_{i(c)}, \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \pi_v(c' + c) &= \pi_v(c' + c_\alpha + a_{i(c)}) \\ &= \sigma(c' + c_\alpha) + b_{i(c)} \\ &= \sigma(c') + \sigma(c_\alpha) + b_{i(c)} \\ &= \sigma(c') + \pi_v(c) \end{aligned}$$

for each $v \in V(D)$ and $c \in A$. Thus

$$\begin{aligned} (\pi_v \rho(\alpha_T(u,v) - g)\pi_u^{-1})(c) &= \pi_v(\pi_u^{-1}(c) + \alpha_T(u,v) - g) \\ &= \sigma(\alpha_T(u,v) - g) + c \\ &= \rho(\sigma(\alpha_T(u,v) - g))(c) \\ &= \rho(\beta_{\gamma T}^{\gamma}(u,v) - h)(c) \end{aligned}$$

for each $(u, v) \in A(D)$ and $c \in A$. That is, we have

$$\rho(\beta_{\gamma T}^{\gamma}(u,v) - h) = \pi_v \rho(\alpha_T(u,v) - g)\pi_u^{-1} \text{ for each } (u,v) \in A(D).$$

Therefore it follows that $D_g(\alpha_T) \cong_\Gamma D_g(\beta_T)$, which completes the proof. □

Let D be a connected symmetric digraph, G its underlying graph and A a finite abelian group. The set of ordinary voltage assignments of G with voltages in A is denoted by $C^1(G; A)$. Note that $C(D) = C^1(G; A)$. Furthermore, let $C^0(G; A)$ be the set of functions from $V(G)$ into A. We consider $C^0(G; A)$ and $C^1(G; A)$ as additive groups. The homomorphism $\delta: C^0(G; A) \to C^1(G; A)$ is defined by $(\delta s)(x, y) = s(x) - s(y)$ for $s \in C^0(G; A)$ and $(x, y) \in A(D)$. For each $\alpha \in C^1(G; A)$, let $[\alpha]$ be the element of $C^1(G; A)/Im\delta$ which contains α.

The automorphism group Aut$A$ acts on $C^0(G; A)$ and $C^1(G; A)$ as follows:

$$(\sigma s)(x) = \sigma(s(x)) \text{ for } x \in V(D),$$

$$(\sigma \alpha)(x, y) = \sigma(\alpha(x, y)) \text{ for } (x, y) \in A(D),$$

where $s \in C^0(G; A)$, $\alpha \in C^1(G; A)$ and $\sigma \in$ Aut$A$. An finite group $\mathcal{B}$ is said to have the *isomorphism extension property (IEP)*, if every isomorphism between any two isomorphic subgroups $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{B}$ can be extented to an automorphism of $\mathcal{B}$ (see [9]). For example, the cyclic group $\mathbf{Z}_n$ for any

$n \in N$, the dihedral group D_n for odd $n \geq 3$, and the direct sum of m copies of $\mathbf{Z}_p$ have the IEP.

Corollary 2. *Let D be a connected symmetric digraph, G its underlying graph,. A a finite abelian group with the IEP, $\alpha, \beta \in C(D)$, $g, h \in A$ and $\Gamma \leq AutD$. Suppose that the orders of g and h are equal and odd. Then the following are equivalent:*

1. $D_g(\alpha) \cong_\Gamma D_h(\beta)$.

2. *There exist $\sigma \in AutA$, $\gamma \in \Gamma$ and $s \in C^0(G; A)$ such that*

$$\beta = \sigma\alpha^\gamma + \delta s \text{ and } \sigma(g) = h.$$

Proof: By Theorem 2 and the definition of T-voltages, $D_g(\alpha) \cong_\Gamma D_h(\beta)$ if and only if there exist $\sigma \in \mathrm{Aut}A$, $\gamma \in \Gamma$ and $s \in C^0(G; A)$ such that

$$\beta(u, v) - h = \sigma(\alpha^\gamma(u, v) - g) + (\delta s)(u, v) \text{ for each } (u, v) \in A(D).$$

But we have $\beta(v, u) - h = \sigma(\alpha^\gamma(v, u) - g) + (\delta s)(v, u)$. Thus we have $-2h = -2\sigma(g)$. Since $ord(g) = ord(h)$ is odd, it follows that $\sigma(g) = h$. □

We state a characterization for two g-cyclic A-covers to be Γ-isomorphic, for any finite abelian group A with the IEP, and any $g \in A$ of odd order.

Corollary 3. *Let D be a connected symmetric digraph, G its underlying graph, A a finite abelian group, $\alpha, \beta \in C(D)$, $g \in A$ and $\Gamma \leq AutD$. Suppose that $ord(g)$ is odd and A has the IEP. Then the following result holds: $D_g(\alpha) \cong_\Gamma D_g(\beta)$ if and only if $\beta = \sigma\alpha^\gamma + \delta s$ and $\sigma(g) = g$ for some $\sigma \in AutA$, some $\gamma \in \Gamma$ and some $s \in C^0(G; A)$.*

Now we consider the number of Γ-isomorphism classes of cyclic A-covers of a connected symmetric digraph D. Let G be the underlying graph of D, A a finite abelian group with the IEP and $\Pi = \mathrm{Aut}A$. For any $g \in A$, set

$$\Pi_g = \{\sigma \in \Pi \mid \sigma(g) = g\}.$$

Then Π_g is a subgroup of Π.

Let $\Gamma \leq \mathrm{Aut}D$. Set $H^1(G; A) = C^1(G; A)/Im\delta$. Let $\Pi_g \times \Gamma$ act on $H^1(G; A)$ as follows:

$$(\sigma, \gamma)[\alpha] = [\sigma\alpha^\gamma] = \{\sigma\alpha^\gamma + \delta s \mid s \in C^0(G; A)\},$$

where $\sigma \in \Pi_g$, $\gamma \in \Gamma$ and $\alpha \in C^1(G; A)$. By Corollary 3, the number of Γ-isomorphism classes of g-cyclic A covers of D is equal to that of $\Pi_g \times \Gamma$-orbits on $H^1(G; A)$ for any $g \in A$ of odd order. Let $isc(D, A, g, \Gamma)$ be the number of Γ-isomorphism classes of g-cyclic A-covers of D.

Theorem 3. *Let D be a connected symmetric digraph, G its underlying graph, A a finite abelian group with the IEP, $g, h \in A$ and $\Gamma \leq AutD$. Assume that the orders of g and h are odd, and $\phi(g) = h$ for some $\phi \in \Pi = AutA$. Then*

$$isc(D, A, g, \Gamma) = isc(D, A, h, \Gamma).$$

Proof: By the preceding remark and Burnside's Lemma, we have

$$isc(D, A, g, \Gamma) = \frac{1}{|\Pi_g| \cdot |\Gamma|} \sum_{(\sigma,\gamma)\in\Pi_g\times\Gamma} |H^1(G; A)^{(\sigma,\gamma)}|,$$

where $U^{(\sigma,\gamma)}$ is the set consisting of the elements of U fixed by (σ, γ).

Let ϕ be an automorphism of A such that $\phi(g) = h$. Then we have $\Pi_h = \phi\Pi_g\phi^{-1}$. Let $(\sigma, \gamma) \in \Pi_g \times \Gamma$. Then $[\alpha] \in H^1(G; A)^{(\sigma,\gamma)}$ if and only if $\sigma\alpha^\gamma = \alpha + \delta s$ for some $s \in C^0(G; A)$. But, $\sigma\alpha^\gamma = \alpha + \delta s$ if and only if

$$\phi\sigma\phi^{-1}(\phi\alpha^\gamma) = \phi\alpha + \delta(\phi s).$$

Note that $\phi\sigma\phi^{-1} \in \Pi_h$ and $(\phi\alpha)^\gamma = \phi\alpha^\gamma$. Thus we have

$$|H^1(G; A)^{(\sigma,\gamma)}| = |H^1(G; A)^{(\phi\sigma\phi^{-1},\gamma)}| \text{ for each } (\sigma, \gamma) \in \Pi_g \times \Gamma.$$

Furthermore, we have $|\Pi_g| = |\Pi_h|$. Therefore it follows that

$$isc(D, A, g, \Gamma) = isc(D, A, h, \Gamma).$$

□

Let D be a connected symmetric digraph, p (> 2) prime and $\mathbf{F}_p = GF(p)$ the finite field with p elements. Let $\mathbf{F}_p^\tau$ be the τ-dimensional vector space over $\mathbf{F}_p$. Then the additive group $\mathbf{F}_p^\tau$ has the IEP and the general linear group $GL_\tau(\mathbf{F}_p)$ is the automorphism group of $\mathbf{F}_p^\tau$. Furthermore, $GL_\tau(\mathbf{F}_p)$ acts transitively on $\mathbf{F}_p^\tau \setminus \{0\}$.

Corollary 4. *Let D be a connected symmetric digraph, p (> 2) prime and $\Gamma \leq AutD$. Let g, h be any two elements of $\mathbf{F}_p^\tau \setminus \{0\}$. Then*

$$isc(D, \mathbf{F}_p^\tau, g, \Gamma) = isc(D, \mathbf{F}_p^\tau, h, \Gamma).$$

3 Isomorphisms of orbit-cyclic abelian covers

Let D be a connected symmetric digraph, A a finite abelian group with the IEP and $\Pi = \mathrm{Aut}A$. For an element g of A with odd order, the Π-orbit on A containing g is denoted by $\Pi(g)$. A cyclic A-cover $D_h(\alpha)$ of D is called $\Pi(g)$-*cyclic* if $h \in \Pi(g)$. Let $\mathcal{D}_k$ be the set of all k-cyclic A-covers of D for any $k \in A$, and let $\mathcal{D} = \bigcup_{h\in\Pi(g)} \mathcal{D}_h$. Then $\mathcal{D}$ is the set of all $\Pi(g)$-cyclic A-covers of D. Let $\mathcal{D}/\cong_\Gamma$ and $\mathcal{D}_h/\cong_\Gamma$ be the set of all Γ-isomorphism classes over $\mathcal{D}$ and $\mathcal{D}_h$, respectively. The Γ-isomorphism class of $\mathcal{D}_h$ containing $D_h(\alpha)$ is denoted by $[D_h(\alpha)]$.

Theorem 4. *Let D be a connected symmetric digraph, A a finite abelian group with the IEP, $\Gamma \leq AutD$ and $\Pi = AutA$. Furthermore, let g be an element of A with odd order. Then*

$$|\mathcal{D}/\cong_\Gamma| = isc(D, A, h, \Gamma) \text{ for each } h \in \Pi(g).$$

Proof: Let $g \in A$ have odd order. For any $h \neq g \in \Pi(g)$ and any $\tau \in C(D)$, let

$$\beta = \sigma^{-1}(\tau^\gamma + \delta s),\ \gamma \in \Gamma,\ s \in C^0(G; A),$$

where σ is an automorphism of A such that $\sigma(g) = h$, and G is the underlying graph of D. By Corollary 2, we have $D_g(\beta) \cong_\Gamma D_h(\tau)$.

For each $h \neq g \in \Pi(g)$, we define a map $\Phi_h \colon \mathcal{D}_g/\cong_\Gamma \to \mathcal{D}_h/\cong_\Gamma$ by

$$\Phi_h([D_g(\beta)]) = [D_h(\tau)],$$

where $D_g(\beta) \cong_\Gamma D_h(\tau)$. Since $\cong_\Gamma$ is an eqnivalence relation over $\mathcal{D}$, Φ_h is injective. By Theorem 3, we have

$$|\mathcal{D}_g/\cong_\Gamma| = |\mathcal{D}_h/\cong_\Gamma| < \infty.$$

Thus Φ_h is a bijection. Therefore it follows that

$$|\mathcal{D}/\cong_\Gamma| = isc(D, A, h, \Gamma).$$

□

Let $A = \mathbf{F}_p^\tau$. Then a cyclic $\mathbf{F}_p^\tau$-covers $D_g(\alpha)$ is called *nonzero-cyclic* if g is not equal to the unit 0 of $\mathbf{F}_p^\tau$. The set $\mathcal{D}$ is the set of all nonzero-cyclic $\mathbf{F}_p^\tau$-covers.

Corollary 5. *Let D be a connected symmetric digraph, p (> 2) prime and $\Gamma \leq AutD$. Furthermore, let g be any element of $\mathbf{F}_p^\tau \setminus \{0\}$. Then*

$$|\mathcal{D}/\cong_\Gamma| = isc(D, \mathbf{F}_p^\tau, g, \Gamma).$$

Now, we state the structure of Γ-isomorphism classes of $\Pi(g)$-cyclic A-covers of D.

Theorem 5. *Let D be a connected symmetric digraph, A a finite abelian group with the IEP, $\Gamma \leq AutD$ and $\Pi = AutA$. Suppose that $g \in A$ has odd order. Let σ_h be a fixed automorphism of A such that $\sigma_h(g) = h$ for*

$h \in \Pi(g)$. *Then any* Γ*-isomorphism class of* $\Pi(g)$*-cyclic* A*-covers of* D *is of the form*

$$\bigcup_{h \in \Pi(g)} \{D_h(\sigma_h \beta) \mid \beta = \sigma\alpha^\gamma + \delta s, \sigma \in \Pi_g, \gamma \in \Gamma, s \in C^0(G; A)\},$$

where $\alpha \in C(D)$ *and* G *is the underlying graph of* D.

Proof: Let $\tau \in C(D)$, $h \neq g \in \Pi(g)$ and $[[D_h(\tau)]]$ the Γ-isomorphism class of $\mathcal{D}$ containing $D_h(\tau)$. By the first half of the proof of Theorem 4, there exists a g-cyclic A-cover $D_g(\alpha)$ such that $[[D_h(\tau)]] = [[D_g(\alpha)]]$.

In the proof of Theorem 4, the map Φ_h is a bijection from $\mathcal{D}_g/\cong_\Gamma$ into $\mathcal{D}_h/\cong_\Gamma$ for any $h \neq g \in \Pi(g)$. Thus there exists a h-cyclic A-cover $D_h(\beta)$ such that $D_g(\alpha) \cong_\Gamma D_h(\beta)$ for any $h \neq g \in \Pi(g)$. We define a map $\Psi_h : [D_g(\alpha)] \to [D_h(\beta)]$ by

$$\Psi_h(D_g(\alpha')) = D_h(\sigma_h \alpha'), \ \alpha' = \sigma\alpha^\gamma + \delta s,$$

where $\sigma \in \Pi(g)$, $\gamma \in \Gamma$, $s \in C^0(G; A)$ and $\sigma_h(g) = h$. It is clear that Ψ_h is injective.

Now, let $D_h(\eta)$ be any element of $[D_h(\beta)]$. Then we have $D_h(\eta) \cong_\Gamma D_g(\alpha)$. By Corollary 2, there exist $\sigma' \in \Pi$, $\nu \in \Gamma$ and $t \in C^0(G; A)$ such that $\eta = \sigma'\alpha^\nu + \delta t$ and $\sigma'(g) = h$. Let $\mu = \sigma_h^{-1}\sigma'\alpha^\nu + \delta(\sigma_h^{-1}t)$. Then we have

$$\sigma_h^{-1}\sigma' \in \Pi_g \text{ and } \Psi_h(D_g(\mu)) = D_h(\eta).$$

Therefore Ψ_h is surjective, i.e., bijective. Hence it follows that

$$[D_h(\beta)] = \{D_h(\sigma_h\alpha') \mid \alpha' = \sigma\alpha^\gamma + \delta s, \sigma \in \Pi_g, \gamma \in \Gamma, s \in C^0(G; A)\},$$

and so the result follows. □

Corollary 6. *Let* D *be a connected symmetric digragh,* G *its underlying graph,* A *a finite abelian group with the IEP, and* $g \in A$ *have odd order. Then any* I*-isomorphism class of* $\Pi(g)$*-cyclic* A*-covers of* D *is of the form*

$$\bigcup_{h \in \Pi(g)} \{D_h(\sigma_h \beta) \mid \beta = \sigma\alpha + \delta s, \sigma \in \Pi_g, s \in C^0(G; A)\},$$

where $\alpha \in C(D)$.

Corollary 7. *Let* D *be a connected symmetric digraph,* p (> 2) *prime and* $\Gamma \leq AutD$. *Set* $e = (10\ldots0)^t \in \mathbf{F}_p^\tau$. *Furthermore,* $\mathbf{A}_g$ *be a fixed element of* $GL_\tau(\mathbf{F}_p)$ *such that* $\mathbf{A}_g e = g$ *for each* $g \neq 0 \in \mathbf{F}_p^\tau$. *Then any* Γ*-isomorphism class of nonzero-cyclic* $\mathbf{F}_p^\tau$*-covers of* D *is of the form*

$$\bigcup_{g \neq 0} \{D_g(\mathbf{A}_g\beta) \mid \beta = \mathbf{B}\alpha^\gamma + \delta s, \ \mathbf{B} \in \Pi_e, \ \gamma \in \Gamma, \ s \in C^0(G; \mathbf{F}_p^\tau)\},$$

where $\alpha \in C(D)$.

4 Isomorphisms of cyclic $\mathbf{Z}_n$-covers

Let $\mathbf{Z}_n$ be the cyclic group of order n. Then $\mathbf{Z}_n$ has the IEP.

Let D be a connected symmetric digraph, G its underlying graph and T a spanning tree of G. Then, set $C_T(D) = C_T^1(G;\mathbf{Z}_n) = \{\alpha_T \mid \alpha \in C(D) = C^1(G;\mathbf{Z}_n)\}$.

Lemma 1. *Let D be a connected symmetric digraph, G the underlying graph of D, T a spanning tree of G, n odd, $\alpha, \beta \in C(D)$ and $g \in \mathbf{Z}_n$. then the following are equivalent.*

1. $D_g(\alpha) \cong_I D_g(\beta)$.
2. *There exists a $\sigma \in \mathrm{Aut}\mathbf{Z}_n$ such that*

$$\beta_T = \sigma\alpha_T \text{ and } \sigma(g) = g.$$

Proof: By Corollaries 1,3. □

We shall consider the number of I-isomorphism classes of g-cyclic $\mathbf{Z}_{p^m}$-covers of D, for any $g \in \mathbf{Z}_{p^m}$. Set $\Pi_g = \{\sigma \in \mathrm{Aut}\mathbf{Z}_{p^m} \mid \sigma(g) = g\}$. By Lemma 1, the number of I-isomorphism classes of g-cyclic $\mathbf{Z}_{p^m}$-covers of D is equal to that of Π_g-orbits on $C_T^1(G;\mathbf{Z}_{p^m})$. Let $B(D) = m - n + 1$ be the Betti-number of D, where $m = |A(D)|/2$ and $n = |V(D)|$.

Theorem 6. *Let D be a connected symmetric digraph and $n = p^m$ ($p > 2$: prime). Let $g \in \mathbf{Z}_{p^m}$ and $ord(g) = p^{m-\mu}$ the order of g. Set $B = B(D)$. Then the number of I-isomorphism classes of g-cyclic $\mathbf{Z}_{p^m}$-covers of D is*

$$isc(D,\mathbf{Z}_{p^m},g,I) =$$
$$\begin{cases} p^{mB-\mu} + p^{(m-\mu)B-1}(p-1)(p^{\mu(B-1)}-1)/(p^{B-1}-1) & \text{if } \mu \neq m \text{ and } B > 1, \\ p^{m-\mu-1}\{(\mu+1)p-\mu\} & \text{if } \mu \neq m \text{ and } B = 1, \\ (p^{m(B-1)+1}-1)/(p-1) + (p^{m(B-1)}-1)/(p^{B-1}-1) & \text{if } \mu = m \text{ and } B > 1, \\ m+1 & \text{otherwise,} \end{cases}$$

Proof: In the case of $\mu = m$, g-cyclic $\mathbf{Z}_{p^m}$-covers of D are $\mathbf{Z}_{p^m}$-coverings of the underlying graph of D, and so the result is given in Theorem 7 of [9].

Suppose that $\mu < m$. By the above note and Burnside's Lemma, we have

$$isc(D,\mathbf{Z}_{p^m},g,I) = \frac{1}{|\Pi_g|}\sum_{\sigma\in\Pi_g}|C_T(D)^\sigma|,$$

where U^σ is the set consisting of the elements of U fixed by σ. Let $F(\sigma) = \{h \in \mathbf{Z}_{p^m} \mid \sigma(h) = h\}$. Then, by Corollary 3 of [9], we have $|C_T(D)^\sigma| = |F(\sigma)|^{B(D)}$.

But we have

$$\Pi_g = \{\lambda \in \mathbf{Z}_{p^m} \mid (\lambda, p^m) = 1 \text{ and } \lambda g = g\}.$$

Then

$$\lambda \in \Pi_g \leftrightarrow \lambda g \equiv g \pmod{p^m} \leftrightarrow g(\lambda - 1) \equiv 0 \pmod{p^m} \leftrightarrow \lambda - 1 \in \langle ord(g) \rangle.$$

Thus we have $|\Pi_g| = n/ord(g)$. That is, $|\Pi_g| = p^\mu$ if $g \in K_\mu(m)$, where $K_\mu(m) = \{k \in \mathbf{Z}_{p^m} \mid k \in \langle p^\mu \rangle, k \notin \langle p^{\mu+1} \rangle\}$. If $ord(g) = 1$, then $g = p^m$. Otherwise $\Pi_g = \{p^{m-\mu}\nu + 1 \mid \nu = 0, 1, \ldots, p^\mu - 1\}$.

By Lemma 3 of [9], $|F(\sigma)| = p^\mu$ if $\sigma - 1 \in K_\mu(m)$. Thus we have

$$|\{\lambda \in \Pi_g \mid |F(\lambda)| = p^{m-\mu+t}\}| = p^{\mu-t-1}(p-1)(0 \le t \le \mu - 1),$$
$$|\{\lambda \in \Pi_g \mid |F(\lambda)| = p^m\}| = 1.$$

Therefore the result follows. □

In Table 1, we give some values of $isc(D, \mathbf{Z}_{3^b}, g, I)$ ($\mu < 6$).

$\mu \backslash B$	1	2	3	4	5
1	405	216513	138706101	9647701761	69195236437845
2	189	76545	46589661	32184598401	23067403335549
3	81	26001	15543009	10728553761	7689144011121
4	33	8721	5181489	3576188961	2563048043073
5	13	2913	1727181	1192063041	854349347853

Table 1

5 Isomorphisms of cyclic $\mathbf{F}_p$-covers

Let p (> 2) be prime. Then $\mathbf{F}_p$ has the IEP.

Let D be a connected symmetric digraph, G its underlying graph, $g \in \mathbf{F}_p$ and $\Gamma \le \mathrm{Aut}D$.

Let $\gamma \in \Gamma$. A $\langle\gamma\rangle$-orbit σ of length k on $E(G)$ is called *diagonal* if $\sigma = \langle\gamma\rangle\{x, \gamma^k(x)\}$ for some $x \in V(G)$.

For $\gamma \in \Gamma$, let G_γ be the graph whose vertices are the $\langle\gamma\rangle$-orbits on $V(G)$, with two vertices adjacent in G_γ if and only if some two of their representatives are in G. The *kth p-level* of G_γ is the induced subgraph of G_γ on the vertices ω such that $\theta_p(|\omega|) = p^k$, where $\theta_p(i)$ is the largest power of p dividing i, A *p-level component* H of G_γ is a connected component of some p-level of G_γ, where H is considered as a subset of $V(G_\gamma)$. A p-level component H is called *minimal* if there exists no vertex σ of H which is adjacent to a vertex ω such that $\theta_p(|\sigma|) > \theta_p(|\omega|)$ (see [8, 15])

Theorem 7. *Let D be a connected symmetric digraph, G its underlying graph, p (> 2) prime, $g \in \mathbf{F}_p \setminus \{0\}$ and $\Gamma \leq AutD$. For $\gamma \in \Gamma$, let $\epsilon(\gamma)$ and $\nu(\gamma)$ be the number of $\langle\gamma\rangle$-orbits on $E(G)$ and $V(G)$, respectively. Furthermore, let $\xi(\gamma)$ and $\rho(\gamma)$ be the number of minimal p-level components on G_γ, and diagonal $\langle\gamma\rangle$-orbits on $E(G)$, respectively. Then the number of Γ-isomorphism classes of g-cyclic $\mathbf{F}_p$-covers of D is*

$$isc(D, \mathbf{F}_p, g, \Gamma) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} p^{\epsilon(\gamma)-\nu(\gamma)+\xi(\gamma)-\rho(\gamma)}.$$

Proof: Let G be the underlying graph and $\Pi = \mathrm{Aut}\mathbf{F}_p$. For $g \in \mathbf{F}_p \setminus \{0\}$, let $\Pi_g = \{\sigma \in \Pi \mid \sigma(g) = g\}$. By the remark before Theorem 3 and Burnside's Lemma, we have

$$isc(D, \mathbf{F}_p, g, \Gamma) = \frac{1}{|\Pi_g| \cdot |\Gamma|} \sum_{(\sigma,\gamma)\in\Pi_g\times\Gamma} |H^1(G; \mathbf{F}_p)^{(\sigma,\gamma)}|.$$

But we have

$$\Pi_g = \{\lambda \in \mathbf{F}_p \mid (\lambda, p) = 1 \text{ and } \lambda g = g\}.$$

Then

$$\lambda \in \Pi_g \leftrightarrow \lambda g \equiv g \pmod p \leftrightarrow \lambda \equiv 1 \pmod p.$$

Thus we have $\Pi_g = \{1\}$. Therefore it follows that

$$isc(D, \mathbf{F}_p, g, \Gamma) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} |H^1(G; \mathbf{F}_p)^\gamma|,$$

where $|H^1(G; \mathbf{F}_p)^\gamma| = \{[\alpha] \in H^1(G; \mathbf{F}_p) \mid \alpha^\gamma = \alpha + \delta s, s \in C^0(G; \mathbf{F}_p)\}$.

Theorem 5 of [8] implies that

$$|H^1(G; \mathbf{F}_p)^\gamma| = p^{\epsilon(\gamma)-\nu(\gamma)+\xi(\gamma)-\rho(\gamma)}.$$

□

The element of $H^1(G; \mathbf{F}_p)$ is called *switching equivalence classes.*

Corollary 8. *Let D be a connected symmetric digraph, G its underlying graph, p (> 2) prime, $g \in \mathbf{F}_p \setminus \{0\}$ and $\Gamma = AutD$. Then the number of Γ-isomorphism classes of g-cyclic $\mathbf{F}_p$-covers of D is equal to that of nonisomorphic switching equivalence classes of G.*

Proof: By Theorem 5 of [8]. □

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