k-equitable labelings of complete bipartite and multipartite graphs

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Abstract

Let $G = G(V, E)$ be a graph. A labeling of G is a function $f : V \longrightarrow \{0, 1, ..., n\}$ such that for each edge $e = (u, v) \in E$, $f(e) = |f(u) - f(v)|$. Such a labeling is said to be $k-$ *equitable* if it is a labeling of the vertices with the numbers 0 through $k-1$ such that, if v^i is the number of vertices labeled i, and e^i is the number of edges labeled i, then $\left|v^i - v^j\right| \leq 1$ and $\left|e^i - e^j\right| \leq 1$ for all i, j. A graph is said to be k-equitable if it has a k-equitable labeling. In this paper we characterize the k-equitability of complete bipartite graphs and consider the equitability of complete multipartite graphs.

1 Introduction:

The concept of graph labelings was first introduced by Rosa [Ro] in 1967 as a way of decomposing a complete graph into isomorphic subgraphs. Since then, the field has expanded greatly, and there are now many different types of graph labelings being studied.

In 1990, Cahit [C2] proposed distributing the vertex and edge labels of a labeling as evenly as possible, calling such a labeling a k-equitable labeling. k-equitability, as defined by Cahit, has been applied to several classes of graphs. Cahit [C1] has proved various results concering 3-equitability of graphs, including wheels, friendship graphs, and Eulerian graphs. He conjectures that all trees are k-equitable [C3]. Szaniszlo [Sz] has shown that paths and stars are k-equitable, as well as providing necessary and

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sufficient conditions for a cycle to be k-equitable. There has also been a significant amount of work done concerning 2-equitable labelings, also known as cordial labelings.

Labeled graphs serve as useful models for such fields as coding theory, x-ray crystallography, astronomy, circuit design, and communication network design [BG1], [BG2]. For a good summary of known results, see [Ga].

In considering the k-equitability of bipartite and multipartite graphs, it proves advantageous to define a special labeling called an even labeling. We can show that for a labeling to be k-equitable, it must be even. The problem is then reduced to checking the k-equitability of this much simpler labeling.

2 Complete Bipartite Graphs:

Let $K_{m,n}$ be a complete bipartite graph with partitions U and V. Unless otherwise stated, we write $m = kq+r$ and $n = ks+t$, with $0 \leq r, t \leq k-1$.

It has previously been shown that: $K_{m,n}$ is 2-equitable, or cordial, for all m, n [C1]; that $K_{1,n}$ is k-equitable for all k [Sz]; that $K_{2,n}$ is k-equitable when $n \equiv 0, 1, ..., \lfloor \frac{k}{2} \rfloor - 1 \mod k$, $n \equiv k-1 \mod k$, and $n = \lfloor \frac{k}{2} \rfloor$ for k odd [Sz]; and that $K_{m,n}$ is graceful, i.e. $mn+1$-equitable [Go],[Ro]. We will show that for $m \geq 3$ and $k \geq 3$, $K_{m,n}$ is k-equitable if and only if it is one of the following graphs: $K_{4,4}$ for $k = 3$; $K_{3,k-1}$ for all k; or $K_{m,n}$ for $k > mn$.

We first introduce a definition:

Definition 1 *Even labeling: A labeling of the vertices of a bipartite graph on groups of vertices U, V is said to be even if $\left|v^i - v^j\right| \leq 1$ and $\left|u^i - u^j\right| \leq 1$ for all i, j, where v^i denotes the number of vertices labeled i in V and u^i denotes the number of vertices labeled i in U.*

The idea of the characterization is that an even labeling is the only labeling for which there are a sufficient number of edges labeled 0 for the graph to be equitable. Once we have shown this, then we can easily count the number of edges labeled $k-1$, which will almost always prove to be insufficient in terms of equitability.

We need the following lemma:

Lemma 1 *The maximum number of edges labeled 0 in an even labeling is $\lfloor \frac{mn}{k} \rfloor$.*

Proof. Without loss of generality, we let $v^0 = v^1 = \cdots = v^{r-1} = q+1$, $v^r = v^{r+1} = \cdots = v^{k-1} = q$, and $u^0 = u^1 = \cdots = u^{k-t-1} = s$, $u^{k-t} = u^{k-t+1} = \cdots = u^{k-1} = s+1$.

We count the number of edges labeled 0. If $r+t \le k$, we have:

$$r(q+1)s + (k-r-t)qs + tq(s+1) = kqs + rs + tq \qquad (1)$$

Since $\lfloor \frac{mn}{k} \rfloor = \lfloor \frac{(kq+r)(ks+t)}{k} \rfloor = kqs + qt + rs + \lfloor \frac{rt}{k} \rfloor \ge kqs + rs + qt$, the maximum value e^0 can attain is $\lfloor \frac{mn}{k} \rfloor$.

If $r+t > k$, we have:

$$(k-t)(q+1)s + (r+t-k)(q+1)(s+1) + (k-r)q(s+1) = kqs + rs + tq + r + t - k \qquad (2)$$

We would like to show that $e^0 \le \lfloor \frac{mn}{k} \rfloor$. Since $\lfloor \frac{mn}{k} \rfloor = kqs + qt + rs + \lfloor \frac{rt}{k} \rfloor$, it is sufficient to show that $r+t-k \le \lfloor \frac{rt}{k} \rfloor$. For any fixed sum $r+t$, we minimize rt by maximizing $|t-r|$, and thus $\lfloor \frac{rt}{k} \rfloor$ is minimal when $t = k-1$. We then have the inequality $r - 1 \le r + \lfloor -\frac{r}{k} \rfloor$, and since $\lfloor -\frac{r}{k} \rfloor \ge -1$, the maximum value of e^0 in an evenly labeled graph is $\lfloor \frac{mn}{k} \rfloor$. ■

We can now show that only even labelings can be equitable:

Lemma 2 *An equitable labeling of the vertices of the complete bipartite graph $K_{m,n}$ produces at most $\lfloor \frac{mn}{k} \rfloor$ edges labeled 0, with equality possible only when $K_{m,n}$ is evenly labeled.*

Proof. Let v_0^i and u_0^i be the number of vertices labeled i in parts V, U in an even labeling of $K_{m,n}$. Assign values to the v_0^i's and u_0^i's as we did in the proof of Lemma 1. For any given labeling, let v^i and u^i denote the number of vertices labeled i in parts V and U of $K_{m,n}$. We write $v^i = v_0^i - a_i$ and $u^i = u_0^i + a_i$, with $-u_0^i \le a_i \le v_0^i$. Since v^i can take on any value from 0 to $v_0^i + u_0^i$, any labeling of $K_{m,n}$ can be written in this form. Also, since $|V| = \sum v^i = \sum v_0^i = \sum (v^i + a_i) = \sum v^i + \sum a_i$, we have $\sum a_i = 0$.

Case 1: $r+t \le k$.

For any given graph, the number of edges labeled 0 is:

$$\begin{aligned}
& \sum_{i=0}^{r-1} (q+1-a_i)(s+a_i) + \sum_{i=r}^{k-t-1} (q-a_i)(s+a_i) + \\
& \sum_{i=k-t}^{k-1} (q-a_i)(s+1+a_i) \\
= \; & r(q+1)s + (k-r-t)qs + tq(s+1) + \\
& q\sum_{i=0}^{k-1} a_i - s\sum_{i=0}^{k-1} a_i + \sum_{i=0}^{r-1} a_i - \sum_{i=k-t}^{k-1} a_i - \sum_{i=0}^{k-1} (a_i)^2 \\
= \; & kqs + rs + tq + \sum_{i=0}^{r-1} a_i - \sum_{i=k-t}^{k-1} a_i - \sum_{i=0}^{k-1} (a_i)^2
\end{aligned}$$

We want to show that this expression is less than or equal to (1), which is equivalent to proving:

$$\begin{aligned} kqs + rs + tq + \sum_{i=0}^{r-1} a_i - \sum_{i=k-t}^{k-1} a_i - \sum_{i=0}^{k-1}(a_i)^2 &\leq kqs + rs + tq \\ \sum_{i=0}^{r-1} a_i - \sum_{i=k-t}^{k-1} a_i &\leq \sum_{i=0}^{k-1}(a_i)^2 \end{aligned}$$

Since $x^2 \geq x$ for all integers x, this inequality holds. Moreover, equality holds if and only if: $a_i = 0$ or 1 for $0 \leq i \leq r-1$, $a_i = 0$ for $r \leq i \leq k-t-1$, and $a_i = 0$ or -1 for $k-t \leq i \leq k-1$. But for $0 \leq i \leq r-1$, we have $v_0^i - 1 = q + 1 - 1 = q$ and $u_0^i + 1 = s + 1$, and for $k - t \leq i \leq k-1$ we have $v_0^i - (-1) = q + 1$ and $u_0^i - 1 = s + 1 - 1 = s$. Thus, any values of a_i for which equality holds correspond to an even labeling.

Case 2: $r + t > k$.

We now have:

$$\begin{aligned} &\sum_{i=0}^{k-t-1} (q+1-a_i)(s+a_i) + \sum_{i=k-t}^{r-1} (q+1-a_i)(s+1+a_i) + \\ &\sum_{i=r}^{k-1}(q-a_i)(s+1+a_i) \\ = \quad & kqs + rs + tq + r + t - k + \\ & q\sum_{i=0}^{k-1} a_i - s\sum_{i=0}^{k-1} a_i + \sum_{i=0}^{r-1} a_i - \sum_{i=k-t}^{k-1} a_i - \sum_{i=0}^{k-1}(a_i)^2 \\ = \quad & kqs + rs + tq + r + t - k + \sum_{i=0}^{r-1} a_i - \sum_{i=k-t}^{k-1} a_i - \sum_{i=0}^{k-1}(a_i)^2 \end{aligned}$$

Comparing to (2), we get the inequality:

$$\begin{aligned} \sum_{i=0}^{r-1} a_i - \sum_{i=k-t}^{k-1} a_i &\leq \sum_{i=0}^{k-1}(a_i)^2 \\ \sum_{i=0}^{k-t-1} a_i + \sum_{i=k-t}^{r-1} a_i - \sum_{i=k=t}^{r-1} a_i - \sum_{i=r}^{k-1} a_i &\leq \sum_{i=0}^{k-1}(a_i)^2 \\ \sum_{i=0}^{k-t-1} a_i - \sum_{i=r}^{k-1} a_i &\leq \sum_{i=0}^{k-1}(a_i)^2 \end{aligned}$$

This inequality holds as above, with equality holding only when we have a rearrangement of an even labeling.

Thus, for any $K_{m,n}$, $e^0 \leq \lfloor \frac{mn}{k} \rfloor$, with equality only when $K_{m,n}$ is labeled evenly. ■

Theorem 3 *Any labeling which is not even is not k-equitable.*

Proof. From Lemma 2, a graph which is not labeled evenly has fewer than $\lfloor \frac{mn}{k} \rfloor$ edges labeled 0. However, for a labeling to be k-equitable, it must have at least $\lfloor \frac{mn}{k} \rfloor$ of each edge label. Thus, if a labeling is not even, then it is not k-equitable. ■

Now consider the number of edges labeled $k-1$ in an even labeling, again writing $m = kq + r$ and $n = ks + t$.

Proposition 4 *For $q, s > 0$ and $k \geq 3$, the only complete bipartite graph which is equitable is $K_{4,4}$ for $k = 3$.*

Proof. We consider various values of r and t.

Case 1: $r = t = 0$

The maximum number of edges labeled $k-1$ is $2qs$. If $\lfloor \frac{mn}{k} \rfloor = \lfloor \frac{(kq)(ks)}{k} \rfloor = kqs$ is greater than $2qs$, then the graph is not k-equitable; $kqs > 2qs$ for $k \geq 3$.

Case 2: $r = 0$, $t > 0$

We have at most $2q(s+1) = 2qs + 2q$ edges labeled $k-1$. We need $\lfloor \frac{mn}{k} \rfloor = kqs + qt$ edges of each type, but $kqs + qt > 2qs + 2q$ for $k \geq 3$, $t \geq 2$. If $r = 0$ and $t = 1$, we have at most $qs + q(s+1)$ edges labeled k-1, but need $\lfloor \frac{mn}{k} \rfloor = kqs + q$; $kqs + q > 2qs + q$ for $k \geq 3$.

Case 3: $r, t > 0$

If $r + t \leq k$, we have at most $(q+1)(s+1) + qs$ edges labeled $k-1$, and $\lfloor \frac{mn}{k} \rfloor = kqs + qt + rs + \lfloor \frac{rt}{k} \rfloor$. For $k \geq 4$ and $r, t > 0$, the inequality $kqs + qt + rs + \lfloor \frac{rt}{k} \rfloor > 2qs + q + s$ holds. For $k = 3$ the inequality holds unless $q = s = r = t = 1$. In this case, we have the graph $K_{4,4}$ with $k = 3$, which can be labeled equitably: $m = \{0, 0, 1, 2\}$, $n = \{0, 1, 2, 2\}$.

If $r + t > k$, we have at most $2(q+1)(s+1)$ edges labeled $k-1$, with $\lfloor \frac{mn}{k} \rfloor = kqs + qt + rs + \lfloor \frac{rt}{k} \rfloor$. This implies that if the inequality $2 < (k-2)qs + (t-2)q + (r-2)s + \lfloor \frac{rt}{k} \rfloor$ holds, then the specified graph is not k-equitable. Since $r + t > k$ and $r, t < k$, both r and t are at least 2, which means that $t - 2$ and $r - 2$ are non-negative. Also, $\lfloor \frac{rt}{k} \rfloor \geq \lfloor \frac{2(k-1)}{k} \rfloor = 2 + \lfloor -\frac{2}{k} \rfloor$, which is at least one for $k \geq 2$. Thus, we have $(k-2)qs + (t-2)q + (r-2)s + \lfloor \frac{rt}{k} \rfloor \geq k - 2 + 1 > 2$, which holds for $k \geq 4$. If $k = 3$ we have $qs + (t-2)q + (r-2)s + \lfloor \frac{rt}{k} \rfloor > 2$, which is true unless $q = s = 1$ and $t = r = 2$. This gives us the graph $K_{5,5}$ with $k = 3$.

However, for this graph we have at most $(q+1)(s+1)+q(s+1)=6$ edges labeled $k-1$, and $\lfloor \frac{mn}{k} \rfloor = 8$.

Thus, for $k \geq 3$, $q, \dot{s} > 0$, the only complete bipartite graph which is k-equitable is $K_{4,4}$ for $k=3$. ■

Now suppose that at least one of q, s equals 0, say q. We then have $m=r$ and $m<k$. We assume that $m, n \geq 3$.

Proposition 5 *$K_{m,n}$ is not k-equitable if $m+n \leq k \leq mn$.*

Proof. If $k \geq m+n$, then there are no repeated vertex labels, and thus no edges labeled 0. If $k \leq mn$, then there must be at least one of each edge label. This gives us a contradiction for $m+n \leq k \leq mn$. ■

Proposition 6 *If $0 \leq t \leq k-m$, and $s=0$, then $K_{m,n}$ is not k-equitable.*

Proof. As we showed above, V must be labeled evenly for an equitable labeling. This means that there is at most one vertex labeled 0 and one vertex labeled $k-1$ in V. Since $m+n \leq k(s+1)$, this implies that there are exactly 0, s, $s+1$, $2s-1$, $2s-2$, or $2s$ edges labeled $k-1$. The inequalities $2s < ms$, $s+1 < ms$, and $0 < ms$ hold for $m \geq 3$ and $s=0$, and $K_{m,n}$ is not k-equitable. ■

Proposition 7 *If $0 \leq t \leq k-m$, and $s=0$, then $K_{m,n}$ is k-equitable if and only if $k > mn$.*

Proof. We have $n+m \leq (ks+k-m)+m = k(s+1) = k$ vertices, so if $k \leq mn$ then we have $m+n \leq k \leq mn$, which means that $K_{m,n}$ is not k-equitable by Proposition 4. If $k > mn$, then we label $V = \{0, 1, 2, ..., m-1\}$ and $U = \{m, 2m, ..., nm\}$(when $k = mn+1$, this is the graceful labeling as given in [Ro] and [Go]). ■

Proposition 8 *If $k-m+1 \leq t \leq k-1$, then $K_{m,n}$ is k-equitable if and only if $m=3$ and $n=k-1$.*

Proof. Write $t = k-p$, where $1 \leq p \leq m-1$. Then there are a total of $m(ks+k-p)$ edges, and at least $ms+m+\lfloor -\frac{mp}{k} \rfloor$ of each edge label. Since $k \geq m$, we have $\lfloor -\frac{mp}{k} \rfloor \geq -p$, and so we have at least $ms+m-p$ of each edge label. We also have $ks+k-p+m$ total vertices, which implies that we have $m-p$ vertex labels appearing $s+2$ times. As above, we know that the vertex labels in V are distinct. This means that the number of edges labeled $k-1$ is $0, s, s+1, 2s, 2s+1$, or $2s+2$. Thus, if

$$2s+2 < ms+m-p \tag{3}$$

then $K_{m,n}$ is not k-equitable.

Case 1: $s = 0$

When $s = 0$, (3) becomes $2 < m - p$ or $p < m - 2$.

Case 1.1: If $p = m - 2$, we have $n = k - m + 2$. We then have $k + 2$ vertices and $(k - m + 2)m$ edges, or $m + \lfloor \frac{(2-m)m}{k} \rfloor$ of each edge label. This implies that in an even labeling there are at most 2 edges labeled 0 and 2 edges labeled $k-1$. If $m = k-1$, we have $n = 3$; if $m = 3$, we have $n = k-1$. This graph, i.e. $K_{3,k-1}$, can be labeled k-equitably: $V = \{0, k-2, k-1\}$ and $U = \{0, 1, 2, ..., k-3, k-1\}$. If m and n are less than $k - 1$, we have $k \geq m + 2$. In order to show that $K_{m,n}$ is not k-equitable, we show that $m + \lfloor \frac{(2-m)m}{k} \rfloor > 2$. Since $(2 - m)$ is negative, we minimize the left hand side by setting $k = m + 2$. After simplifying, our inequality becomes $4 + \left\lfloor -\frac{8}{m+2} \right\rfloor > 2$, which holds for $m \geq 6$. For $m = 4$ we have the inequality true except when $k = 6$ or 7, and for $m = 5$ it is true except for $k = 7$. These three cases correspond to two graphs: $K_{4,4}$ with $k = 6$, and $K_{4,5}$ with $k = 7$. In both cases, in order to get 2 each of the edge labels 0 and $k - 1$, we had to make the two duplicate vertex labels 0 and $k - 1$. Since the labels in U and V are distinct, there is one 0 and one $k - 1$ in each. This implies that there are only two edges labeled $k-2$, since $(0, k-2)$ and $(1, k-1)$ are the only two ways to get such an edge label. This means that for $K_{4,4}$ with $k = 6$, there are at least $16 - 6 = 10$ edges which must be labeled with 3 labels, which implies that one of these labels must be used at least 4 times. Similarly, for $K_{4,5}$, $k = 7$, there are 14 edges labeled with 4 labels, again implying at least 4 of one label. Thus, these graphs are not k-equitable.

Case 1.2: If $p = m - 1$, then $n = k - m + 1$, and there are $2k + 1$ vertices and $(k - m + 1)m$ edges. Each edge label must appear at least $m + \lfloor \frac{(1-m)m}{k} \rfloor$ times, and we have at most one edge labeled 0 and one edge labeled $k - 1$. If $k = m + 1$, then we have $n = 2$. If $k \geq m + 2$, we have $m + \lfloor \frac{(1-m)m}{k} \rfloor \geq 3 + \left\lfloor -\frac{6}{m+2} \right\rfloor > 1$ for $m \geq 4$. If $m = 3$, we have $n = k - 2$ and $3k - 6$ edges. There are then $3k - 8$ edges not labeled 0 or $k - 1$, and $k - 2$ labels that are not 0 or $k - 1$. Since $k \geq m + 2 = 5$, the inequality $\frac{3k-8}{k-2} > 2$ holds, and the graph is not k-equitable.

Case 2: $s = 1$

When $s = 1$, (3) becomes $4 < 2m - p$ or $p < 2m - 4$. Since $p \leq m - 1$, we have $m - 1 < 2m - 4$, which holds for $m \geq 4$. If $m = 3$ then the inequality holds unless $p = 2$. If $m = 3$ and $p = 2$, then we have $n = 2k - 2$, giving us $2k + 1$ vertices and $6k - 6$ edges. Hence, we need $\lfloor \frac{(6k-6)}{k} \rfloor = k + \lfloor -\frac{6}{k} \rfloor \geq 4$ of each edge label for $k \geq 3$. An evenly labeled graph gives us at most three edges labeled $k - 1$, so this graph is not k-equitable.

Case 3: $s \geq 2$

When $s \geq 2$, (3) becomes $p < (s + 1)(m - 2)$. Since $p \leq m - 1$ and

$(s+1)(m-2) \geq 3(m-2)$, the (3) holds when $m-1 < 3(m-2)$ or $m \leq 3$. ■

Combining the preceding, we get:

Theorem 9 *For the bipartite graph $K_{m,n}$, $m, n \geq 3$, $k \geq 3$, the only graphs which are k-equitable are $K_{4,4}$ for $k = 3$; $K_{3,k-1}$ for all k; and $K_{m,n}$ for $k > mn$.*

Proof. Theorem 9 immediately follow from Propositions 4 through 8. ■

3 Complete Multipartite Graphs

Let G be the complete multipartite graph with vertex set $V = \{V_1 \cup V_2 \cup \ldots \cup V_n\}$, where $|V_i| = m_i = kq_i + r_i$. We will consider multipartite graphs in much the same way we considered bipartite graphs: we will first provide a necessary condition for these graphs to be equitable and then count the number of edges labeled $k-1$. It has been shown that a complete multipartite graph is 2-equitable if and only if the number of parts of odd size is at most three [LLC]. We will show that, when k is less than or equal to the number of edges in our graph, $k \geq 3$, the only equitable complete multipartite graphs are $K_{kn+k-1,2,1}$ and $K_{kn+k-1,1,1}$. In order to do this, we generalize the definition of an even labeling:

Definition 2 *An n-partite graph is said to be labeled evenly if $|v_i^a - v_i^b| \leq 1$ for all a, b, and i, where v_i^a denotes the number of vertices with label a in part V_i.*

When we attempt to generalize the previous proof for $n = 2$, we find that counting the number of edges labeled 0 becomes very difficult. Thus, we will use a different approach.

Again, as in the definition of even labeling, let v_i^a denote the number of vertices labeled a in part V_i. Then, the number of edges in our graph is

$$\sum_{i<j}\left(\left(\sum_{a=0}^{k-1} v_i^a\right)\left(\sum_{a=0}^{k-1} v_j^a\right)\right),$$

the number of each edge label required is

$$\left\lfloor \frac{\sum_{i<j}\left(\left(\sum_{a=0}^{k-1} v_i^a\right)\left(\sum_{a=0}^{k-1} v_j^a\right)\right)}{k} \right\rfloor,$$

and the number of edges labeled 0 is

$$\sum_{i<j}\left(\sum_{a=0}^{k-1} v_i^a v_j^a\right).$$

Also, we know that $\sum_{i=1}^{n} v_i^a = p$ or $p+1$ for some p (i.e. our graph has an equitable labeling on the vertices). We first want to show that the maximum possible number of 0 edge labels is the minimum required. For $n = 2$ this is equivalent to showing that

$$\left\lfloor \frac{\left(\sum_{a=0}^{k-1} v_1^a\right)\left(\sum_{a=0}^{k-1} v_2^a\right)}{k} \right\rfloor \geq \sum_{a=0}^{k-1} v_1^a v_2^a$$

This should look familiar; it is simply Chebyshev's inequality with a floor function surrounding the left side. However, the right side is always an integer, so it is enough to prove the inequality holds without the floor.

The statement of Chebyshev's inequality requires that the v_i^a's be arranged so that if $v_1^0 \geq v_1^1 \geq ... \geq v_1^{k-1}$ then $v_2^0 \leq v_2^1 \leq ... \leq v_2^{k-1}$. We will show that this condition is implied by $v_1^a + v_2^a = p$ or $p+1$. Without loss of generality, we can assume that the v_1^a's are in non-decreasing order. We want to show that if $v_1^a < v_1^b$ then $v_2^a \geq v_2^b$. Suppose $v_1^a < v_1^b$ and $v_2^a < v_2^b$. Then $v_1^a + 1 \leq v_1^b$ and $v_2^a + 1 \leq v_2^b$ imply $v_1^a + v_2^a + 2 \leq v_1^b + v_2^b$. But this is a contradiction since the vertices are labeled equitably.

Note that this approach is somewhat more elegant than the counting method used above. Moreover, we can use this idea to generalize our result. We now wish to show a natural generalization of Chebyshev's inequality:

$$\left\lfloor \frac{\sum_{i<j}\left(\left(\sum_{a=0}^{k-1} v_i^a\right)\left(\sum_{a=0}^{k-1} v_j^a\right)\right)}{k} \right\rfloor \geq \sum_{i<j}\left(\sum_{a=0}^{k-1} v_i^a v_j^a\right) \tag{4}$$

with $\sum_{i=1}^{n} v_i^a = p$ or $p+1$ for some p.

Proof of (4). As before, we can ignore the floor function since the right side is always an integer. In order to prove this inequality, we rearrange the right hand side:

$$\sum_{i<j}\left(\sum_{a=0}^{k-1} v_i^a v_j^a\right) = \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{a=0}^{k-1}\left(v_i^a \sum_{j=i} v_j^a\right)\right)$$

Also,

$$\begin{aligned}
\sum_{i<j}\left(\left(\sum_{a=0}^{k-1} v_i^a\right)\left(\sum_{a=0}^{k-1} v_j^a\right)\right) &= \frac{1}{2}\sum_{i=1}^{n}\left(\left(\sum_{a=0}^{k-1} v_i^a\right)\left(\sum_{j=i}\left[\sum_{a=0}^{k-1} v_j^a\right]\right)\right) \\
&= \frac{1}{2}\sum_{i=1}^{n}\left(\left(\sum_{a=0}^{k-1} v_i^a\right)\left(\sum_{a=0}^{k-1}\left[\sum_{j=i} v_j^a\right]\right)\right)
\end{aligned}$$

We can now break the larger inequality down into n smaller inequalities, each of the form

$$\frac{\left(\sum_{a=0}^{k-1} v_i^a\right)\left(\sum_{a=0}^{k-1}\left(\sum_{j=i} v_j^a\right)\right)}{k} \geq \sum_{a=0}^{k-1}\left(v_i^a \sum_{j=i} v_j^a\right)$$

We know that $\sum_{i=1}^{n} v_i^a = p$ or $p+1$ or $v_i^a + \sum_{j=i} v_j^a = p$ or $p+1$. Now, each of these smaller inequalities is equivalent to the statement of Chebyshev's inequality with $v_1^a = v_i^a$ and $v_2^a = \sum_{j=i} v_j^a$. Thus, (4) also holds. ∎

We have shown that the maximum number of 0 edge labels possible is equal to the minimum required. For $n = 2$, we counted the number of 0 edges and compared that number to the number of 0 edge labels in an even labeling. This becomes very messy for large n, so we again turn to a slightly different approach.

Suppose that the vertices of a complete n-partite graph are labeled equitably but not evenly. We show that for any such labeling, there is another labeling with the same number of each vertex label but more edges labeled 0.

Lemma 10 *Suppose $v_i^a - v_i^b \geq 2$ for some a, b, and i. Then there is some j for which $v_j^a < v_j^b$.*

Proof. Suppose $v_i^a - v_i^b \geq 2$ for some i and $v_j^a \geq v_j^b$ for all j. Then, adding the inequalities over j $=i$, we get $\sum_{l=0}^{n} v_l^a \geq \sum_{l=1}^{n} v_l^a + 2$, which contradicts with the requirement that the vertices are labeled equitably. ∎

We now construct a new labeling by "switching" one vertex labeled a in V_i with one vertex labeled b in V_j. We want to show that such a switch increases the number of edges labeled 0. We first notice that a switch only affects edges between V_i and V_j. Thus, what we wish to show is the inequality $(v_i^a - 1)(v_j^a + 1) + (v_i^b + 1)(v_j^b - 1) - v_i^a v_j^a - v_i^b v_j^b > 0$. This reduces to $v_i^a - v_i^b + v_j^b - v_j^a > 2$, which holds since $v_i^a - v_i^b \geq 2$ and $v_j^a < v_j^b$, and thus each switch increases the number of edges labeled 0. Furthermore, if a and b are chosen so that v_i^a and v_i^b are maximal and minimal, respectively, within V_i, then the radius of V_i, i.e. the difference between the maximum and minimum value of v_i^a and v_i^b over all a and b, is non-increasing. Finally, if the radius of V_i is at least 2, then the radius can be decreased after a finite number of steps by repeatedly picking v_i^a and v_i^b to be extremal within V_i. Thus, after a finite number of steps, we can make the radius of each V_i at most 1, which means that we have obtained an even labeling. Since at each step we increased the number of edges labeled 0, our original labeling has fewer than the required number of zeros, and thus such a labeling cannot be k-equitable.

Notice that this argument provides another way of showing that k-equitable bipartite labelings must be even.

We have shown:

Theorem 11 *If a labeling of a complete n-partite graph is equitable, then it is also even.*

Now that we are restricted to even labelings, we can easily count the number of edges labeled $k-1$.

The number of each edge label required is

$$\sum_{i=j} kq_iq_j + \sum_{i=j} q_ir_j + \left\lfloor \frac{\sum_{i=j} r_ir_j}{k} \right\rfloor$$

Consider all the r_i labels in V_i that appear more times than the others, i.e. $p+1$ times. Let $R_i \subset V_i$ denote a collection of vertices with these vertex labels, each label represented by one vertex ($|R_i| = r_i$). Let $r_i = x_i + y_i$, where x_i is the number of vertices in R_i labeled either 0 or $k-1$. Also, let $p = \sum_{i=1}^{n} x_i$ (note that $p \leq 2n$), and let P be the number of edges labeled $k-1$ among edges connecting vertices among the R_i's. Then the number of edges labeled $k-1$ is $\sum_{i=j} 2q_iq_j + \sum_{i=j} q_ix_j + P$, . Thus, if we show that

$$(k-2)\sum_{i=j} q_iq_j + \sum_{i=j} q_iy_j + \left\lfloor \frac{\sum_{i=j} r_ir_j}{k} \right\rfloor > P \tag{5}$$

then we have shown that the corresponding graph is not equitable.

First, we make a few easy observations:

Fact 1: Suppose p is odd. Then there must be at least $(\frac{p-1}{2})$ "groups" of k vertices plus one additional vertex, giving us $\sum_{i=1}^{n} r_i \geq (\frac{p-1}{2})k + 1$. Similarly, for p even, $\sum_{i=1}^{n} r_i \geq \left(\frac{p}{2} - 1\right) k + 2$.

Fact 2: For $p \geq 5$ and $k \geq 3$, we may assume that all of the r_i's are non-zero. In fact, for $p \geq 5$ and $k \geq 3$, adding an $r_i = 0$ increases the number of edges labeled 1 by at least 2 more than it increases the number of edges labeled 0, and since the number of edges labeled 0 is always minimal, this means that our new graph is not equitable.

We will consider the cases when $k = 2$ and $p \leq 4$ separately; elsewhere we will assume that the r_i's are non-zero.

We now show that, without loss of generality, we may assume that $p \geq n$. Suppose that $p < n$. Then there are at least $n - p$ R_i's which do not have a vertex labeled either 0 or $k-1$. Also, since we can assume that in the collection of vertices $\{R_1 \cup R_2 \cup ... \cup R_n\}$ the only "extra" vertex labels are 0 and/or $k-1$, then there are at most $(\frac{p-1}{2})k+1$ or $\left(\frac{p}{2} - 1\right) k+2$

vertices, or at most $p(k-1)$ vertices for $k \geq 2$. This implies that we can redistribute the vertices of the $n-p$ R_i's containing neither a 0 nor a $k-1$ so that $|R_i| = 0$ for each of these parts. In doing so, we lose edges but do not lose any edges labeled 0 or $k-1$. Thus, if we can show that our new graph with p parts is not equitable, then we have shown that our original graph with $n > p$ parts is not equitable. For the rest of the paper, we will assume that $p \geq n$.

Case 1: p odd, $p \geq 5$, $k \geq 3$

For $k = 3$, we will show that there are too many edges labeled 1. For $k \geq 4$, we will show that there are too few edges labeled $k-1$.

Case1.1: $k = 3$

We will show that there are too many edges labeled 1 for the graphs under consideration to be 3-equitable. The idea is to show that $\left\lceil \frac{\sum_{i=j} r_i r_j}{k} \right\rceil <$ I, where I is the minimum number of edges labeled 1.

Suppose p is odd. Then there are either $\frac{p+1}{2}$ or $\frac{p-1}{2}$ vertices labeled 1.

If there are $\left(\frac{p-1}{2}\right)$ vertices labeled 1, we have $I \geq \left(\frac{p-1}{2}\right)(p-1)$ since each 1 vertex label has edges connecting it to at least $p-1$ vertices labeled 0 or 2. In order to maximize $\left\lceil \frac{\sum_{i=j} r_i r_j}{k} \right\rceil$, we maximize the number of edges. We have $\frac{3p-1}{2}$ vertices, and thus the upper bound for our fraction is

$$\left\lceil \frac{\binom{n}{2}\left(\frac{3p-1}{2n}\right)^2}{3} \right\rceil = \left\lceil \frac{(n-1)(3p-1)^2}{24n} \right\rceil$$

since the number of edges is maximized when the parts have the same size. In order to analyze the inequality

$$\left\lceil \frac{(n-1)(3p-1)^2}{24n} \right\rceil < \frac{(p-1)^2}{2}$$

we recognize that the right hand side is maximized when n is large. But n is at most $\frac{3p-1}{2}$(since each r_i is at least 1), so we have the inequality $\left\lceil \frac{(3p-3)(3p-1)}{24} \right\rceil < \frac{(p-1)^2}{2}$, which holds for $p \geq 5$.

If there are $\left(\frac{p+1}{2}\right)$ vertices labeled 1, then we have $R \geq \left(\frac{p+1}{2}\right)(p-1)$, and at most

$$\left\lceil \frac{\binom{n}{2}\left(\frac{3p+1}{2n}\right)^2}{3} \right\rceil = \left\lceil \frac{(n-1)(3p+1)^2}{24n} \right\rceil$$

of each edge label. n is at most $\frac{3p+1}{2}$, so we have the inequality

$$\left\lceil \frac{9p^2-1}{24} \right\rceil < \frac{p^2-1}{2}$$

which holds for $p \geq 5$.

Now suppose that $k \geq 4$. We wish to show that (5) holds. To do this, it is sufficient to show that $\lfloor \frac{\sum_{i=j} r_i r_j}{k} \rfloor > P$. P is maximized when the x_i's are distributed as evenly as possible; for $p \geq n$ this gives us $P \leq \frac{p^2-1}{4} + n - p$ for p odd. In order to minimize $\lfloor \frac{\sum_{i=j} r_i r_j}{k} \rfloor$, we want to make the r_i's as uneven as possible. We wish to show that in the most uneven distribution of r_i's there are enough vertices so that there are either $(\frac{p-1}{2})$ or $(\frac{p}{2} - 1)$ r_i's which are equal to $k - 1$. For p odd, this is equivalent to showing $(\frac{p-1}{2})k + 1 \geq (\frac{p-1}{2})(k-1) + (n - (\frac{p-1}{2}))$ which simplifies to $p \geq n$. Thus, we can calculate the lower bound for $\lfloor \frac{\sum_{i=j} r_i r_j}{k} \rfloor$:

$$\left\lfloor \frac{\binom{\frac{p-1}{2}}{2}(k-1)^2 + (\frac{p-1}{2})\left(n - \frac{p-1}{2}\right)(k-1) + \binom{n-\frac{p-1}{2}}{2}}{k} \right\rfloor$$
$$= \left(\frac{p-1}{2}\right)\left((k-2)\left(\frac{p-3}{4}\right) + n - \left(\frac{p-1}{2}\right)\right) + \left\lfloor \frac{(n-p+1)^2 - n}{2k} \right\rfloor$$

We know $p \geq n$. To show that our graph is not equitable, we need:

$$\left(\frac{p-1}{2}\right)\left((k-2)\left(\frac{p-3}{4}\right) + n - \left(\frac{p-1}{2}\right)\right) + \left\lfloor \frac{(n-p+1)^2 - n}{2k} \right\rfloor$$
$$> \frac{p^2-1}{4} + n - p$$

$$\left\lfloor \frac{2k(\frac{p-1}{2})((k-2)(\frac{p-3}{4}) + n - p) + (n-p+1)^2 - n - 2k - 2k(n-p)}{2k} \right\rfloor$$
$$> -1$$

$$2k\left(\frac{p-1}{2}\right)\left((k-2)\left(\frac{p-3}{4}\right) + n - p\right) + (n-p+1)^2 - n - 2k - 2k(n-p) \geq 0 \tag{6}$$

Case 1.2: $k = 4$

Suppose $k = 4$. Then there must be at least $4(\frac{p-1}{2}) + 1$ vertices among the R_i's, but there can be at most $3n$ vertices. Thus, we have $2p-2+1 \leq 3n$ or $n \geq \frac{2p-1}{3}$. Also, (6) becomes

$$2(p-1)(p-3+2n-2p) + (n-p+1)^2 - n - 8n + 8p - 8 \geq 0$$

The left hand side increases with n for $p \geq 5$. Substituting the minimum value of n, we get $\frac{7}{9}p^2 - \frac{58}{9}p + \frac{25}{9} \geq 0$ which holds for $p \geq 9$.

If $p = 7$ then our inequality becomes $15n - 72 + (n-6)^2$, which holds for $n \geq 5$, or all possible values of n.

For $p = 5$, our inequality becomes $7n - 32 + (n-4)^2$, which holds for $n \geq 5$.

Thus, we have two cases to check: when $n = 4$ and $p = 5$ and when $n = 3$ and $p = 5$. When $n = 4$, $p = 5$, and $k = 4$, we have at least 9 vertices among the R_i's, which implies that there are at least 23 edges. Spreading the 0's and 3's out as evenly as possible, we get 5 edges labeled 3. However, we know that there can only be 5 edges labeled 0, so there are at least 13 edges labeled either 1 or 2. This implies that at least one of these labels is used 7 times, and the graph is not equitable.

When $n = 3$, $p = 5$, and $k = 4$, we have exactly 9 vertices and 27 edges. There are only 4 edges labeled 3, and our graph is not equitable.

Case 1.3: $k = 5$

Suppose $k = 5$. Then (5) becomes

$$5(p-1)\left(n - \frac{p}{4} - \frac{9}{4}\right) + (n-p+1)^2 - n - 10 - 10(n-p) \geq 0$$

The left hand side increases with n for $p \geq 5$, so we substitute the minimum value of n, $n = \frac{p+1}{2}$ and get $\frac{3}{2}p^2 - 7p - \frac{9}{2} \geq 0$ which holds for $p \geq 7$. If $p = 5$, then we have the inequality $9n - 30 + (n-4)^2 \geq 0$, which holds for $n \geq 4$. If $p = 5$ and $n = 3$, then there are exactly 9 vertices, 27 edges, but only 4 edges labeled 3. Thus, none of these graphs are equitable.

Case 1.4: $k \geq 6$

Suppose $k \geq 6$. Unless $p = 5$ and $n = 3$, (5) increases with k, and so it is sufficient to show

$$2k\left(\frac{p-1}{2}\right)(n-3) + (n-p+1)^2 - n - 2k - 2k(n-p) \geq 0$$

Substituting $n = \frac{p+1}{2}$, we have

$$3(p-1)(p-5) + \left(\frac{p-3}{2}\right)^2 - \frac{p+1}{2} - 12 + 6(p-1) > 0$$

$$\Longrightarrow p \geq 5$$

If $p = 5$ and $n = 3$, then (5) becomes

$$2k(k-6) - 2 + 2k \geq 0$$

which holds for $k \geq 6$, and again there are no equitable labelings.

Case 2: p even, $p \geq 6$, $k \geq 3$

We will use the same ideas as above.

Case 2.1: $k = 3$

Suppose $k = 3$. Then there are $\frac{p}{2}$ vertices labeled 1, and $I \geq \frac{p}{2}(p-1)$. There are $\frac{3p}{2}$ vertices among the , and at most

$$\left\lceil \frac{\binom{n}{2}\left(\frac{3p}{2n}\right)^2}{3} \right\rceil = \left\lceil \frac{3p^2(n-1)}{8n} \right\rceil$$

of each edge label. $n = \frac{3p}{2}$, and so our inequality is

$$\left\lceil \frac{p(3p-2)}{8} \right\rceil < \frac{p}{2}(p-1)$$

This inequality holds for $p \geq 6$, and we are done.

Suppose $k \geq 4$. Then $P \leq \frac{p^2}{4} + n - p$. As above, it is easily shown that there must be at least $\left(\frac{p}{2} - 1\right)$ parts of size $k-1$ in the most uneven distribution of vertices, giving us a lower bound for $\left\lfloor \frac{\sum_{i=j} r_i r_j}{k} \right\rfloor$:

$$\left(\frac{p}{2} - 1\right)\left[(k-2)\left(\frac{p}{4} - 1\right) + n - \left(\frac{p}{2} - 1\right)\right] + \left\lfloor \frac{(n-p+2)^2 - n}{2k} \right\rfloor$$

Again, we have $p \geq n$, which gives us the inequality

$$2k\left(\frac{p}{2} - 1\right)\left((k-2)\left(\frac{p}{4} - 1\right) + n - p\right) + (n-p+2)^2 - n - 4k - 2k(n-p) \geq 0 \tag{7}$$

Case 2.2: $k = 4$

Suppose $k = 4$. Then $n \geq \frac{2p-2}{3}$ must hold. Also, (6) becomes

$$2(p-2)(-p+2n-4) + (n-p+2)^2 - n - 16 - 8n + 8p \geq 0$$

The left hand side increases with n when $n \geq 6$. We substitute the minimum value of $n = \frac{2p-2}{3}$ and get $\frac{7}{9}p^2 - \frac{98}{9}p + \frac{118}{9} \geq 0$, which holds for $p \geq 14$. When $p = 12$, we have $n \geq 8$ and the inequality $31n - 240 + (n-10)^2 \geq 0$ which holds for $n \geq 8$; when $p = 10$, we have $n \geq 6$ and the inequality $23n - 160 + (n-8)^2 \geq 0$ which holds for $n \geq 7$; when $p = 8$, we have $n \geq 5$ and the inequality $15n - 96 + (n-6)^2 \geq 0$ which holds for $n \geq 7$; and when $p = 6$, we have $n \geq 4$ and the inequality $7n - 48 + (n-4)^2$ which holds for $n \geq 7$. Thus, we need to check the following cases: $p = 10$, $n = 6$; $p = 8$, $n = 5$ or 6; and $p = 6$, $n = 4$, 5, or 6.

If $p = 10$, $n = 6$, and $k = 4$, then there are exactly 18 vertices and 135 edges. We can get at most 25 edges labeled 3, and we are done. If $p = 8$, $n = 6$, and $k = 4$, there are at least 14 vertices and at least 79 edges. We can get at most 14 edges labeled 3, and we are done. If $p = 8$, $n = 5$, and $k = 4$, there are at least 14 vertices and at least 78 edges. We can get at most 13 edges labeled 3, and we are done. If $p = 6$, $n = 4$, and $k = 4$, there are at least 10 vertices and at least 36 edges. We can get at most 7 edges labeled 3, and we are done. If $p = 6$, $n = 5$, and $k = 4$, there are at least 10 vertices and at least 38 edges. We can get at most 8 edges labeled 3, and we are done. If $p = 6$, $n = 6$, and $k = 4$, there are at least 10 vertices and at least 39 edges. We can get at most 9 edges labeled 3. We also know that we can get at most 9 edges labeled 0, which means that there are at least 11 edges labeled either 1 or 2.

Case 2.3: $k = 5$

(6) becomes

$$5(p-2)(-\frac{p}{4}+n-3)+(n-p+2)^2-n-20-10(n-p)\geq 0$$

The right hand side increases with n for $p \geq 6$, so we substitute $n = \frac{p}{2}$ and get $\frac{3}{2}p^2 - 15p + 14 \geq 0$, which holds for $p \geq 10$. If $p = 8$, then we have the inequality $19n - 90 + (n-6)^2 \geq 0$, which holds for $n \geq 5$. If $p = 6$, then we have the inequality $9n - 50 + (n-4)^2 \geq 0$, which holds for $n \geq 6$.

If $p = 8$, $n = 4$, and $k = 4$, then we must have 14 vertices but can only have 12 vertices, a contradiction. If $p = 6$, $n = 3$, and $k = 4$, then we must have 10 vertices but can only have 9. If $p = 6$, $n = 4$, and $k = 4$, then we must have at least 10 vertices and at least 36 edges. We can have at most 7 edges labeled 3, and we are done. If $p = 6$, $n = 5$, and $k = 4$, then we must have at least 10 vertices and at least 38 edges. We can only have 8 edges labeled 3, and we are done.

Case 2.4: $k \geq 6$

Suppose $k \geq 6$. The left hand side of (6) increases with k unless $p = 6$ and $n = 4$ or 5, so we have

$$6(p-2)(n-4)+(n-p+2)^2-n-24-12n+12p\geq 0$$

The left hand side increases with n for $p \geq 6$. Substituting $n = \frac{p}{2}$, we get:

$$3(p-2)(p-8)+(\frac{p}{2}-2)^2-\frac{p}{2}-24-6p+12p\geq 0$$
$$p\geq 8$$

If we substitute $n = \frac{p}{2} + 2$, we get:

$$3(p-2)(p-4)+(\frac{p}{2}-4)^2-\frac{p}{2}-26-6p-24+12p\geq 0$$
$$p\geq 6$$

So, we must check the following cases: $p = 6$, $n = 3$ and $p = 6$, $n = 4$. Checking $k \geq 7$ with these two cases, we get the inequalities:

$$\begin{aligned} 2\left(-\frac{1}{2}\right) + \left\lfloor -\frac{1}{k} \right\rfloor &> 1 + 3 - 6 \\ 2\left(\frac{1}{2}\right) + \left\lfloor -\frac{2}{k} \right\rfloor &> 1 + 4 - 6 \end{aligned}$$

where the second holds.
If $k \geq 8$, the first becomes:

$$2\,(0) + \left\lfloor -\frac{2}{k} \right\rfloor > 1 + 3 - 6$$

which holds. This leaves the following cases: $p = 6$, $n = 3$ and $k = 6$ or 7; and $p = 6$, $n = 4$ with $k = 6$.

If $p = 6$, $n = 3$, and $k = 6$, then there must be at least 14 vertices in the graph. Thus, the minimum number of edges is $(5)\,(5)+(5)\,(4)+(5)\,(4) = 65$, requiring at least 10 of each edge label. But we can only get 6 edges labeled 5, and so this case is done. If $p = 6$, $n = 3$, and $k = 7$, there must be at least 16 vertices, at least $(6)\,(6) + (6)\,(4) + (6)\,(4) = 84$ edges, and at least 12 of each type. Again, we can only get 6 edges labeled 6. If $p = 6$, $n = 4$ with $k = 6$, then there must be at least 14 vertices, at least 68 edges, and at least 11 of each type. However, we can only have 7 edges labeled 5.

Case 3: $k = 2$

Lee, Lee, and Chang [LLC] have given necessary and sufficient conditions for multipartite graphs being 2-equitable(cordial): A complete n-partite graph K is cordial if and only if the number of parts with an odd size is less than or equal to 3.

Case 4: $p = 3$

Let $p = 3$ and $n \geq 3$. Until noted otherwise, assume that each part of our graph has size less than k.

Case 4.1: Suppose there are at least $k + 2$ vertices total. Then there are at least $3k - 1$ edges, and since we can only have 2 edges labeled $k - 1$ and 2 edges labeled 0, we are done. If there are more than $k + 2$ vertices or there are $k + 2$ vertices distributed more equally, then there are at least $3k$ edges, and we are done since there can only be two edges labeled $k - 1$.

Case 4.2: Suppose there are $k + 1$ vertices total. Then there is only one edge labeled 0, and so we must have the most unequal graph, $K_{k-1,1,1}$. This graph has an equitable labeling: $\{0, 1, ..., k-2\}, \{0\}, \{k-1\}$. Assuming that $k \geq 3$, we see that: adding a group to the first part maintains equitability by adding two of each edge label, and thus $K_{kn+k-1,1,1}$ is equitable; adding a group to the second part adds $2k - 2$ edges labeled 1 but only k edges labeled 0; that adding a group to the third part produces $2k-2$ edges labeled 1 but only k edges labeled 0; and that adding a group to make a fourth part adds $2k-1$ edges labeled 1 but only $k+1$ edges labeled 0.

Thus, for $p=3$, only graphs of the form $K_{kn+k-1,1,1}$ are equitable.

Case 5: $p=4$

Let $p=4$ and $n \geq 3$. Again, assume that each part is of size less than k.

Case 5.1: Suppose that there are at least $k+4$ vertices total. Then there are at least $5k-1$ edges, but at most 4 edges labeled $k-1$ and 4 edges labeled 0. If there are more than $k+4$ vertices or the $k+4$ vertices are distributed more equally, then there are at least $5k$ edges and we are done.

Case 5.2: Suppose there are $k+3$ vertices. Then there are at most 3 edges labeled 0. Again, we have $4k-1$ edges if the vertices are distributed as unequally as possible and more edges if they are distributed more equally. Thus, we consider the graph $K_{k-1,3,1}$. There must be exactly 4 of each edge label other than 0. There are three cases.

Case 5.2.1: Suppose that one part has no vertices labeled either 0 or $k-1$. Then there are at most 2 edges labeled $k-1$ and we are done.

Case 5.2.2: Suppose that each part has at least one vertex labeled either 0 or $k-1$ and that the part which has both a vertex labeled 0 and a vertex labeled $k-1$ is the part with size 3. Then the third vertex in the part of size 3 connects to one vertex label of each type(since this vertex must have the third duplicated vertex label). Thus, we have 5 edges labeled 1: 2 from this special vertex, and 1 from each of the other three vertices in the two smaller parts.

Case 5.2.3: Suppose that each part has at least one vertex labeled 0 or $k-1$ and that the part with two such labels is the part of size $k-1$. Then, since one of the labels in the size of part three is a duplicate label which is not 0 or $k-1$, there are again 5 edges labeled 1. (Note that since there is only one label missing from the part of size $k-1$ that the non-duplicate, non-0 or $k-1$ vertex label in the part of size 3 is guaranteed to produce an edge labeled 1.)

Case 5.3: Suppose there are $k+2$ vertices, the minimum possible. Then there are 2 edges labeled 0, and so the only possibility for an equitable graph is $K_{k-1,2,1}$ because this graph has $3k-1$ edges. We know that there must be 3 of each edge label other than 0, which implies that each part must have a vertex labeled either 0 or $k-1$. There are now two cases.

Case 5.3.1: Suppose that the part which contains both a vertex labeled 0 and a vertex labeled $k-1$ is the part with size $k-1$. Without loss of generality, suppose that the label in the part of size two is a 0. Then we know that the non-0, non-$k-1$ vertex label which appears in the part of size 2 connects to at least two edges labeled $k-1$. The only way to compensate for this is to label the graph so that the vertex labeled 0 in the part of size 2 connects to no edges labeled 1. The only way to do this is to put a label of 1 in the part of size 2, which gives us the labeling $\{0, 2, ..., k-1\}, \{0, 1\}, \{k-1\}$, giving us the equitable graph $K_{k-1,2,1}$. We check what happens when we add groups: when adding to the first part, edges labeled 1 increases by 4, edges labeled 0 increases by 3; when adding to the second part edges labeled 1 increases by $2k-3$, edges labeled 0 increases by k(note that equitability is maintained for $k=3$ but that for $k=3$ we have the graph labeled as below); if we add a group to the third part we increases edges labeled 1 by $2k-1$ and edges labeled 0 by $k+1$; and if we add a group to for a new part we add $2k$ edges labeled 1 and $k+2$ edges labeled 0.

Case 5.3.2: Suppose that the part with both an vertex labeled 0 and a vertex labeled $k-1$ is the part of size 2. This leads to the labeling $\{1, 2, ..., k-1\}, \{0, k-1\}, \{0\}$. Adding a group to the first part, we maintain equitability by adding 3 of each edge label, giving us the equitable graph $K_{kn+k-1,2,1}$; adding a group to the second part, we increase edges labeled 1 by $2k-2$ and edges labeled 0 by k; adding a group to the third part, we increases edges labeled 1 by $2k-1$ and edges labeled 0 by $k+1$; and adding a group to form a new part, we increase edges labeled 1 by $2k$ and edges labeled 0 by $k+2$.

Thus, for $p=4$, the only graphs which are equitable are those of the form $K_{kn+k-1,2,1}$.

Case 6: $p \leq 2$

Without loss of generality, we can assume that there are no duplicate vertex labels; if there were then we would arrange the labels so $p > 2$. Thus, there are no edges labeled 0, and so we have k larger than the number of edges in our graph.

We note that if we show that a certain graph with $k = l$ is equitable, then that same graph will be k-equitable for $k > l$. However, the problem of finding this lower bound l is quite difficult. It can be easily shown that such an l exists. However, finding the least l for which a graph is l-equitable involves considering a large variety of graphs. For instance, complete graphs are a subcase: a complete subgraph of size 4, for example, is simply $K_{1,1,1,1}$. [Go] has shown that the complete graph K_n is graceful only for $n \leq 4$, but the question as to the lowest l such that K_n is l-equitable seems quite difficult. In terms of the more general case, the only results seem to be that [AM] has shown that the graph $K_{1,m,n}$ is graceful, and [Gn, p. 25-31] has shown that $K_{1,1,m,n}$ and $K_{2,m,n}$ are graceful.

We can now formulate:

Theorem 12 *For $k \geq 3$, k less than or equal to the number of edges in our graph, the only k-equitable complete multipartite graphs $K_{n_1,n_2,...,n_l}$ are*

$K_{kn+k-1,2,1}$ *and* $K_{kn+k-1,1,1}$.

Proof. The theorem follows from the preceding arguments. ■

4 Conclusion:

Complete bipartite and multipartite graphs have enough structure that, for most cases, it is possible to determine whether they are k-equitable. However, even given the rigid structure of complete multipartite graphs, the k-equitability of complete multipartite graphs when k exceeds the number of edges is as of yet undetermined. The k-equitability of many types of well-structured graphs, such as wheels, ladders, and other symmetric graphs, is also unknown; however, the strong structural requirements of these graphs suggests that the question of their k-equitability may soon be settled.

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