A large collection of designs from a wreath product on 21 points

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Abstract

The total of 4079 2-designs and two 3-designs on 21 points have been found. All these designs have the same group as an automorphism group. This group can be represented as the wreath product of G and H, where G denotes the subgroup of order 3 of $PSL(2,2)$ and H denotes the transitive subgroup of order 21 of $PSL(3,2)$.

In particular, 1, 20, 101, 93, 173, 824 and 2867 values of λ for 2-(21,k,λ) designs have been detected, where k takes values from 4 through 10. Up to our knowledge, 2217 of these λ-values are new (14, 76, 65, 122, 587 and 1353, for k equal to 5, 6, ..., 10, respectively). By Alltop's extension [4], 1353 new 2-(21,10,λ) designs can be extended to the same number of new 3-(22,11,λ) designs.

An extensive search with $t > 2$ and $k < 8$ has given only the 3-(21,6,216) design and the 3-(21,7,1260) design with the same automorphism group.

A t-(v,k,λ) design is a collection $\mathcal{B}$ of k-subsets (called *blocks*) of a v-element set Δ of *points*, which satisfies the property that each t-element subset of Δ is in exactly λ blocks. We also require that no block is repeated.

Given a group M acting on Δ, the Kramer-Mesner method searches for t-(v,k,λ) designs having M as an automorphism group. The group M is a subgroup of the full automorphism group and the collection $\mathcal{B}$ is a union of M-orbits of k-subsets (shortly: k-M-orbits).

The method includes a construction of t-M-orbits and k-M-orbits, computation of the orbit incidence matrix $\Lambda(t,k) = (\lambda_{ij})$ (where λ_{ij} denotes the number of blocks from the j-th k-M-orbit, containing a specified set from the i-th t-M-orbit), and design recognition (by finding those proper sets of the column-set of $\Lambda(t,k)$, that have the uniform row sum λ).

In this paper we will be applying the Kramer-Mesner method to the wreath product of some groups. This product will be described and discussed in the following section.

1 Construction

Let be given two groups G and H acting on the ground-sets Γ and Ω respectively. The wreath product $G \wr H$ is the group which acts on $\Gamma \times \Omega$ as follows ([6], Ch.I, Th.15.3.):

$$(i,\ j)\ (\mathbf{f},\ h)\ =\ (i^{\ \mathbf{f}\ (j)},\ j^{\ h})\ ,$$

where $h \in H$, f is a mapping from Ω into G, $(\mathbf{f}, h) \in G \wr H$, $i \in \Gamma$, $j \in \Omega$.

Groups G and H will be defined as some transitive subgroups of $PSL(2,2)$ and $PSL(3,2)$, respectively. The group $PSL(2,2)$ acts 2-transitively on the projective line Γ of order 2 and is isomorphic to the group $GL(2,2)$ of all regular 2×2 matrices over $GF(2)$. Similarly, the group $PSL(3,2)$ acts 2-transitively on the projective plane Ω of order 2 and is isomorphic to the group $GL(3,2)$ of all regular 3×3 matrices over $GF(2)$.

The group $PSL(2,2)$ is also isomorphic to the symmetric group S_3. We choose G to be its alternating subgroup A_3, which is known to act transitively on Γ. We choose H to be the normalizer of a 7-Sylow subgroup of $PSL(3,2)$. This normalizer is known ([6], Ch.II, Th.6.15.) to act transitively on Ω.

The group $PSL(2,2) \wr PSL(3,2)$ of order $6^7 \cdot 168$ is not computationally tractable. Combining the facts that

- the Kramer-Mesner method searches (for) designs as some unions of orbits ;
- orbits by action of a group are partitioned into orbits by action of its subgroups ([2], Lemma 1) ,

it follows that no design arising by action of $PSL(2,2) \wr PSL(3,2)$ will be missed by considering the action of $G \wr H$.

On the other hand, $G \wr H$ acts transitively on $\Gamma \times \Omega$, since the wreath product inherits transitivity from its constituents ([6], Ch.I, Th.15.3.). This transitivity enables design computation by restricting attention to the reduced orbits, [2].

2 Results on 2-designs

Throughout the remaining part of this paper, considerations will be restricted to the automorphism group $G \wr H$. Therefore, the denotation "k-$(G \wr H)$-orbit" will be abbreviated to "k-orbit". It turns out that there exist 2 2-orbits, 6 3-orbits, 11 4-orbits, 21 5-orbits, 38 6-orbits, 56 7-orbits, 76 8-orbits, 96 9-orbits and 104 10-orbits.

Design recognition, i.e. search over the matrices $\Lambda(2,k)$, has been very much facilitated by the facts that there are only two 2-orbits and that the matrices have many repeated columns. We use these repetitions to abbreviate denotations for $\Lambda(2,k)$ by writing down only the non-repeating columns, with the additional third row containing data on multiplicity (on the number of repetitions). The abbreviated tables will be denoted by $T(k)$; the third row in such a table will be separated by a horizontal line.

We have performed the complete search for 2-designs with the automorphism group $G \wr H$.

2.1 Matrices $\Lambda(2,k)$

We list the matrices $\Lambda(2,k)$ in their abbreviated forms $T(k)$ (frequencies of the columns are listed in the third row of the tables):

3	9	2	0
0	0	9	1
2	1	2	1

T(3)
$\lambda_{max} = 19$
6 3-orbits

54	18	15	4	1
0	0	27	18	9
1	2	5	1	2

T(4)
$\lambda_{max} = 171$
11 4-orbits

270	2	7	24	27	81
0	12	27	54	27	81
1	2	5	5	2	6

T(5)
$\lambda_{max} = 969$
21 5-orbits

405	1	11	12	36	117	378
0	6	36	27	81	162	243
1	1	10	4	7	10	5

T(6)
$\lambda_{max} = 3876$
38 6-orbits

243	5	16	51	54	162	513	1620
0	18	45	108	81	243	486	729
1	5	5	20	2	11	10	2

T(7)
$\lambda_{max} = 11628$
56 7-orbits

7	22	69	72	216	675	2187	2106
21	54	135	108	324	729	729	1458
5	10	20	2	21	12	1	5

T(8)
$\lambda_{max} = 27132$
76 8-orbits

1	3	29	30	90	279	891	864	2673	8262
3	9	63	54	162	405	729	972	2187	4374
2	1	20	2	16	30	3	15	6	1

T(9)
$\lambda_{max} = 50388$
96 9-orbits

4	12	37	114	351	1080	3402	3321	10206
9	27	72	189	486	1215	2187	2916	6561
2	6	10	30	25	21	2	7	1

T(10)
$\lambda_{max} = 75582$
104 10-orbits

2.2 Design parameters

All the parameters corresponding to 2-(21,k,λ) designs with $\lambda \leq \lambda_{max}/2$ and with $G \wr H$ as an automorphism group are listed below, for $4 \leq k \leq 10$ (there are no such designs with $k = 3$). An effort has been made to abbreviate denotations by grouping the parameters into suitably chosen families.

1 2-$(21, 4, 3 \cdot s)$ **design** , for $s = 27$.

20 2-$(21, 5, s)$ **designs** , for $s = 27 \cdot t + r$, where

$r \in \{0, 24\}$,

$t \in \{1, ..., 17\}$ for $r = 0$,

$t \in \{15, 16, 17\}$ for $r = 24$.

101 2-$(21, 6, s)$ **designs** , for $s = 27 \cdot t + r$, where r and t are given in the following table (three dots correspond to a subsequence of consecutive values of t):

r	t	
0	17, 18, 20, 21, 24...42, 44...69, 71	50 designs
15	16, 17, 20, 23...45, 47...71	51 designs

93 2-$(21, 7, 9 \cdot s)$ **designs** , for $s = 28 \cdot t + r$, where r and t are given by:

r	t	
0	9...14, 18...23	12 designs
4	5, 6, 8...10, 14...19	11 designs
7	2, 6, 11...16, 20...22	11 designs
11	2, 7, 8, 10...12, 16...21	12 designs
14	4, 8, 9, 13...18, 22	10 designs
18	1, 4, 9...14, 18...22	13 designs
21	6, 9...11, 15...20	10 designs
25	2, 3, 5...7, 11...16, 20...22	14 designs

173 2-$(21, 8, 42 \cdot s)$ **designs** , for $s = 9 \cdot t + r$, where

$r \in \{0, 1, 2, 5, 6, 7\}$,

$t \in \{7, ..., 35\}$ for $r \in \{1, 2, 5, 6, 7\}$,

$t \in \{8, ..., 35\}$ for $r = 0$.

824 $2\text{-}(21, 9, 6 \cdot s)$ **designs** , for $s = 27 \cdot t + r$, where r and t are given by:

r	t	
0	13...15, 18, 20, 21, 24...155	138 designs
1	13, 14, 19, 20, 24...29, 31...155	135 designs
6	11, 13, 14, 19...22, 24...155	139 designs
11	6, 7, 13, 19...22, 24...28, 30...155	138 designs
17	6, 17...21, 24...154	137 designs
22	12...14, 17, 19, 20, 23, 25...154	137 designs

2867 $2\text{-}(21, 10, 9 \cdot s)$ **designs** , for $s = 27 \cdot t + r$, where r and t are given by:

r	t	
0	16, 18, 21...34, 36...155	136 designs
1, 2	17, 20...30, 32, 33, 35...155	2 · 135 designs
3	16, 17, 20...155	138 designs
5, 11	16, 19, 21...155	2 · 137 designs
6, 7	16, 17, 19...155	2 · 139 designs
9, 10, 13, 14	15, 16, 19, 21...155	4 · 138 designs
15, 18, 19	15, 16, 19, 21...154	3 · 137 designs
17, 23	15, 21...33, 35...154	2 · 134 designs
21, 22, 25, 26	15, 20...33, 35...154	4 · 135 designs

Theorem. *Let G denote the subgroup of order 3 of $PSL(2,2)$ and let H denote the transitive subgroup of order 21 of $PSL(3,2)$. There exist 2-(21,k,λ) designs with the automorphism group equal to the wreath product $G \wr H$, with $k \in \{4,5,...,10\}$ and with all 4079 λ-values described in this section. Direct action of this wreath product on the cartesian product of the projective line of order 2 and the projective plane of order 2 (Fano plane) does not give 2-(21,k,λ) designs with other values of λ.*

By using the Alltop's extension [4], the obtained 2-(21,10,λ) designs can be extended to 3-(22,11,λ) designs. This implies the following

Corollary. *There exist 3-(22,11,λ) designs with 2867 λ values described in this section for $k = 10$.*

2.3 Retrieved design parameters

Some of the design parameters listed above are not new and can be found in the paper [8] and in the catalogue contained in [5]. Given a value k from the set $\{3,4,...,10\}$, we define five sets $S(k)$, $A(k)$, $B(k)$,

$N(k)$, $T(k)$. These sets contain λ-values for $2-(21,k,\lambda)$ designs which respectively:

– have $G \wr H$ as an automorphism group (denotation $S(k)$)

– belong to $S(k)$ and are listed in [8] (denotation $A(k)$)

– belong to $S(k)$ and are listed in [5] (denotation $B(k)$)

– are new up to our knowledge (denotation $N(k)$)

– are not greater than $\lambda_{max}/2$ and are theoretically possible by divisibility condition [5] (denotation $T(k)$).

The following table contains the corresponding cardinalities:

k	3	4	5	6	7	8	9	10
$\|S(k)\|$	0	1	20	101	93	173	824	2867
$\|A(k)\|$	0	0	3	21	23	44	203	1438
$\|B(k)\|$	0	1	4	6	9	14	46	155
$\|A(k) \cap B(k)\|$	0	0	1	2	4	7	12	79
$\|N(k)\|$	0	0	14	76	65	122	587	1353
$\|T(k)\|$	9	28	484	1938	1938	969	4199	4199

It might be worth mentioning that more than two thirds (2867 out of 4199) of possible λ-values for 2-(21,10,λ) designs are obtained with the group $G \wr H$.

Remark. The 1353 λ-values that are new with 2-(21,10,λ) designs are also new with 3-(22,11,λ) designs.

3 Results on designs with $t > 2$

A complete search for $t > 2$ and for $k < 8$ gives that the only $t-(21,k,\lambda)$ designs in this class that arise from the considered wreath product have the parameters 3-(21,6,216) and 3-(21,7,1260).

More precisely:

The 3-(21,6,216) design corresponds to the column combination $\{1,7,8,14,15,20,21,25,27,28,29,31,32,33\}$ of the 6×38 matrix $\Lambda(3,6)$ given below.

The 3-(21,7,1260) design corresponds to the column combination $\{6,8,9,14,17,19,20,36,44,51,52,53,3,12,15,27,28,34,41,42\}$ of the 6×56 matrix $\Lambda(3,7)$ given below.

The 7-th row of the following two matrices is used merely to represent the ordinal numbers of the columns.

The matrix $\Lambda(3,6)$:

108	108	81	81	81	81	36	36	18	9	36	18	9
108	81	81	81	108	81	36	18	36	9	18	36	9
108	81	90	90	81	90	12	18	18	4	18	18	4
0	54	81	54	54	27	36	27	36	9	36	45	18
0	54	27	54	54	81	36	45	36	18	36	27	9
0	0	0	0	0	0	0	0	0	81	0	0	81
1	2	3	4	5	6	7	8	9	10	11	12	13
18	9	3	36	18	9	1	8	6	6	0	0	0
18	3	9	0	0	0	0	0	0	0	36	18	9
24	6	6	24	30	7	3	0	0	0	24	30	7
36	9	18	45	27	18	0	6	7	6	27	45	9
36	18	9	27	45	9	9	6	6	7	45	27	18
0	81	81	0	0	81	27	0	27	27	0	0	81
14	15	16	17	18	19	20	21	22	23	24	25	26
0	0	0	0	0	0	0	0	0	0	0	0	
1	8	6	6	0	0	0	0	0	0	0	0	
3	0	0	0	8	2	2	2	2	2	2	0	
9	6	7	6	18	10	7	9	4	6	3	1	
0	6	6	7	18	3	6	4	9	7	10	1	
27	0	27	27	0	27	27	27	27	27	27	6	
27	28	29	30	31	32	33	34	35	36	37	38	

The matrix $\Lambda(3,7)$:

81	486	486	162	162	162	162	54	135	108	108	36	108	108
81	486	486	135	108	162	108	36	108	162	108	36	162	135
81	486	486	126	135	117	135	36	144	135	153	42	135	144
0	243	162	162	135	135	108	54	162	162	135	81	135	108
0	162	243	108	135	135	162	54	108	108	135	27	135	162
0	0	0	0	0	0	0	243	0	0	0	243	0	0
1	2	3	4	5	6	7	8	9	10	11	12	13	14
36	135	36	36	72	36	18	18	36	18	18	9	6	36
36	162	54	36	36	72	18	18	36	9	6	18	18	36
42	126	36	42	24	24	5	5	36	8	9	8	9	36
54	108	54	27	63	72	24	27	63	18	21	21	30	72
54	162	54	81	72	63	27	24	72	33	30	30	21	63
243	0	243	243	0	0	81	81	0	81	81	81	81	0
15	16	17	18	19	20	21	22	23	24	25	26	27	28
18	18	9	6	9	6	24	36	18	18	9	6	12	3
9	6	18	18	6	9	0	0	0	0	0	0	0	0
8	9	8	9	12	12	12	48	11	11	14	15	0	0
30	21	33	30	21	30	27	63	33	30	27	12	11	4
21	30	18	21	30	21	18	72	18	21	24	39	11	4
81	81	81	81	81	81	0	0	81	81	81	81	27	18
29	30	31	32	33	34	35	36	37	38	39	40	41	42
0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	36	18	18	9	6	12	3	0	0	0	0	0	0
12	48	11	11	14	15	0	0	4	4	4	1	1	1
18	72	21	18	24	39	11	4	14	11	8	5	4	3
27	63	30	33	27	12	11	4	8	11	14	3	4	5
0	0	81	81	81	81	27	18	27	27	27	18	18	18
43	44	45	46	47	48	49	50	51	52	53	54	55	56

3.1 Some non-existence results

It has been checked by backtracking that the found 3-(21,6,216) design is not a 4-(21,6,36) design at the same time. To prove that the found 3-(21,7,1260) design is not a 4-(21,7,280) design, we considered the second row of the matrix $\Lambda(4,7)$. One of the entries in that row is equal to 8; all the remaining entries in that row are divisible by 3. Non-existence of the 4-design follows from the fact that neither 280 nor 272 is divisible by 3.

There do not exist 3-(21,7,540) and 3-(21,7,855) designs with the considered wreath product (540 and 855 were the remaining two candidates for λ with $t = 3$ and $k = 7$). The first candidate has not passed the first level of the shortcut search described in Appendix, while the second candidate has not passed the second level (although it gives more than 14000 solutions for the first level).

The candidates for λ corresponding to 3-(21,8,λ) designs belong to the set
$\{3780, 4536, 5292, 7560, 8316, 10584, 11340, 12096\}$. Our search has shown that there are no designs arising from the considered wreath product and having some of the last six values of λ; the question of existence for the values 3780 and 4536 remains open.

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4 Appendix: Search for $t > 2$

The matrices $\Lambda(3,k)$ have six rows. The number of columns is equal to 11, 21, 38, 56, 76, 96, 104, for k equal to 4 through 10, respectively.

It was easy to apply an exhaustive search over *all* column combinations for $k \in \{4,5\}$. The result of this search was that there do not exist 3-designs with these values of k.

Candidates for λ corresponding to 3-$(21,k,\lambda)$ designs have been determined for $k \geq 6$, in accordance with the divisibility conditions and with the found 2-designs. Each such λ should be an integer of the form

$\lambda = \frac{k-2}{19} \cdot \lambda_2$, where λ_2 corresponds to a found 2-$(21,k,\lambda_2)$ design.

An exhaustive backtracking (based on temporary sums of entries in the rows) has been applied for $k = 6$ to all candidates for λ. It has been checked that $\lambda = 216$ is the only value of λ corresponding to a 3-(21,6,λ) design arising from the considered wreath product.

A shortcut algorithm, which can be used for searching 3-$(21,k,\lambda)$ designs having $k \geq 7$ will be described in the continuation.

4.1 Shortcut search for $k \geq 7$:

The following two rules are evident for columns of matrices $(a_{ij}) = \Lambda(3,k)$, $7 \leq k \leq 10$ (see the matrix $\Lambda(3,7)$ above):

1) IF $a_{6j} = a_{6k} \neq 0$, THEN $a_{4j} + a_{5j} = a_{4k} + a_{5k}$.

2) IF $a_{4j} + a_{5j} = a_{4k} + a_{5k}$ AND $a_{3j} = a_{3k}$ THEN $a_{1j} + a_{2j} = a_{1k} + a_{2k}$.

Let E_r denote an equivalence class of columns of (a_{ij}) w.r.t. the relation

two columns j and k are in the same class iff $a_{1j} + a_{2j} = a_{1k} + a_{2k}$, $a_{3j} = a_{3k}$, $a_{4j} + a_{5j} = a_{4k} + a_{5k}$ and $a_{6j} = a_{6k}$.

The rules 1) and 2) justify the following search strategy:

It is primarily attempted to find a vector $B = (b_r)$, where b_r denotes the number of columns of (a_{ij}) which are *chosen* from the class E_r, so that the chosen columns have the row sums $S_{12}(B)$, $S_3(B)$, $S_{45}(B)$,

$S_6(B)$ in the matrix $(a_{1j} + a_{2j},\ a_{3j},\ a_{4j} + a_{5j},\ a_{6j})$ equal to $2 \cdot \lambda$, λ, $2 \cdot \lambda$, λ, respectively.

Using these considerations, the following shortcut algorithm can be made:

Input:
- the matrix $(a_{ij} = \Lambda(3, k)$, for some k, $7 \leq k \leq 10$.
- candidates for λ corresponding to 3-$(21, k, \lambda)$ designs

FOR all candidates for λ **REPEAT**
Find the first part $FP(B)$ of B
over the columns satisfying $a_{6j} \neq 0$
so that $S_6(B) = \lambda$.
REPEAT
Find the second part $SP(B)$ of B
over the columns satisfying $a_{6j} = 0$
so that $S_{45}(B) = 2 \cdot \lambda$.
IF, additionally, the following two conditions are fulfiled:
$S_{12}(B) = 2 \cdot \lambda$; $S_3(B) = \lambda$
THEN an additional backtracking search is made
in order to look whether one can derive from B
a column combination C of (a_{ij}) such that
$S_1(C) = S_2(C) = S_4(C) = S_5(C) = \lambda$,
(where $S_i(C)$ denotes the sum $\sum_{j \in C} a_{ij}$)
UNTIL the possibilities for $SP(B)$ are exhausted or design is found
UNTIL the possibilities for $FP(B)$ are exhausted or design is found