Another Doyen - Wilson Theorem

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Abstract

In this paper we show the necessary and sufficient conditions for a complete graph on n vertices with a hole of size v ($K_n \setminus K_v$) to be decomposed into isomorphic copies of K_3 with a pendant edge.

ARS COMBINATORIA 54(2000), pp. 87-96

1 Introduction

A *G-design on H* is a collection of subgraphs of H each isomorphic to G that partition the edges of H. A *G-design of order n* is a G-design on the complete graph K_n on n vertices. One problem in design theory is the *spectrum problem* for G, i.e. for what values of n is there a G-design of order n? The spectrum problem has been solved for complete graphs on less than six vertices, stars, paths, cycles of length at most 50, and various other small graphs [1, 2, 3, 10, 13, 14, 15, 16]. G-designs on non-complete graphs have also been studied, for example, designs when G and H are both complete bipartite graphs [11]. Another example is when H is a complete graph of order n with a *hole* of size v, $K_n \backslash K_v$. This is a complete graph of order n from which the edges of a complete graph of order v have been removed. Doyen and Wilson first considered such designs with $G = K_3$ [9]. G-designs on graphs with holes have also be found when G is an n-cycle with $n \leq 14$ and stars [12, 5, 6, 7, 8]. We will consider such designs where G is a triangle with a pendant edge:

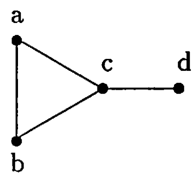

Figure 1: Block $(a, b, c) - d$

We denote the copy of G in Figure 1 by $(a, b, c) - d$ or $(b, a, c) - d$.

2 The Main Theorem

Theorem 2.1 *Let H be a complete graph with a hole of size v where $\nu(H) = d + v$. Let G be $K_3 \cup \{e\}$ (otherwise known as "a triangle and a stick"). Then H has a G-design iff $v \leq \frac{3}{2}(d-1)$ and $8|d(d-1+2v)$.*

Proof Let V denote the vertices of the "hole" and D denote the graph induced by the remaining vertices of H. Since $\epsilon(G) = 4$, the number of edges in H must be divisible by 4, so $8|d(d-1+2v)$. Also, since G is not bipartite, each block, or copy of G, must contain at least one edge in D. Thus the number of blocks must be less than or equal to the number of edges in D; equivalently, $v \leq \frac{3}{2}(d-1)$.

Therefore the theorem's conditions are certainly necessary.

Now we need to show these conditions are sufficient. We will use difference methods (mod d), so D is the ring of integers (mod d).

Case 1: d is odd

When d is odd, $8 | (d-1+2v)$. Let $t = \frac{1}{8}(d-1+2v)$, so the number of blocks needed is td, and the number of orbits of G, base blocks (mod d) of G, needed is t. Let α be a non-negative integer to be specified later. Choose α orbits of G to be entirely contained in D in the following manner:

Let $d = 8n + \epsilon$ with $\epsilon \in \{1, 3, 5, 7\}$. For $n > 0$, the difference triples $(2n+1+i, 4n-i, 2n-1-2i)$, $i = 0, ..., \alpha - 1$ form αd triangles and any α of the remaining differences may be used for the αd sticks. (If $n = 0$ then $\alpha = 0$). We require 4 differences for every orbit so $4\alpha \leq \frac{1}{2}(d-1)$. We need $t - \alpha$ more orbits to complete the design. Each of these orbits of G must have at least one edge of each triangle in D so $t - \alpha \leq \frac{1}{2}(d-1)$. These 2 inequalities imply $t + 3\alpha \leq \frac{1}{2}(d-1)$.

First we will form $d(t-\alpha)$ triangles. Each of these triangles contains at least one edge of D, so let $\widehat{D}$ be the graph induced by $t-\alpha$ of the remaining differences of D. $\widehat{D}$ is regular of degree $2(t-\alpha)$ and therefore, by Vizing's Theorem [4], is edge-colorable with $2(t-\alpha)+1$ colors. Give $\widehat{D}$ a proper coloring using the colors $(1,2,...,v)$; so we will need $v \geq 2(t-\alpha)+1$. Let $\infty_1, \infty_2, ..., \infty_v$ be the vertices of V. Then ∞_i and the vertices incident with each edge colored i will form triangles giving $d(t-\alpha)$ triangles in all.

Now we need to match these triangles with the remaining edges to form the sticks. We will do this by orienting certain edges.

Let the $t-\alpha$ colored differences represent each of the $d(t-\alpha)$ triangles. Orient these differences along with any difference not already used in the following manner. For each of these differences i and each vertex $x \in D$, orient the edge from x to $x+i$. Orient the edges between D and V that are not already in a triangle, so that for each $x \in D$ and each $\infty \in V$ the edge is oriented from x to ∞.

With this orientation, at each vertex of D there are $(t-\alpha)$ colored heads and $\frac{1}{2}(d-1)-(t-\alpha)-4\alpha+v-2(t-\alpha)$, or $t-\alpha$ non-colored tails. Thus for every colored edge (representing a triangle) into a vertex there is a non-colored edge (representing a stick) out of the vertex to be matched with it as the stick for that colored triangle.

This construction works only when

$$2(t-\alpha)+1 \leq v \text{ and } t+3\alpha \leq \tfrac{1}{2}(d-1).$$

So

$$\tfrac{1}{8}(d-2v+3) \leq \alpha \text{ and } \alpha \leq \tfrac{1}{8}\left(d-1-\tfrac{2}{3}v\right)$$

There is certainly such an integer α if

$$\tfrac{1}{8}(d-2v+3)+1 \leq \tfrac{1}{8}\left(d-1-\tfrac{2}{3}v\right)$$

or

$$9 \leq v.$$

Also when v is odd $\alpha = \frac{1}{8}(d - 2v + 3)$ is an integer that satisfies these conditions so we only need to consider the cases $v = 2, 4, 6, 8$.

When $v = 8$ then $d \equiv 1 \pmod 8$ and $\alpha = \frac{1}{8}(d - 9)$ is an integer that satisfies the condition.

When $v = 6$ then $d \equiv 5 \pmod 8$ and $\alpha = \frac{1}{8}(d - 5)$ is an integer that satisfies the condition.

When $v = 4$ then $d = 8n + 1$ and the following is a G-design on the vertex set $\mathbb{Z}_{8n+1} \cup \infty_1, ..., \infty_4$.

Consider $n \geq 3$ (the cases when $n = 1$ and $n = 2$ can be found in the appendix). First we will form $n + 1$ triangles then we will use the following "color and orient" method to match the triangle to the remaining edges as sticks. The difference triples $(2n - 2i - 1,\ 2n + i + 1,\ 4n - i)$ for $i = 0, ..., n - 1$ and $(2, 4, 6)$ form $n + 1$ triangles. Properly color the graph induced by one difference from each triangle, say the differences $4n - i$ for $i = 0, ..., n - 1$ and the difference 2, in order to distinguish triangles at each vertex from sticks at each vertex. Thus there are $2(n + 1)$ colored edges at each vertex. Orient the edges in these differences along with the edges of the $n - 3$ differences that were not used in a triple (these are the possible sticks) so that for each of these differences i and each vertex x in D the edge goes from x to $x + i$. Orient the edges from D to V so that, for each $x \in D$ and each $\infty \in V$ the edge goes from x to ∞. With this orientation $d^-(v) = d^+(v)$ for each $v \in D$ and $d^-(\infty) = 0$ for each $\infty \in V$. In particular, at each $v \in D$ we have $n + 1$ colored heads, $n - 3$ non-colored tails from the $n - 3$ remaining differences and 4 non-colored tails from the ∞ vertices. Thus, at each vertex there are $n + 1$ colored heads (representing triangles) and $n + 1$ non-colored tails (representing possible sticks). So at each vertex we can match the triangles corresponding to the colored heads with any of the non-colored tails as the sticks to complete the design.

When $v = 2$ then $d = 8n + 5$ and the following is a G-design on the vertex set $\mathbb{Z}_{8n+1} \cup \infty_1, ..., \infty_4$.

First consider n odd and $n \geq 3$ (the case for $n = 1$ can be found in the appendix). The difference triples $(2n - 2i, 2n + i + 1, 4n - i + 1)$ for $i = 0, ..., n - 1$ and $(4n + 2, 3n + 1, n + 2)$ form $n + 1$ triangles. Use the "color and orient" method described earlier to match these triangles with the remaining edges as sticks.

For n even and $n \geq 2$ (the case for $n = 0$ can be found in the appendix), the difference triples $(2n - 2i,\ 2n + i + 2,\ 4n - i + 2)$ for $i = 0, ..., n - 1$ and $(3n+2,\ 2n+1,\ n+1)$ form $n + 1$ triangles. Complete the design using the "color and orient" method described earlier.

Case 2: d even

Since d is even, $(d-1+2v)$ is odd and therefore $8|d$. Let $d=8n$ and $v=4m+\epsilon$, $\epsilon \in \{0,1,2,3\}$. There are $n(8n+8m+2\epsilon-1)$ blocks needed. As before $v \leq \frac{3}{2}(d-1)$ so, $4m+\epsilon \leq 12n-2$.

Since $d = 8n$ the graph induced by the differences $n, 2n, 3n$ and $4n$ consists of n components each isomorphic to K_8. Take ϵ of the vertices of V along with each of these components and place a design with $d=8$ and $v=\epsilon$ on the resulting graph (a list of these decompositions may be found in the appendix). Let $\widehat{V}$ be the remaining $4m$ vertices of V. The number of blocks remaining in each case will be $n(8n+8m-8)$ so we will need $t=(n+m-1)$ more orbits.

Let α be a non-negative integer to be specified later. Choose α orbits to be entirely contained in the remaining $4n-4$ differences of D as follows:

For n odd, $n \geq 3$, the difference triples $(2n+i, 4n-i, 2n-2i)$ for $i=1,2,...,\alpha$ form triangles and any α of the remaining differences may be used for the αd sticks. (A list of the designs when $n=1$ can be found in the appendix).

For n even, $n \geq 4$, the difference triples $(2n+1+i, 4n-i, 2n-1-2i)$ for $i=1,2,...,\alpha-1$ and the difference triple $(3n+1, 2n-1, n+2)$ form α triangles and any α of the remaining differences may be used for the αd sticks. (When $n=2$, $\alpha=0$)

We require 4 of the remaining $4n-4$ differences for every orbit so $4\alpha \leq 4n-4$ or $\alpha \leq n-1$.

When $\alpha = n-1$ then we must have $m=0$ and the decomposition described above is a G-design. For $\alpha < n-1$, choose the α orbits so that the difference 1 is not used.

We need $t-\alpha$ more orbits to complete the design. Each of these orbits of G must have at least one edge of each triangle in D so $t-\alpha \leq 4n-4$. This inequality along with the previous one implies $t+3\alpha \leq 4n-4$. We will first form $d(t-\alpha)$ triangles. Each of these triangles must have at least one edge in D so let $\widehat{D}$ be the graph induced by $t-\alpha$ of the remaining differences, including difference 1. Then $\widehat{D}$ is regular of degree $2(t-\alpha)$. Since this set of differences includes the difference 1, and $\frac{8n}{\gcd(1,8n)}$ is even, by Stern and Lenz [4], the graph induced by this set of differences is Class I and is therefore edge-colorable with $2(t-\alpha)$ colors. Give $\widehat{D}$ a proper coloring using the colors $(1,2,...,4m)$; so we will need $4m \geq 2(t-\alpha)$. Let $\infty_1, \infty_2, ..., \infty_{4m}$ be the vertices of $\widehat{V}$. Then ∞_i and the vertices incident with each edge colored i will form $d(t-\alpha)$ triangles in all.

Use the orientation described earlier to match these triangles with the remaining edges to form the necessary blocks.

This construction works only when $\alpha \geq 0$ and $t+3\alpha \leq 4n-4$. There is such an integer α when $m \leq 3(n-1)$.

Since $v \leq \frac{3}{2}(d-1)$, we have $m \leq 3n - \frac{3+2\epsilon}{8}$. So the values of m which satisfy $3n-3 < m \leq 3n - \frac{3+2\epsilon}{8}$ still need to be found. These designs are described below.

Let $v = 4m+\epsilon$ with $\epsilon \in \{0,1,2,3\}$ and $d = 8n$. Then there are 2 values of m to consider, $m = 3n-2$ and $m = 3n-1$. We start with some partial decompositions of $K_{8n} \backslash K_v$.

Decomposition using only differences n and $4n$ with 4 vertices of V:
$(\infty_1, 0, n) - \infty_4$ $(n, 2n, \infty_2) - 0$ $(\infty_1, 3n, 2n) - \infty_4$ $(\infty_2, 4n, 3n) - \infty_4$
$(\infty_1, 5n, 4n) - \infty_4$ $(5n, 6n, \infty_2) - 7n$ $(\infty_1, 6n, 7n) - \infty_4$
$(\infty_3, 4n, 0) - \infty_4$ $(\infty_3, n, 5n) - \infty_4$ $(\infty_3, 2n, 6n) - \infty_4$ $(\infty_3, 3n, 7n) - 0$

Decomposition using only differences $n, 2n$, and $4n$ with 6 vertices of V:
$(\infty_1, 0, n) - \infty_4$ $(\infty_2, n, 2n) - \infty_5$ $(\infty_1, 2n, 3n) - n$ $(\infty_2, 3n, 4n) - \infty_6$
$(\infty_1, 4n, 5n) - \infty_6$ $(\infty_2, 5n, 6n) - \infty_6$ $(\infty_1, 6n, 7n) - \infty_6$
$(\infty_3, n, 7n) - \infty_2$ $(\infty_3, 5n, 3n) - \infty_6$ $(\infty_3, 0, 2n) - 6n$
$(\infty_3, 4n, 6n) - \infty_5$ $(\infty_4, 4n, 2n) - \infty_6$ $(\infty_4, 6n, 0) - \infty_2$
$(\infty_4, 5n, 7n) - 0$ $(\infty_5, 5n, n) - \infty_6$ $(\infty_5, 7n, 3n) - \infty_4$ $(\infty_5, 4n, 0) - \infty_6$

Decomposition with $m = 3n - 2$

Case I: $\epsilon = 0$

In this case we have $v = 12n - 8$.

Use differences n and $4n$ and 4 vertices of V for $11n$ blocks. Use difference $2n$ to create two 1-factors. The first 1-factor along with a 5th vertex of V will form triangles and the second 1-factor will join these triangles as sticks to create $4n$ blocks.

For the remaining $n(8n-1+24n-16-15) = 8n(4n-4)$ blocks properly color $4n-4$ of the remaining $4n-3$ differences with the colors $1, \ldots, 8n-8$. Each edge colored i will join the point $\infty_i \in V$ to form $8n(4n-4)$ triangles. Use the orientation described earlier to match these triangles with the remaining edges to form the necessary blocks.

Case II: $\epsilon = 1$

Use differences $n, 2n$ and $4n$ with 6 vertices of V for $17n$ blocks. For the remaining $n(8n-1+24n-14-17) = 8n(4n-4)$ blocks properly color $4n-4$ of the remaining $4n-3$ differences and continue as in case I.

Case III: $\epsilon = 2$

Use differences n and $4n$ with 4 vertices of V for $11n$ blocks. For the remaining $n(8n-1+24n-12-11) = 8n(4n-3)$ blocks properly color $4n-3$ of the remaining $4n-2$ differences and continue as in case I.

Case IV: $\epsilon = 3$

Place a decomposition of $K_{15} \backslash K_7$ (found in the appendix) on the graph induced by the differences $n, 2n, 3n$ and $4n$ with 7 vertices of V to form $21n$ blocks. For the remaining $n(8n-1+24n-10-21) = 8n(4n-4)$ blocks properly color the remaining $4n - 4$ differences and continue as in case I.

Decomposition with $m = 3n - 1$

Case I: $\epsilon = 0$

Use differences $n, 4n$ and 4 vertices of V for $11n$ blocks. Use difference $2n$ to create two 1-factors. The first 1-factor along with a 5th vertex of V will form triangles and the second 1-factor will join these triangles as sticks to create $4n$ blocks.

For the remaining $n\,(8n - 1 + 24n - 8 - 15) = 8n\,(4n - 3)$ blocks properly color the remaining $4n - 3$ differences with the colors $1, \ldots, 8n - 6$. Each edge colored i will join the point $\infty_i \in V$ to form $8n\,(4n - 3)$ triangles. Use the orientation described earlier to match these triangles with the remaining edges to form the necessary blocks.

Case II: $\epsilon = 1$

Use differences $n, 2n$ and $4n$ with 6 vertices of V for $17n$ blocks. For the remaining $n\,(8n - 1 + 24n - 6 - 17) = 8n\,(4n - 3)$ blocks properly color the remaining $4n - 3$ differences and continue as in case I.

Case III: $\epsilon = 2$

Use differences n and $4n$ with 4 vertices of V for $11n$ blocks. For the remaining $n\,(8n - 1 + 24n - 4 - 11) = 8n\,(4n - 2)$ blocks properly color the remaining $4n - 2$ differences and continue as in case I.

□

References

[1] E. Bell, Decomposition of K_n into cycles of lenght at most fifty. *Ars Combinatoria* 40 (1995) 49–58.

[2] J.-C. Bermond, and J. Schönheim, G-Decomposition of K_n Where G Has Four Vertices or Less. *Discrete Mathematics* 19 (1977) 113–120.

[3] J.-C. Bermond, C. Huang, A. Rosa and D. Sotteau, Decomposition of Complete Graphs into Isomorphic Subgraphs with Five Vertices. *Ars Combinatoria* 10 (1980) 211–254.

[4] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications* North-Holland Publishing Company, New York (1979).

[5] D. E. Bryant, D. G. Hoffman and C. A. Rodger, 5-cycle systems with holes. *Designs, Codes and Cryptography* 8 (1996) 103–108.

[6] D. E. Bryant, C. A. Rodger and E. R. Spicer, Embedding m-cycle systems, and incomplete m-cycle systems: $m \leq 14$. *Discrete Mathematics* to appear.

[7] D. E. Bryant, C. A, Rodger, The Doyen-Wilson Theorem extended to 5-cycles. *Journal of Combinatorial Theory (A)* 68 (1994) 218–225.

[8] D. E. Bryant, C. A. Rodger, On the Doyen-Wilson Theorem for m-cycles. *Journal of Combinatorial Design* 2 (1994) 253–271.

[9] J. Doyen and R. M. Wilson, Embeddings of Steiner Triple Systems. *Discrete Mathematics* 5 (1973) 229-239.

[10] H. Hanani, The existence and construction of Balanced Incomplete Block Designs. *Annals of Mathematical Statistics* 32 (1961) 361–386.

[11] D. G. Hoffman and M. Liatti, Bipartite Designs. *Journal of Combinatorial Designs* 3 (1995) 449–454.

[12] D. G. Hoffman, Unpublished Manuscript.

[13] Rev. T. P. Kirkman, "On a Problem in Combinatorics". *Cambridge and Dublin Mathematical Journal* 2 (1847) 191–204.

[14] A. Kotzig, On the Decomposition of Complete Graphs into k-gons. *Mat. Fyz. Casop.* 15 (1965) 229–233.

[15] M. Tarsi, Decomposition of Complete Multigraphs into Stars. *Discrete Mathematics* 26 (1979) 273–278.

[16] M. Tarsi, Decomposition of a complete multigraph into simple paths. *Journal of Combinatorial Theory* A 34 (1983), no. 1, 60–70.

A G-Designs for $v = 4$, $d = 9$ and 17

- $d = 9$ on the vertex set $\mathbb{Z}_9 \cup \infty_1, \dots, \infty_4$: $(0,4,\infty_1) - 8$ $(3,7,\infty_1) - 2$ $(6,1,\infty_1) - 5$ $(4,8,\infty_2) - 3$ $(7,2,\infty_2) - 6$ $(1,5,\infty_2) - 0$ $(0,1,\infty_3) - 2$ $(3,4,\infty_3) - 5$ $(6,7,\infty_3) - 8$ $(1,2,\infty_4) - 3$ $(4,5,\infty_4) - 6$ $(7,8,\infty_4) - 0$ $(0,3,2) - 5$ $(0,6,8) - 2$ $(0,5,7) - 4$ $(8,3,1) - 7$ $(2,6,4) - 1$ $(6,3,5) - 8$

- $d = 17$ on the vertex set $\mathbb{Z}_{17} \cup \infty_1, \dots, \infty_4$: $(\infty_1,0,4) - 2$ $(\infty_1,8,12) - 14$ $(\infty_1,16,3) - 5$ $(\infty_1,7,11) - 13$ $(\infty_1,15,2) - 0$ $(\infty_1,6,10) - 8$ $(\infty_1,14,1) - 3$ $(\infty_1,5,9) - 7$ $(\infty_2,4,8) - 6$ $(\infty_2,16,12) - 10$ $(\infty_2,3,7) - 5$ $(\infty_2,15,11) - 9$ $(\infty_2,2,6) - 4$ $(\infty_2,10,14) - 16$ $(\infty_2,5,1) - 16$ $(\infty_2,9,13) - \infty_1$ $(13,15,0) - \infty_2$. In addition, here are 2 base blocks to be developed cyclically (mod 17): $(0,6,7) - \infty_3$ $(0,5,8) - \infty_4$

B G-Designs for $v = 2$, $d = 5$ and 13

- $d = 5$ on the vertex set $\mathbb{Z}_5 \cup \infty_1, \infty_2$: $(\infty_1,3,0) - \infty_2$ $(\infty_1,2,4) - 3$ $(\infty_2,1,4) - 0$ $(\infty_2,2,3) - 1$ $(0,2,1) - \infty_1$

- $d = 13$ on the vertex set $\mathbb{Z}_{13} \cup \infty_1, \infty_2$. Here are the base blocks to be developed cyclically (mod 13): $(0,1,4) - \infty_1$ $(0,2,7) - \infty_2$

C G-Designs with $d = 8$

- $v = 0$ on the vertex set $\mathbb{Z}_7 \cup \infty$. Here is a base block to be developed cyclically (mod 7): $(0,1,3) - \infty$.

- $v = 1$ on the vertex set $\mathbb{Z}_9$ where the element 0 is the element of V. Here is a base block to be developed cyclically (mod 9): $(0,1,3) - 7$.

- $v = 2$ on the vertex set $\mathbb{Z}_8 \cup \infty_1, \infty_2$: $(\infty_1,0,4) - \infty_2$ $(\infty_1,1,5) - \infty_2$ $(\infty_1,2,6) - \infty_2$ $(\infty_1,3,7) - \infty_2$ $(3,1,0) - \infty_2$ $(4,2,1) - \infty_2$ $(5,3,2) - \infty_2$ $(6,4,3) - \infty_2$ $(4,5,7) - 0$ $(5,6,0) - 2$ $(1,6,7) - 2$

- $v = 3$ on the vertex set $\mathbb{Z}_8 \cup \infty_1, \infty_2, \infty_3$: $(\infty_1,0,3) - 4$ $(\infty_1,6,1) - 3$ $(\infty_1,4,7) - 5$ $(\infty_1,2,5) - 6$ $(\infty_2,3,6) - 4$ $(\infty_2,1,4) - 5$ $(\infty_2,7,2) - 4$ $(\infty_2,5,0) - 6$ $(\infty_3,4,0) - 7$ $(\infty_3,1,5) - 3$ $(\infty_3,2,6) - 7$ $(\infty_3,3,7) - 1$ $(0,1,2) - 3$

- $v=4$ on the vertex set $\mathbb{Z}_8 \cup \infty_1,\dots,\infty_4$: $(\infty_1,0,3)-\infty_4$ $(\infty_1,1,6)-\infty_4$ $(\infty_1,7,4)-\infty_4$ $(\infty_1,2,5)-\infty_4$ $(3,6,\infty_2)-0$ $(\infty_2,4,1)-\infty_4$ $(7,2,\infty_2)-5$ $(\infty_3,4,0)-\infty_4$ $(\infty_3,1,5)-0$ $(\infty_3,6,2)-\infty_4$ $(\infty_3,3,7)-\infty_4$ $(6,0,7)-1$ $(0,2,1)-3$ $(2,4,3)-5$ $(4,6,5)-7$

- $v=5$ on the vertex set $\mathbb{Z}_8 \cup \infty_1,\dots,\infty_5$: $(\infty_1,0,1)-5$ $(\infty_1,2,3)-7$ $(\infty_1,4,5)-3$ $(\infty_1,6,7)-1$ $(\infty_2,1,2)-6$ $(\infty_2,4,3)-\infty_5$ $(\infty_2,5,6)-0$ $(\infty_2,7,0)-\infty_5$ $(\infty_3,0,3)-1$ $(\infty_3,6,1)-\infty_5$ $(\infty_3,4,7)-\infty_5$ $(\infty_3,2,5)-7$ $(\infty_4,3,6)-\infty_5$ $(\infty_4,1,4)-\infty_5$ $(\infty_4,7,2)-\infty_5$ $(\infty_4,0,5)-\infty_5$ $(0,2,4)-6$

- $v=6$ on the vertex set $\mathbb{Z}_8 \cup \infty_1,\dots,\infty_6$: $(\infty_1,2i,2i+1)-\infty_5$ $(\infty_2,2i+1,2i+2)-\infty_5$ for $i=0,1,2,3$ and $(\infty_3,0,3)-\infty_6$ $(\infty_3,6,1)-3$ $(\infty_3,4,7)-\infty_6$ $(\infty_3,2,5)-3$ $(\infty_4,3,6)-0$ $(\infty_4,4,1)-7$ $(\infty_4,2,7)-3$ $(\infty_4,5,0)-2$ $(\infty_6,0,4)-2$ $(\infty_6,1,5)-7$ $(\infty_6,2,6)-4$

- $v=7$ on the vertex set $\mathbb{Z}_8 \cup \infty_1,\dots,\infty_7$: $(\infty_1,2i,2i+1)-\infty_6$ $(\infty_2,2i+1,2i+2)-\infty_6$ $(\infty_3,2i,2i+3)-\infty_7$ for $i=0,1,2,3$ and $(\infty_4,3,6)-\infty_7$ $(\infty_4,1,4)-\infty_7$ $(\infty_4,2,7)-1$ $(\infty_4,0,5)-3$ $(\infty_5,0,4)-6$ $(\infty_5,2,6)-0$ $(\infty_5,5,1)-3$ $(\infty_5,3,7)-5$ $(\infty_7,0,2)-4$

- $v=8$ on the vertex set $\mathbb{Z}_8 \cup \infty_1,\dots,\infty_8$: $(\infty_1,2i,2i+1)-\infty_7$ $(\infty_2,2i+1,2i+2)-\infty_7$ $(\infty_3,2i,2i+3)-\infty_8$ for $i=0,1,2,3$ and $(\infty_4,6,3)-7$ $(\infty_4,4,1)-\infty_6$ $(\infty_4,7,2)-\infty_8$ $(\infty_4,5,0)-\infty_8$ $(\infty_5,0,2)-6$ $(\infty_5,4,6)-\infty_8$ $(\infty_5,3,1)-7$ $(\infty_5,5,7)-\infty_6$ $(\infty_6,2,4)-\infty_8$ $(\infty_6,6,0)-4$ $(\infty_6,3,5)-1$

- $v=9$ on the vertex set $\mathbb{Z}_8 \cup \infty_1,\dots,\infty_9$: $(\infty_1,2i,2i+1)-\infty_8$ $(\infty_2,2i+1,2i+2)-\infty_8$ for $i=0,1,2,3$ and $(\infty_3,2,0)-\infty_9$ $(\infty_3,4,6)-\infty_9$ $(\infty_3,3,1)-5$ $(\infty_3,7,5)-\infty_9$ $(\infty_4,4,2)-\infty_9$ $(\infty_4,0,6)-\infty_7$ $(\infty_4,5,3)-7$ $(\infty_4,1,7)-\infty_7$ $(\infty_5,0,3)-\infty_9$ $(\infty_5,6,1)-\infty_9$ $(\infty_5,4,7)-\infty_9$ $(\infty_5,5,2)-6$ $(\infty_6,6,3)-\infty_7$ $(\infty_6,4,1)-\infty_7$ $(\infty_6,7,2)-\infty_7$ $(\infty_6,0,5)-\infty_7$ $(\infty_7,0,4)-\infty_9$

- $v=10$ on the vertex set $\mathbb{Z}_8 \cup \infty_1,\dots,\infty_{10}$: $(\infty_1,2i,2i+1)-\infty_{10}$ $(\infty_2,2i+1,2i+2)-\infty_{10}$ $(\infty_5,2i,2i+3)-\infty_8$ for $i=0,1,2,3$ and $(\infty_3,0,2)-\infty_9$ $(\infty_3,4,6)-\infty_9$ $(\infty_3,5,7)-\infty_9$ $(\infty_3,1,3)-\infty_9$ $(\infty_4,2,4)-\infty_9$ $(\infty_4,0,6)-\infty_8$ $(\infty_4,5,3)-7$ $(\infty_4,7,1)-\infty_9$ $(\infty_6,6,3)-\infty_7$ $(\infty_6,1,4)-\infty_8$ $(\infty_6,2,7)-\infty_7$ $(\infty_6,5,0)-\infty_8$ $(\infty_7,1,5)-\infty_9$ $(\infty_7,6,2)-\infty_8$ $(\infty_7,4,0)-\infty_9$