New orthogonal designs and sequences with two and three variables in order 28

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Abstract

We give new sets of sequences with zero autocorrelation function and entries from the set $\{0, \pm a, \pm b, \pm c\}$ where a, b and c are commuting variables (which may also be set zero). Then we use these sequences to construct some new orthogonal designs.

We show the known necessary conditions for the existence of an $OD(28; s_1, s_2, s_3)$ plus the condition that $(s_1, s_2, s_3) \neq (1, 5, 20)$ are sufficient conditions for the existence of an $OD(28; s_1, s_2, s_3)$. We also show the known necessary conditions for the existence of an $OD(28; s_1, s_2, s_3)$ constructed using four circulant matrices are sufficient conditions for the existence of 4-$NPAF(s_1, s_2, s_3)$ of length n for all lengths $n \geq 7$.

We establish asymptotic existence results for $OD(4N; s_1, s_2)$ for $3 \leq s_1 + s_2 \leq 28$. This leaves no cases undecided for $1 \leq s_1 + s_2 \leq 28$. We show the known necessary conditions for the existence of an $OD(28; s_1, s_2)$ with $25 \leq s_1 + s_2 \leq 28$, constructed

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using four circulant matrices, plus the condition that $(s_1, s_2) \neq (1,26)$, $(2,25)$, $(7,19)$, $(8,19)$ or $(13,14)$, are sufficient conditions for the existence of 4-$NPAF(s_1, s_2)$ of length n for all lengths $n \geq 7$.

Key words and phrases: Autocorrelation, construction, sequence, orthogonal design.

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1 Introduction

Throughought this paper we will use the definitions and notation of Koukouvinos, Mitrouli, Seberry and Karabelas [4]. Parts (i) and (ii) of the next theorem appear in Geramita and Seberry [2]. Part (iii) appears in Eades [1].

Theorem 1 (Necessary Conditions) *Write $(s_i, s_j)_p$ for the Hilbert norm residue symbol. The following conditions are necessary for the existence of an $OD(4n; s_1, s_2, \cdots, s_\ell)$ in orders with n odd:*

i) for $\ell = 2$, $(-1, s_1 s_2)_p (s_1, s_2)_p = 1$ for all primes p;

ii) for $\ell = 3$, $(s_1, s_2)_p (s_1, s_3)_p (s_2, s_3)_p (-1, s_1 s_2 s_3)_p = 1$ for all primes p; if $s_1 + s_2 + s_3 = n - 1$ then $s_1 s_2 s_3$ is a square and $(s_1, s_2)_p (s_1, s_3)_p (s_2, s_3)_p = 1$ for all primes p;

The Goethals-Seidel construction can only be used if

iii) there exists a $t \times 4$ integer matrix P (called the sum-fill matrix), with all entries of modulus $\leq n$ which satisfies $PP^T = diag(s_1, s_2, \cdots, s_t)$, $t \leq 4$.

In this paper there are no 2- or 3-tuples which satisfy (i) and (ii) which do not also satisfy (iii). However in others orders, such as 20, this does happen. For example the following matrix

$$P = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 1 \\ 0 & 2 & -2 & 0 \end{pmatrix}$$

is a 3×4 integer matrix satisfying $PP^T = diag(3, 7, 8)$ and yet an exhaustive search has not found suitable sequences to construct the $OD(20; 3, 7, 8)$ using the Goethals-Seidel array neither is any other $OD(20; 3, 7, 8)$ known.

2 New orthogonal designs

2.1 Four variable designs

We note that the $OD(28;4,4,9,9)$ is known but not constructed from circulant matrices. We have carried out a computer search, which we believe was exhaustive, to find:

Lemma 1 *The following orthogonal designs*

$$OD(28;1,4,9,9) \quad OD(28;1,8,8,9) \quad OD(28;2,8,9,9)$$
$$OD(28;3,6,8,9) \quad OD(28;4,4,4,9) \quad OD(28;4,4,9,9)$$

cannot be constructed using four circulant matrices in the Goethals-Seidel array.

2.2 Three variable designs

In the next theorem the designs for (3,6,16), (3,8,15), (4,6,11), (8,8,9), and (8,9,9) are not new but giveb in [3]. We quote the sequences to construct them in Appendix C for completeness. Thus we have

Theorem 2 *There exist orthogonal designs* $OD(4n;s_1,s_2,s_3)$ *where* (s_1,s_2,s_3) *is one of the 3-tuples*

(1,1,17)	(1,3,14)	(1,6,11)	(1,8,11)	(1,8,16)
(1,9,16)	(2,5,7)	(2,7,10)	(2,7,13)	(2,8,18)
(3,4,14)	(3,6,8)	(3,6,16)	(3,7,11)	(3,8,10)
(3,8,15)	(3,9,14)	(4,4,13)	(4,5,14)	(4,6,11)
(5,5,13)	(5,10,10)	(7,8,13)	(8,8,9)	(8,9,9)

for all $n \geq 7$, *constructed using four circulant matrices in the Goethals-Seidel array.*

Proof. We use the 4-$NPAF$ given in Appendices A and C, as the first rows of circulant matrices in the Goethals-Seidel array to obtain the required orthogonal designs. Since these sequences have zero non-periodic autocorrelation function, the sufficient zeros are added to the end to make their length $n \geq 7$.

□

Using the design for $(4,9,13)$ which is given in Koukouvinos [3] we find:

Theorem 3 *There exist orthogonal designs $OD(28; s_1, s_2, s_3)$, constructed using four circulant matrices in the Goethals-Seidel array, where (s_1, s_2, s_3) is one of the 3-tuples*

$(1,1,17)$	$(1,3,14)$	$(1,6,11)$	$(1,8,11)$	$(1,8,16)$
$(1,8,17)$	$(1,9,16)$	$(2,4,22)$	$(2,5,7)$	$(2,7,10)$
$(2,7,13)$	$(2,8,18)$	$(2,9,11)$	$(2,9,17)$	$(3,4,14)$
$(3,6,8)$	$(3,6,16)$	$(3,6,17)$	$(3,7,11)$	$(3,7,15)$
$(3,8,10)$	$(3,8,15)$	$(3,9,14)$	$(3,10,15)$	$(4,4,13)$
$(4,5,14)$	$(4,6,11)$	$(4,9,13)$	$(4,10,11)$	$(5,5,13)$
$(5,5,18)$	$(5,9,14)$	$(5,10,10)$	$(6,7,8)$	$(6,9,11)$
$(7,8,10)$	$(7,8,13)$	$(8,8,9)$	$(8,9,9)$	$(8,9,11)$
$(9,9,10)$.				

Proof. We use the sequences given in Appendices A, B and C, which have zero periodic and non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain the required orthogonal designs.

□

We have carried out a computer search, which we believe was exhaustive, to find:

Lemma 2 *$OD(28; 1, 5, 20)$ and $OD(20; 3, 7, 8)$ cannot be constructed using four circulant matrices in the Goethals-Seidel array.*

Theorem 4 *There are no orthogonal designs $OD(4n; s_1, s_2, s_3)$, for any odd $n \geq 7$, constructed using four circulant matrices in the Goethals-Seidel array, where (s_1, s_2, s_3) is one of the 3-tuples*

$(1,3,22)$	$(1,5,19)$	$(2,5,15)$	$(2,6,11)$	$(2,6,17)$
$(2,11,11)$	$(2,11,13)$	$(2,11,15)$	$(3,7,10)$	$(3,11,14)$
$(4,5,19)$	$(5,6,14)$	$(5,6,15)$	$(5,7,10)$	$(5,7,14)$
$(6,8,11)$	$(7,10,11)$,			

Proof. There is no integer sum-fill matrix P as described in Theorem 1(iii). □

An exhaustive computer search showed that the the following 3-tuples do not correspond to 4-$NPAF$.

Theorem 5 *There are no 4-$NPAF(s_1, s_2, s_3)$ or 4-$NPAF(s_1, s_2, s_3, s_4)$ of length 5 for the following 3- and 4-tuples*

$(1,3,14)$ $(1,4,13)$ $(1,5,20)$ $(3,7,8)$ $(1,3,6,8)$ $(1,4,4,9)$

Proof.

The existence of 4-$NPAF(1,3,6,8)$ of length 5, after equating the last two variables would prove the existence of 4-$NPAF(1,3,14)$ of length 5; hence the nonexistence of 4-$NPAF(1,3,14)$ of length 5 gives the nonexistence of 4-$NPAF(1,3,6,8)$.

Also, the existence of 4-$NPAF(1,4,4,9)$, after equating the last two variables would prove the existence of 4-$NPAF(1,4,13)$ of length 5: the nonexistence of 4-$NPAF(1,4,13)$ of length 5 gives the nonexistence of 4-$NPAF(1,4,4,9)$. □

We have carried out a computer search, which we believe was exhaustive, to find:

Lemma 3 *The necessary conditions given by Theorem* 1 *plus the condition that* $(s_1, s_2, s_3) \neq (1,5,20)$ *are sufficient conditions for the existence of an* $OD(28; s_1, s_2, s_3)$.

Proof. See Table 1. □

Theorem 6 *There are no* 4-$NPAF(s_1, s_2, s_3)$ *of length* 7 *for the following* 3-*tuples*

(1,1,25)	(1,1,26)	(1,2,25)	(1,8,17)	(1,8,18)	(1,8,19)
(1,9,13)	(1,10,14)	(1,13,13)	(1,13,14)	(2,4,22)	(2,7,19)
(2,9,17)	(2,13,13)	(3,6,17)	(3,7,15)	(3,10,15)	(4,4,18)
(4,9,13)	(4,10,11)	(5,5,18)	(5,9,9)	(5,9,14)	(6,7,8)
(6,9,11)	(7,8,10)	(8,9,11)	(9,9,9)	(9,9,10).	

Proof. We have carried out a computer search, which we believe was exhaustive, to find that there are no 4-$NPAF(s_1, s_2)$ for the following 2-tuples

(1,26) (2,25) (7,19) (8,19) (13,14).

Hence, there can be no 4-$NPAF(s_1, s_2, s_3)$ of length 7 for the following 3-tuples

(1,1,25)	(1,1,26)	(1,2,25)	(1,8,18)	(1,8,19)	(1,13,13)
(1,13,14)	(2,7,19).				

A complete computer search has also given that there are no 4-$NPAF(s_1, s_2, s_3)$ of length 7 for the following 3-tuples

(1,8,17)	(1,9,13)	(1,10,14)	(2,4,22)	(2,9,17)	(2,13,13)
(3,6,17)	(3,7,15)	(3,10,15)	(4,4,18)	(4,9,13)	(4,10,11)
(5,5,18)	(5,9,9)	(5,9,14)	(6,7,8)	(6,9,11)	(7,8,10)
(8,9,11)	(9,9,9)	(9,9,10)			

□

Remark 1 The 236 3-tuples given in Table 1 are possible types of orthogonal designs in order 28 which might be made using four sequences. There are no cases unresolved. We use

n if they are made from 4-sequences with zero NPAF and length n;

P if they are made from 4-sequences with zero PAF;

X if they do not exist because the integer sum-fill matrix does not exist;

Y as it does not exist by what we believe was an exhaustive computer search for length 7.

In the Table a design marked by * is known in all orders $4n \geq 24$; a † indicates the design is known for $4n \geq 28$ from [3]. □

2.3 Two variable designs

Lemma 4 *The existence of an* $OD(28; s_1, s_2)$ *plus condition (iii) from Theorem 1 is equivalent to the existence of* 4-$NPAF(s_1, s_2)$, *with the possible exception of* $(s_1, s_2) = (1,26), (2,25), (7,19), (8,19)$, *and* $(13,14)$.

Proof. The theoretical necessary conditions of Theorem 1 are used to prove the following 2-tuples (s_1, s_2) do not give orthogonal designs $OD(28; s_1, s_2)$:

(1,7)	(3,5)	(4,7)	(5,12)	(7,9)	(9,15)	(12,13)
(1,15)	(3,13)	(4,15)	(5,19)	(7,16)	(10,17)	(12,15)
(1,23)	(3,20)	(4,23)	(5,22)	(7,17)	(11,13)	
(2,14)	(3,21)	(5,11)	(6,10)	(8,14)	(11,16).	

The specific cases mentioned in the enunciation have been eliminated by exhaustive computer search.

The 4-$NPAF(s_1, s_2)$ can easiy by used in the Goethals-Seidel construction to form $OD(28; s_1, s_2)$ giving the sufficiency. □

3 Asymptotic Results

Tables 2, 3 and 4 indicate the smallest known length, ℓ, such that 4-$NPAF(4\ell; s_1, s_2)$, with $\sigma = s_1 + s_2 \leq 28$ exist for every length $\geq \ell$. Every case for $1 \leq \sigma \leq 28$ is resolved.

All the results not given in this paper may be found in Geramita and Seberry [2, p168] and [4, 5].

(1,1,1)	1	(1,5,16)	7	(2,4,9)	5	(3,4,18)	7	(4,9,10)	7
(1,1,2)	1	(1,5,19)	X	(2,4,11)	5	(3,6,6)	5	(4,9,13)†	7
(1,1,4)	2	(1,5,20)	Y	(2,4,12)	7	(3,6,8)	5	(4,10,10)	7
(1,1,5)	3	(1,6,8)	5	(2,4,16)	7	(3,6,9)	5	(4,10,11)	P
(1,1,8)	3	(1,6,11)	5	(2,4,17)	7	(3,6,11)	5	(4,10,14)	7
(1,1,9)	7	(1,6,12)	7	(2,4,18)	7	(3,6,12)	7	(5,5,5)	7
(1,1,10)	3	(1,6,14)	7	(2,4,19)	7	(3,6,16)†	7	(5,5,8)	7
(1,1,13)	5	(1,6,18)	7	(2,4,22)	P	(3,6,17)	P	(5,5,9)	5
(1,1,16)	7	(1,6,21)	7	(2,5,5)	3	(3,6,18)	7	(5,5,10)	5
(1,1,17)*	P	(1,8,8)	7	(2,5,7)	5	(3,6,19)	7	(5,5,13)	P
(1,1,18)	6	(1,8,9)	5	(2,5,8)	5	(3,7,8)	6	(5,5,16)	7
(1,1,20)	6	(1,8,11)	5	(2,5,13)	6	(3,7,10)	X	(5,5,18)	P
(1,1,25)	P	(1,8,12)	7	(2,5,15)	X	(3,7,11)	7	(5,6,9)	7
(1,1,26)	P	(1,8,16)	7	(2,5,18)	7	(3,7,15)	P	(5,6,14)	X
(1,2,2)	2	(1,8,17)	P	(2,6,7)	5	(3,7,18)	7	(5,6,15)	X
(1,2,3)	2	(1,8,18)	P	(2,6,9)	5	(3,8,9)	7	(5,7,8)	7
(1,2,4)	2	(1,8,19)	P	(2,6,11)	X	(3,8,10)*	P	(5,7,10)	X
(1,2,6)	3	(1,9,9)	7	(2,6,12)	6	(3,8,15)†	7	(5,7,14)	X
(1,2,8)	3	(1,9,10)	5	(2,6,13)	7	(3,9,14)	7	(5,8,8)	7
(1,2,9)	3	(1,9,13)*	P	(2,6,16)	7	(3,10,15)	P	(5,8,13)	7
(1,2,11)	5	(1,9,16)	7	(2,6,17)	X	(3,11,14)	X	(5,9,9)*	P
(1,2,12)	5	(1,9,18)	7	(2,7,10)	7	(4,4,4)	3	(5,9,10)*	P
(1,2,16)	7	(1,10,10)	7	(2,7,12)	7	(4,4,5)	5	(5,9,14)	P
(1,2,17)	5	(1,10,11)	7	(2,7,13)	7	(4,4,8)	7	(5,10,10)	7
(1,2,18)	6	(1,10,14)	P	(2,7,19)	P	(4,4,9)	5	(6,6,6)	7
(1,2,19)	6	(1,13,13)	P	(2,8,8)	5	(4,4,10)	5	(6,6,12)	7
(1,2,22)	7	(1,13,14)	P	(2,8,9)	5	(4,4,13)	7	(6,7,8)	P
(1,2,25)	P	(2,2,2)	2	(2,8,10)	5	(4,4,16)	7	(6,8,9)	7
(1,3,6)	3	(2,2,4)	2	(2,8,13)	7	(4,4,17)	7	(6,8,11)	X
(1,3,8)	3	(2,2,5)	3	(2,8,16)	7	(4,4,18)	P	(6,8,12)	P
(1,3,14)	6	(2,2,8)	3	(2,8,18)	7	(4,4,20)	7	(6,9,11)	P
(1,3,18)	6	(2,2,9)	5	(2,9,9)	5	(4,5,5)	5	(7,7,7)	7
(1,3,22)	X	(2,2,10)	5	(2,9,11)	6	(4,5,6)	5	(7,7,14)	7
(1,3,24)	7	(2,2,13)	5	(2,9,12)	7	(4,5,9)	5	(7,8,10)	P
(1,4,4)	5	(2,2,16)	7	(2,9,17)	P	(4,5,14)*	P	(7,8,13)	7
(1,4,5)	5	(2,2,17)	7	(2,10,10)	6	(4,5,16)	7	(7,10,11)	X
(1,4,8)	5	(2,2,18)	6	(2,10,12)	6	(4,5,19)	X	(8,8,8)	7
(1,4,9)	5	(2,2,20)	7	(2,11,11)	X	(4,6,8)	5	(8,8,9)†	7
(1,4,10)	5	(2,3,4)	3	(2,11,13)	X	(4,6,11)†	7	(8,8,10)	7
(1,4,13)	7	(2,3,6)	3	(2,11,15)	X	(4,6,12)	7	(8,9,9)†	7
(1,4,16)	7	(2,3,7)	3	(2,13,13)	P	(4,6,14)	7	(8,9,11)	P
(1,4,17)	7	(2,3,9)	5	(3,3,3)	3	(4,6,18)	7	(8,10,10)	7
(1,4,18)	7	(2,3,10)	7	(3,3,6)	3	(4,8,8)	7	(9,9,9)	P
(1,4,20)	7	(2,3,15)	7	(3,3,12)	7	(4,8,9)	7	(9,9,10)	P
(1,5,5)	3	(2,3,16)	7	(3,3,15)	7	(4,8,11)	7		
(1,5,6)	3	(2,4,4)	3	(3,4,6)	5	(4,8,12)	7		
(1,5,9)	5	(2,4,6)	3	(3,4,8)	5	(4,8,16)	7		
(1,5,14)	5	(2,4,8)	5	(3,4,14)*	P	(4,9,9)	6		

Table 1: Census of 3-variable designs in order 28.

Theorem 7 *Suppose s_1, s_2 and N are such that the necessary conditions of Theorem 1 are satisfied. Then an $OD(4N; s_1, s_2)$ exists for*

(i) $N \geq 2$ *for* $2 \leq s_1 + s_2 \leq 8$;

(ii) $N \geq 4$ *for* $9 \leq s_1 + s_2 \leq 16$;

(iii) $N \geq 5$ *for* $17 \leq s_1 + s_2 \leq 20$ *except possibly for the 2-tuples* $(3,16)$, $(6,13)$, $(7,11)$, $(7,12)$ *which are the types of orthogonal designs for orders* $4N$, $N \geq 6$;

(iv) $N \geq 6$ *for* $21 \leq s_1 + s_2 \leq 24$;

(v) $N \geq 7$ *for* $25 \leq s_1 + s_2 \leq 28$.

Proof. Every 2-tuple with $2 \leq s_1 + s_2 \leq 8$ is the type of an $OD(8; s_1, s_2)$ from [2, p.368]. This combined with Table 2 gives the result of (i).

For (ii), every 2-tuple with $9 \leq s_1 + s_2 \leq 16$ is the type of an $OD(16; s_1, s_2)$ from [2, p.389]. This combined with Table 3 gives the result. Geramita and Seberry give the existence of $OD(24; s_1, s_2)$ for $(s_1, s_2) = (3,16)$, $(6,13)$, $(7,11)$ and $(7,12)$. Hence, using Table 2 these are the types of orthogonal designs in all orders $4N$, $N \geq 6$. The remainder of the 2-tuples for which $17 \leq s_1 + s_2 \leq 20$ are given in Table 3. This establishes (iii).

All 2-tuples $2 \leq s_1 + s_2 \leq 24$ are the types of an $OD(24; s_1, s_2)$ from [2, p.391]. This combined with Tables 3 and 4 gives the result for (iv).

Table 4 gives the result for all 2-tuples except $(1,26)$, $(2,25)$, $(7,19)$ $(8,19)$ and $(13,14)$. Table 1 shows an $OD(28; s_1, s_2)$ exists for each of these 2-tuples. [2, p.394 and p.395] shows each of these 5 2-tuples is the type of an orthogonal design in orders 32 and 40. Now [5] gives $OD(4m; s_1, s_2)$, for these (s_1, s_2) with $m \geq 9$ and so we have that these designs exist for all $N \geq 7$, giving (v). □

Table 2: The indicated $4\text{-}NPAF(s_1, s_2)$ with $1 \leq s_1 + s_2 \leq 10$ exist for every length $N \geq \ell$.
The cases marked *no* have been excluded by Theorem 1.

s_1, s_2	σ	ℓ	s_1, s_2	σ	ℓ	s_1, s_2	σ	ℓ	s_1, s_2	σ	ℓ	s_1, s_2	σ	ℓ
(1,1)	2	1	(1,2)	3	1	(1,4)	5	2	(1,6)	7	3	(1,8)	9	3
			(1,3)	4	1	(1,5)	6	2	(1,7)		*no*	(1,9)	10	3
			(2,2)	4	1	(2,3)	5	2	(2,5)	7	3	(2,7)	9	3
						(2,4)	6	2	(2,6)	8	2	(2,8)	10	3
						(3,3)	6	2	(3,4)	7	3	(3,6)	9	3
									(3,5)		*no*	(3,7)	10	3
									(4,4)	8	2	(4,5)	9	3
												(4,6)	10	3
												(5,5)	10	3

Table 3: The indicated 4-$NPAF(s_1, s_2)$ with $11 \leq s_1 + s_2 \leq 20$ exist for every length $N \geq \ell$.
The cases marked *no* have been excluded by Theorem 1.

s_1, s_2	σ	ℓ	s_1, s_2	σ	ℓ	s_1, s_2	σ	ℓ	s_1, s_2	σ	ℓ	s_1, s_2	σ	ℓ
(1,10)	11	3	(1,12)	13	4	(1,14)	15	5	(1,16)	17	5	(1,18)	19	5
(1,11)	12	3	(1,13)	14	5	(1,15)		*no*	(1,17)	18	5	(1,19)	20	5
(2,9)	11	5	(2,11)	13	5	(2,13)	15	5	(2,15)	17	5	(2,17)	19	5
(2,10)	12	3	(2,12)	14	5	(2,14)		*no*	(2,16)	18	5	(2,18)	20	5
(3,8)	11	3	(3,10)	13	5	(3,12)	15	5	(3,14)	17	5	(3,16)	19	7
(3,9)	12	3	(3,11)	14	5	(3,13)		*no*	(3,15)	18	5	(3,17)	20	5
(4,7)		*no*	(4,9)	13	5	(4,11)	15	5	(4,13)	17	5	(4,15)		*no*
(4,8)	12	3	(4,10)	14	5	(4,12)	16	5	(4,14)	18	5	(4,16)	20	5
(5,6)	11	3	(5,8)	13	5	(5,10)	15	5	(5,12)		*no*	(5,14)	19	5
(5,7)	12	3	(5,9)	14	5	(5,11)		*no*	(5,13)	18	5	(5,15)	20	5
(6,6)	12	3	(6,7)	13	5	(6,9)	15	5	(6,11)	17	5	(6,13)	19	7
			(6,8)	14	5	(6,10)		*no*	(6,12)	18	5	(6,14)	20	5
			(7,7)	14	4	(7,8)	15	5	(7,10)	17	5	(7,12)	19	7
						(7,9)		*no*	(7,11)	18	5	(7,13)	20	5
						(8,8)	16	5	(8,9)	17	5	(8,11)	19	5
									(8,10)	18	5	(8,12)	20	5
									(9,9)	18	5	(9,10)	19	5
												(9,11)	20	5
												(10,10)	20	5

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Table 4: The indicated 4-$NPAF(s_1, s_2)$ with $21 \leq s_1 + s_2 \leq 28$ exist for every length $N \geq \ell$.
The cases marked *no* have been excluded by Theorem 1.

s_1, s_2	σ	ℓ	s_1, s_2	σ	ℓ	s_1, s_2	σ	ℓ	s_1, s_2	σ	ℓ
(1, 20)	21	6	(1, 22)	23	7	(1, 24)	25	7	(1, 26)	27	9
(1, 21)	22	6	(1, 23)		*no*	(1, 25)	26	7	(1, 27)	28	7
(2, 19)	21	6	(2, 21)	23	7	(2, 23)	25	7	(2, 25)	27	9
(2, 20)	22	6	(2, 22)	24	6	(2, 24)	26	7	(2, 26)	28	7
(3, 18)	21	6	(3, 20)		*no*	(3, 22)	25	7	(3, 24)	27	7
(3, 19)	22	7	(3, 21)		*no*	(3, 23)	26	7	(3, 25)	28	7
(4, 17)	21	7	(4, 19)	23	7	(4, 21)	25	7	(4, 23)		*no*
(4, 18)	22	6	(4, 20)	24	6	(4, 22)	26	7	(4, 24)	28	7
(5, 16)	21	7	(5, 18)	23	7	(5, 20)	25	7	(5, 22)		*no*
(5, 17)	22	7	(5, 19)		*no*	(5, 21)	26	7	(5, 23)	28	7
(6, 15)	21	7	(6, 17)	23	7	(6, 19)	25	7	(6, 21)	27	7
(6, 16)	22	6	(6, 18)	24	6	(6, 20)	26	7	(6, 22)	28	7
(7, 14)	21	7	(7, 16)		*no*	(7, 18)	25	7	(7, 20)		*no*
(7, 15)	22	7	(7, 17)		*no*	(7, 19)	26	9	(7, 21)	28	7
(8, 13)	21	7	(8, 15)	23	7	(8, 17)	25	7	(8, 19)	27	9
(8, 14)		*no*	(8, 16)	24	6	(8, 18)	26	7	(8, 20)	28	7
(9, 12)	21	7	(9, 14)	23	7	(9, 16)	25	7	(9, 18)	27	7
(9, 13)	22	6	(9, 15)		*no*	(9, 17)	26	7	(9, 19)	28	7
(10, 11)	21	6	(10, 13)	23	7	(10, 15)	25	7	(10, 17)		*no*
(10, 12)	22	6	(10, 14)	24	6	(10, 16)	26	7	(10, 18)	28	7
(11, 11)	22	6	(11, 12)	23	7	(11, 14)	25	7	(11, 16)		*no*
			(11, 13)		*no*	(11, 15)	26	7	(11, 17)	28	7
			(12, 12)	24	6	(12, 13)		*no*	(12, 15)		*no*
						(12, 14)	26	7	(12, 16)	28	7
						(13, 13)	26	7	(13, 14)	27	9
									(13, 15)	28	7
									(14, 14)	28	7

Appendix A: Order 20 (Sequences with zero nonperiodic autocorrelation function)

Design	A_1					A_2					A_3					A_4				
(1, 3, 14)	do not exist with length 5																			
(1, 4, 13)	do not exist with length 5																			
(1, 6, 11)	$-c$	b	a	$-b$	c	b	c	c	0	c	b	c	$-c$	0	$-c$	$-c$	b	c	b	$-c$
(1, 8, 11)	c	b	a	$-b$	$-c$	c	b	b	$-c$	c	b	$-c$	b	c	$-c$	c	c	c	$-b$	b
(2, 5, 7)	b	a	$-b$	0	c	$-c$	0	$-c$	a	c	c	0	b	0	c	$-c$	0	b	0	b
(3, 6, 8)	b	$-c$	b	c	a	$-b$	c	0	c	a	0	$-c$	a	$-c$	$-b$	$-b$	$-c$	b	c	0
(7, 10)	$-b$	b	b	0	a	a	0	$-b$	b	b	a	b	a	b	$-a$	a	$-b$	0	$-b$	a

Appendix B: Order 28 (Sequences with zero periodic autocorrelation function)

Design	A_1							A_2							A_3							A_4						
(1, 8, 17)	a	$-c$	$-c$	c	$-c$	c	c	b	b	0	$-c$	c	c	c	b	b	$-c$	$-c$	c	$-c$	0	$-b$	b	$-b$	c	b	c	c
(2, 4, 22)	c	$-c$	$-c$	c	c	b	$-a$	$-c$	c	$-c$	c	a	b	c	c	$-c$	c	c	c	$-b$	c	$-c$	c	c	c	c	b	$-c$
(2, 9, 11)	a	b	0	$-c$	$-b$	0	0	a	$-b$	c	c	b	$-c$	0	b	b	$-c$	c	b	0	0	c	$-c$	c	c	c	b	$-b$
(2, 9, 17)	a	$-b$	c	$-b$	b	c	b	a	c	$-c$	c	$-c$	$-c$	$-c$	b	b	$-c$	c	b	$-c$	c	$-b$	$-c$	b	c	c	c	c
(3, 6, 17)	a	b	$-c$	c	[illegible]	0	c	a	$-c$	b	c	$-c$	0	$-c$	a	$-b$	$-b$	$-c$	$-c$	c	c	$-b$	b	c	$-c$	c	c	c
(3, 7, 11)	a	b	0	c	$-c$	c	0	a	b	0	c	0	$-c$	0	c	c	b	$-a$	b	$-c$	0	c	b	$-b$	$-b$	c	c	0
(3, 7, 15)	a	$-b$	b	b	c	c	0	a	b	$-c$	$-c$	c	c	$-c$	a	c	$-b$	$-c$	$-b$	$-c$	0	b	$-c$	c	$-c$	0	$-c$	$-c$
(3, 10, 15)	a	$-b$	b	b	c	b	c	a	b	$-b$	$-b$	c	$-c$	$-c$	a	$-c$	$-b$	$-c$	$-c$	c	c	b	$-c$	b	$-c$	c	$-c$	$-c$
(4, 9, 13)	a	a	$-b$	$-c$	c	b	0	b	b	$-c$	c	b	$-c$	c	c	c	c	c	$-c$	b	$-b$	a	$-a$	$-b$	$-c$	$-c$	b	0
(4, 10, 11)	a	a	$-b$	b	0	$-c$	c	$-a$	a	$-c$	b	c	c	$-b$	b	b	b	c	$-c$	0	$-c$	$-b$	b	c	c	0	b	c
(5, 5, 18)	a	b	$-c$	b	c	$-c$	c	a	$-c$	a	c	b	$-b$	$-b$	a	$-c$	$-a$	$-c$	$-c$	c	$-c$	$-c$	$-c$	c	c	c	c	c
(5, 9, 14)	a	$-b$	b	$-c$	c	c	c	a	b	$-c$	a	c	$-b$	$-c$	b	$-b$	$-b$	$-b$	$-c$	$-b$	c	$-a$	c	c	a	c	$-c$	c
(5, 10, 10)	$-a$	a	a	b	c	b	c	a	0	a	$-b$	$-c$	$-c$	c	b	0	$-b$	$-b$	c	c	$-c$	$-b$	b	b	$-c$	0	$-c$	b
(6, 7, 8)	$-c$	c	c	0	c	b	a	c	$-c$	$-c$	0	$-c$	b	a	b	a	0	$-a$	b	0	0	$-b$	a	$-b$	a	b	0	0
(6, 9, 11)	a	$-c$	c	c	c	0	c	a	$-b$	b	c	$-c$	0	$-c$	a	b	$-c$	a	$-b$	$-b$	b	$-a$	b	b	a	c	b	$-c$
(7, 8, 10)	$-a$	a	a	b	b	$-c$	0	$-a$	b	b	c	$-c$	c	0	a	$-b$	$-c$	$-c$	b	$-b$	b	a	$-c$	a	c	c	0	c
(8, 9, 11)	$-c$	b	$-b$	a	c	$-c$	a	b	a	a	c	a	$-a$	$-b$	c	$-a$	b	b	b	$-c$	a	$-c$	c	c	c	b	c	$-b$
(9, 9, 10)	a	a	a	$-b$	b	$-c$	c	$-b$	b	c	b	b	b	$-c$	a	b	$-a$	$-b$	c	$-c$	$-c$	$-a$	a	$-a$	c	a	c	c

Appendix C: Order 28 (Sequences with zero nonperiodic autocorrelation function)

Design	A_1							A_2							A_3							A_4						
(1, 1, 17)	$-c$	a	c	0	0	0	0	$-c$	b	c	0	0	0	0	c	c	c	$-c$	$-c$	c	$-c$	c	c	c	0	c	$-c$	c
(1, 3, 14)	$-c$	0	$-c$	a	c	0	c	c	0	0	0	b	$-c$	c	c	c	$-c$	0	0	c	b	$-c$	$-c$	0	0	$-c$	0	b
(1, 8, 16)	$-c$	$-b$	c	a	$-c$	b	c	b	b	0	c	$-c$	c	$-c$	b	$-c$	$-c$	0	b	c	c	$-c$	$-b$	b	$-c$	0	$-c$	$-c$
(1, 9, 16)	$-c$	$-b$	c	a	$-c$	b	c	c	b	$-c$	b	$-c$	b	c	$-c$	0	$-c$	b	$-c$	$-b$	$-c$	c	0	c	b	$-c$	$-b$	$-c$
(2, 7, 10)	$-c$	0	a	0	c	b	$-c$	b	a	$-b$	0	$-b$	c	0	$-c$	0	0	b	$-c$	0	0	$-c$	b	c	0	c	b	c
(2, 7, 13)	a	b	$-c$	0	$-c$	0	0	a	$-b$	c	0	c	0	0	c	$-c$	$-b$	b	b	c	c	c	c	b	$-c$	b	c	$-c$
(2, 8, 18)	b	c	c	a	$-c$	$-c$	b	$-b$	$-c$	c	a	$-c$	c	$-b$	b	c	$-c$	c	c	c	$-b$	b	c	c	c	$-c$	c	$-b$
(3, 4, 14)	c	c	a	$-c$	c	$-c$	0	c	b	0	a	0	$-b$	c	$-c$	c	a	$-c$	$-c$	$-c$	0	c	b	0	0	0	b	$-c$
(3, 6, 16)	c	b	c	a	$-c$	0	$-c$	c	b	$-c$	0	c	a	$-c$	c	b	$-c$	b	$-c$	$-a$	c	c	0	c	$-b$	c	b	c
(3, 7, 11)	c	0	a	b	0	c	$-c$	c	0	$-c$	0	a	b	0	$-c$	$-c$	$-b$	a	$-b$	c	0	$-c$	0	$-c$	$-b$	b	b	$-c$
(3, 8, 10)	b	c	c	0	a	$-c$	b	$-b$	$-c$	c	0	a	c	$-b$	b	$-c$	0	a	0	$-c$	$-b$	b	c	0	0	0	c	$-b$
(3, 8, 15)	b	c	c	0	a	$-c$	b	b	c	$-c$	0	$-a$	$-c$	b	b	$-c$	$-c$	a	c	$-c$	$-b$	b	c	c	c	$-c$	c	$-b$
(3, 9, 14)	c	$-b$	0	b	c	c	a	$-c$	b	$-c$	a	$-c$	$-b$	c	$-c$	b	0	$-b$	$-c$	c	a	$-c$	b	c	b	c	b	$-c$
(4, 4, 13)	a	$-c$	$-c$	0	c	c	a	b	$-c$	c	0	$-c$	c	b	a	c	0	0	0	c	$-a$	b	c	0	c	0	c	$-b$
(4, 5, 14)	a	0	$-b$	0	b	0	a	c	0	$-c$	b	c	0	c	a	$-c$	b	c	b	$-c$	$-a$	c	c	c	c	$-c$	c	$-c$
(4, 6, 11)	b	0	a	0	a	0	$-b$	c	c	c	b	$-c$	c	0	c	$-c$	c	b	$-c$	$-c$	0	b	0	$-a$	$-c$	a	0	b
(5, 5, 13)	b	c	c	a	$-c$	$-c$	b	a	$-c$	c	$-b$	$-c$	c	a	a	c	0	0	0	c	$-a$	b	c	0	c	0	c	$-b$
(5, 10, 10)	b	c	c	a	c	$-c$	b	$-b$	0	$-b$	a	$-c$	a	b	c	$-c$	$-c$	0	$-b$	b	b	b	0	$-c$	$-a$	b	a	$-c$
(7, 8, 13)	$-c$	b	$-a$	a	a	b	$-c$	$-a$	c	b	$-c$	c	b	c	a	$-c$	$-b$	c	c	b	c	$-c$	b	a	c	a	$-b$	c
(8, 8, 9)	b	$-c$	a	c	b	0	a	b	$-c$	b	c	$-a$	0	$-a$	b	$-c$	$-a$	$-c$	$-b$	$-c$	a	b	$-c$	$-b$	0	a	c	$-a$
(8, 9, 9)	a	c	c	0	c	$-c$	a	a	b	$-c$	0	$-c$	$-b$	a	a	b	$-b$	b	b	b	$-a$	a	c	b	$-c$	$-b$	c	$-a$
(4, 22)	b	0	a	a	$-b$	$-b$	b	$-b$	0	$-a$	a	b	b	b	b	b	$-b$	b	$-b$	b	b	$-b$	b	b	b	$-b$	b	b
(5, 23)	b	$-b$	$-a$	a	a	b	b	$-b$	$-b$	a	$-b$	a	b	b	b	b	$-b$	b	$-b$	b	b	$-b$	b	b	b	$-b$	b	b
(11, 14)	$-a$	a	a	$-b$	$-b$	b	b	b	b	a	0	b	$-a$	a	a	0	a	a	$-b$	b	$-b$	$-b$	b	a	b	b	0	$-a$
(11, 15)	a	b	$-b$	a	b	b	$-a$	a	b	b	0	$-a$	$-b$	a	a	b	$-b$	0	a	$-b$	a	a	b	b	b	$-b$	b	$-a$.
(11, 17)	$-a$	a	a	$-b$	b	b	b	b	a	$-b$	$-b$	$-a$	a	$-b$	$-b$	b	a	a	b	a	$-b$	a	b	$-b$	b	$-a$	b	b