E_3-Cordial Graphs

I. Cahit and R. Yilmaz
Department of Mathematics and Computer Science
Eastern Mediterranean University
G. Magosa, North Cyprus

ABSTRACT. In this paper generalization of edge-cordial labellings are introduced and studied for special classes of trees and graphs.

1 Introduction

Edge-graceful graphs have been attracting the attention of graph theorists for the last decade, a number of conjectures have been proposed, and many problems related to the topic remain unsolved [9]. A graph $G(V, E)$ is said to be edge-graceful if there exists a bijection $f : E \longrightarrow \{1, 2, ..., |E|\}$ so that the induced mapping $f^+ : V \longrightarrow \{0, 1, ..., |V| - 1\}$ given by $f^+(x) = \Sigma\{f(xy)|xy \in E\}(\mod |V|)$ is a bijection [9], [10]

One of the well known conjectures came from Lee in 1989 [6] :

Conjecture (Lee). *Every tree with an odd number of vertices is edge-graceful.*

He showed that every tree with an odd number of vertices and with at most one vertex with degree 2 is edge-graceful, and he also gave several classes of trees for which the conjecture holds. For further results on edge-graceful regular graphs and trees see Cabaniss et.al. [7]. However Lee's conjecture has not been proved yet.

In [8] the authors have adopted cordial labelling of graphs [1], [5] to edge-cordial graphs. In this paper we generalize edge-cordial (simply e-cordial) labellings to E_k-cordial labellings of graphs. We focus our attention on the case $k = 3$. We hope that a study of E_k-cordial labellings of graphs may give us a better understanding of edge-graceful graphs. This expectation is reasonable for at least two reasons. Firstly the case for $k = 3$ is the least difficult case if we would compare with the cases $k > 3$ and secondly the study of simplest case other than $k = 2$ e.g., edge-cordial labelling may relieve further clue for the general edge-graceful labelling of graphs [8].

Note that the notion of edge-cordial and its generalization developed here is different than the labellings [2],[3] although they also rely on the cordial labellings of graphs [1]. For the undefined terms the reader is reffered to [2].

Definition 1. *Let f be an edge labelling of graph $G = \{V, E\}$, such that $f : E(G) \longrightarrow \{0, 1, 2, ..., k-1\}$, and the induced vertex labelling is given as $f(v) = \sum_{\forall u} f(u, v) \pmod k$, where $v \in V$ and $\{u, v\} \in E$. f is called an E_k-cordial labelling of G, if the following conditions are satisfied for $i, j = 0, 1, \ldots, k-1$, $i \neq j$.*

1) $| e_f(i) - e_f(j) | \leq 1$,

2) $| v_f(i) - v_f(j) | \leq 1$;

where $e_f(i)$, $e_f(j)$ denote the number of edges, and $v_f(i)$, $v_f(j)$ denote the number of vertices labelled with i's and j's respectively.

The graph G is called E_k-cordial if it admits an E_k-cordial labelling.

The case $k = 2$ is the *E-cordial* case which was introduced and discussed in detail in [8]. In this paper, we investigate the E_3-cordiality of some special classes of graphs.

2 E_3-Cordial Graphs

Theorem 1. *Every path P_n, $n \geq 2$, is E_3-cordial, where n is the number of edges in P_n.*

Proof: Label P_n as follows:

(i) If $n \equiv 0 \pmod 3$ label the edges in the following order :

$$1, 2, 0, 1, 2, 0, 1, 2, 0, \ldots$$

such a labelling will result in $e_f(0) = e_f(1) = e_f(2) = \frac{n}{3}$, and $v_f(0) = \frac{n}{3} + 1$, $v_f(1) = v_f(2) = \frac{n}{3}$.

(ii) If $n \equiv 1 \pmod 3$label the edges in the following order :

$$1, 0, 2, 0, 1, 2, 0, 1, 2, 0, \ldots$$

such a labelling will result in $e_f(0) = \frac{n-1}{3} + 1$, $e_f(1) = e_f(2) = \frac{n-1}{3}$, and $v_f(0) = \frac{n-1}{3}$, $v_f(1) = v_f(2) = \frac{n-1}{3} + 1$.

(iii) If $n \equiv 2 \pmod 3$ label the edges in the following order :

$$1, 2, 0, 1, 2, 0, 1, 2, \ldots$$

such a labelling will result in $e_f(0) = \frac{n+1}{3-1}$, $e_f(1) = e_f(2) = \frac{n+1}{3}$, and $v_f(0) = v_f(1) = v_f(2) = \frac{n+1}{3}$.

Hence P_n is always E_3-cordial. □

Theorem 2. *A star S_n, $n \geq 2$, is E_3-cordial iff $n \not\equiv 1(\mod 3)$.*

Proof:

(i) If $n \equiv 0(\mod 3)$, label the edges of S_n such that $e_f(0) = e_f(1) = e_f(2) = \frac{n}{3}$ and the resulting vertex labelling will give $v_f(0) = \frac{n}{3+1}$, $v_f(1) = v_f(2) = \frac{n}{3}$.

(ii) Let $n \equiv 1(\mod 3)$, i.e., $n = 3k+1, k \geq 1$ and let v_c be the central vertex of S_n. In this case, for any E_3-cordial labeling f we must have at least k edges labelled with 0's, 1's and 2's. Let e be the unlabeled edge of S_n. Then we have

$$v_f(v_c) = 3k + f(e) + 1 = f(e) + 1(\mod 3).$$

If we put $f(e) = 0$ then $v_f(0) = v_f(i) + 2, i = 1, 2$. If we put $f(e) = 1$ then $v_f(1) = v_f(i) + 2, i = 0, 2$. Finally if we put $f(e) = 2$ then $v_f(2) = v_f(i) + 2, i = 0, 1$. Hence there exists no E_3-cordial labeling of S_n for $n \equiv 1(\mod 3)$.

(iii) If $n \equiv 2(\mod 3)$, label the edges of S_n such that $e_f(0) = \frac{n+1}{3-1}$, $e_f(1) = e_f(2) = \frac{n+1}{3}$, and the resulting vertex labelling will give $v_f(0) = v_f(1) = v_f(2) = \frac{n+1}{3}$.

Theorem 3. *Every complete graph K_n, $n \geq 3$, is E_3-cordial.*

Proof: We use induction on n. The induction step is given as follows :

(i) Let f be the E_3-cordial labelling of K_n, when $n \equiv 0(\mod 3)$, i.e. $n = 3k$. The weights of vertices and edges are necessarily $v_f(0) = v_f(1) = v_f(2)$, and $e_f(0) = e_f(1) = e_f(2)$. Let

$$f(v_i) = \begin{cases} 0 & i = 1, 2, \ldots, \frac{n}{3} \\ 1 & i = \frac{n}{3} + 1, \ldots, \frac{2n}{3} \\ 2 & i = \frac{2n}{3} + 1, \ldots, n \end{cases}$$

Add a new vertex v_{n+1}, adjacent to each vertex of K_n, thus obtaining K_{n+1}. Let f' be a labelling of K_{n+1}, such that :

$$f'(v_i, v_{n+1}) = \begin{cases} 0 & i = 1, 2, \ldots, \frac{n}{3} \\ 1 & i = \frac{n}{3} + 1, \ldots, \frac{2n}{3} \\ 2 & i = \frac{2n}{3} + 1, \ldots, n \end{cases}$$

and

$$f'(v_i) = \begin{cases} 0 & i = 1, 2, \dots, \frac{n}{3} \\ 2 & i = \frac{n}{3} + 1, \dots, \frac{2n}{3} \\ 1 & i = \frac{2n}{3} + 1, \dots, n \end{cases}$$

and it follows that $f'(n+1) = 0$, and $v_{f'}(0) = \frac{n}{3+1}$, $v_{f'}(1) = v_{f'}(2) = \frac{n}{3}$, and $e_{f'}(0) = e_{f'}(1) = e_{f'}(2) = \frac{n(n+1)}{6}$. Therefore $f_{\prime}$ is an E_3-cordial labelling of K_{n+1} where $n + 1 \equiv 1(\mod 3)$.

(ii) Let f be the E_3-cordial labelling of K_n, when $n \equiv 1(\mod 3)$, i.e. $n = 3k+1$. From the previous step we have $v_f(1) = v_f(2) = v_f(0)-1$, and $e_f(0) = e_f(1) = e_f(2)$. Let

$$f(v_i) = \begin{cases} 0 & i = 1, 2, \dots, \frac{n+2}{3} \\ 1 & i = \frac{n+2}{3} + 1, \dots, \frac{2(n+2)}{3} - 1 \\ 2 & i = \frac{2(n+2)}{3}, \dots, n \end{cases}$$

Add a new vertex v_{n+1}, adjacent to each vertex of K_n, thus obtaining K_{n+1}. Let f' be a labelling of K_{n+1}, such that :

$$f'(v_i, v_{n+1}) = \begin{cases} 0 & i = 1, 2, \dots, \frac{n-1}{3} \\ 1 & i = \frac{n+2}{3}, \dots, \frac{2(n-1)}{3} \\ 2 & i = \frac{2n+1}{3}, \dots, n \end{cases}$$

and

$$f'(v_i) = \begin{cases} 0 & i = 1, 2, \dots, \frac{n-1}{3}; i = \frac{2n+1}{3} \\ 1 & i = \frac{n+2}{3}; i = \frac{2(n+2)}{3}, \dots, \\ 2 & i = \frac{n+5}{3} + 1, \dots, \frac{2(n-1)}{3} \end{cases}$$

and it follows that $f'(n+1) = 2$, and $v_{f'}(0) = v_{f'}(1) = v_{f'}(2)+1$, and $e_{f'}(0) = e_{f'}(1) = e_{f'}(2) - 1$. Therefore $f_{\prime}$ is an E_3-cordial labelling of K_{n+1} where $n + 1 \equiv 2(\mod 3)$.

(iii) Let f be the E_3-cordial labelling of K_n, when $n \equiv 2(\mod 3)$, i.e. $n = 3k+2$. From the previous step we have $v_f(0) = v_f(1) = v_f(2)+1$, and $e_f(0) = e_f(1) = e_f(2) - 1$. Let

$$f(v_i) = \begin{cases} 0 & i = 1, 2, \dots, \frac{n+1}{3} \\ 1 & i = \frac{n+1}{3} + 1, \dots, \frac{2(n+1)}{3} \\ 2 & i = \frac{2(n+1)}{3} + 1, \dots, n \end{cases}$$

Add a new vertex v_{n+1}, adjacent to each vertex of K_n, thus obtaining K_{n+1}. Let f' be a labelling of K_{n+1}, such that :

$$f'(v_i, v_{n+1}) = \begin{cases} 0 & i = 1, 2, \dots, \frac{n+1}{3} \\ 1 & i = \frac{n+1}{3} + 1, \dots, \frac{2(n+1)}{3} \\ 2 & i = \frac{2(n+1)}{3} + 1, \dots, n \end{cases}$$

and

$$f'(v_i) = \begin{cases} 0 & i = 1, 2, \dots, \frac{n+1}{3} \\ 2 & i = \frac{n+1}{3} + 1, \dots, \frac{2(n+1)}{3} \\ 1 & i = \frac{2(n+1)}{3} + 1, \dots, n \end{cases}$$

and it follows that $f'(n+1) = 1$, and $v_{f'}(0) = v_{f'}(1) = v_{f'}(2)$, and $e_{f'}(0) = e_{f'}(1) = e_{f'}(2)$. Therefore f, is an E_3-cordial labelling of K_{n+1} where $n + 1 \equiv 1(\mod 3)$.

This completes the induction step and thus the proof. □

Theorem 4. *Every cycle C_n, $n \geq 3$, is E_3-cordial.*

Proof: The following labelling procedure results in E_3-cordial C_n:

(i) If $n \equiv 0(\mod 3)$ label the edges as:

$$f(e_i) = \begin{cases} 0 & i = 1, 2, \dots, \frac{n}{3} \\ 1 & i = \frac{n}{3} + 1, \dots, \frac{2n}{3} \\ 2 & i = \frac{2n}{3} + 1, \dots, n \end{cases}$$

This will result in $e_f(0) = e_f(1) = e_f(2) = \frac{n}{3}$, and $v_f(0) = v_f(1) = v_f(2) = \frac{n}{3}$.

(ii) If $n \equiv 1(\mod 3)$ label the edges as:

$$f(e_i) = \begin{cases} 0 & i = 1, 2, \dots, \frac{n+2}{3} \\ 1 & i = \frac{n+2}{3} + 1, \dots, \frac{2(n-1)}{3} + 1 \\ 2 & i = \frac{2(n-1)}{3} + 2, \dots, n \end{cases}$$

This will result in $e_f(0) = \frac{n-1}{3+1}$, $e_f(1) = e_f(2) = \frac{n-1}{3}$, and $v_f(0) = \frac{n-1}{3+1}$, $v_f(1) = v_f(2) = \frac{n-1}{3}$.

(iii) If $n \equiv 2(\mod 3)$ label the edges as:

$$f(e_i) = \begin{cases} 0 & i = 1, 2, \dots, \frac{n+1}{3} - 1 \\ 1 & i = \frac{n+1}{3}, \dots, \frac{2(n+1)}{3} - 1 \\ 2 & i = \frac{2(n+1)}{3}, \dots, n \end{cases}$$

This will result in $e_f(0) = \frac{n+1}{3-1}$, $e_f(1) = e_f(2) = \frac{n+1}{3}$, and $v_f(0) = \frac{n+1}{3-1}$, $v_f(1) = v_f(2) = \frac{n+1}{3}$. □

Theorem 5. *A regular graph of degree 1 on $2n$ vertices, $L(2n)$, $n \geq 1$, is E_3 -cordial iff $n \equiv 0(\mod 3)$.*

Proof: Proof of this theorem is trivial. Each edge label brings together two vertex labels of the same type. Therefore the number of edges labelled with 0, 1, and 2 have to be equal for E_3-cordiality of $L(2n)$. □

Theorem 6. *Every friendship graph F_n, $n \geq 1$, is E_3-cordial.*

Proof: Label the edges of each triangle in F_n, starting from its left edge adjacent to the center, and in clockwise order, as follows : $1,0,2,0,1,2,1,2,0,$ $1,0,2,0,1,2,1,2,0,\ldots$

Such a labelling will result in an E_3-cordial F_n. □

Theorem 7. *Every wheel W_n, $n \geq 3$, is E_3-cordial.*

Proof: The following labelling procedure results in E_3-cordial W_n:

(i) For $n \equiv 0(\mod 3)$, label the edges on the outer cycle as :

$$0,1,2, \quad 0,1,2, \quad 0,1,2,\ldots$$

and let the spoke edges be labelled with the summation (mod 3) of labels of the adjacent edges. This will result in $e_f(0) = e_f(1) = e_f(2) = \frac{2n}{3}$, and $v_f(0) = \frac{n}{3+1}$, $v_f(1) = v_f(2) = \frac{n}{3}$.

(ii) For $n \equiv 1(\mod 3)$, we will use W_{n-1} labelled as in (1).

Take two adjacent vertices labelled 0 and 1 in W_{n-1}. Between these, add a new vertex with degree 3. Label the edge incident with the vertex 0 with 2, the edge incident with the vertex 1 with 0, and the edge incident with the center vertex with 1. This will result in $e_f(0) = e_f(1) = \frac{2n+1}{3}$, $e_f(2) = \frac{2n+1}{3-1}$, and $v_f(0) = v_f(2) = \frac{n-1}{3+1}$, $v_f(1) = \frac{n-1}{3}$.

(iii) For $n \equiv 2(\mod 3)$, we will use W_{n-1} labelled as in (ii).

Take two adjacent vertices labelled 0 in W_{n-1}. Between these, add a new vertex with degree 3. Label the edge incident with the center vertex with 0, and the others with 2. This will result in $e_f(0) = \frac{2n-1}{3+1}$, $e_f(1) = e_f(2) = \frac{2n-1}{3}$, and $v_f(0) = v_f(1) = v_f(2) = \frac{n+1}{3}$. □

Theorem 8. *Every fan f_n, $n \geq 3$, is E_3-cordial.*

Proof: The following labelling procedure results in E_3-cordial f_n:

(i) If $n \equiv 0(\mod 3)$, label the edges on the path as :

$$1,2,0, \quad 1,2,0, \quad \ldots \quad 1,2$$

and the edges incident with the center vertex as:

$$0,2,1, \quad 0,2,1, \quad 0,2,1, \quad \ldots$$

This will result in $e_f(0) = \frac{2n}{3-1}$, $e_f(1) = e_f(2) = \frac{2n}{3}$, and $v_f(0) = \frac{n}{3+1}$, $v_f(1) = v_f(2) = \frac{n}{3}$.

(ii) If $n \equiv 1(\mod 3)$, label the edges on the path as :

$$1,2,2, \quad 0,1,2, \quad 0,1,2, \quad \ldots$$

and the edges incident with the center vertex as:

$$0,2,0,1, \quad 0,2,1, \quad 0,2,1, \quad \ldots$$

This will result in $e_f(0) = e_f(1) = \frac{2(n-1)}{3}$, $e_f(2) = \frac{2(n-1)}{3+1}$, and $v_f(0) = v_f(1) = \frac{n-1}{3+1}$, $v_f(2) = \frac{n-1}{3}$.

(iii) If $n \equiv 2(\mod 3)$, label the edges on the path as :

$$1,2,0, \quad 1,2,0, \quad \ldots \quad 1$$

and the edges incident with the center vertex as:

$$0,2,1, \quad 0,2,1, \quad \ldots \quad 0,2$$

This will result in $e_f(0) = e_f(1) = e_f(2) = \frac{2n-1}{3}$, and $v_f(0) = v_f(1) = v_f(2) = \frac{n+1}{3}$. □

The following theorem generalizes Theorem 3.

Theorem 9. *A star S_n, $n \geq 2$ is E_k-cordial iff*

$$n \not\equiv \begin{cases} 1(\mod k) & \text{for } k \equiv 1(\mod 2) \\ 1(\mod 2k) & \text{for } k \equiv 0(\mod 2)(k \neq 2). \end{cases}$$

Proof: Necessity: Let $n \equiv 0(\mod k)$. Then for $k \equiv 1(\mod 2)$ any E_k-cordial labelling of S_n must satisfy $e_f(0) = e_f(1) = ... = e_f(k-1) = \frac{n}{k}$ and since $\sum_{i=0}^{k-1} i \equiv 0(\mod k)$ for $k \equiv 1(\mod 2)$ the induce vertex label of v_c is 0 and the resulting vertex labelling will give $v_f(0) = \frac{n}{k+1}, v_f(1) = v_f(2) = ... = v_f(k-1) = \frac{n}{k}$.

Assume that we add a new vertex to obtain $S_n, n \equiv 1(\mod 2), k \equiv 1(\mod 2)$. Now whatever label we give to this new edge, both the new vertex and the center will take the same label and this violate the condition for E_k-cordiality.

Let $n \equiv 0(\mod k)$ and $k \equiv 0(\mod 2)$ and let v_c be the central vertex of S_n. Since $\sum_{i=0}^{k-1} i \equiv \frac{k}{2}(\mod k)$ the induce vertex label is $f(v_c) = 0$ only if $n = 2k$. That is for $k \equiv 0(\mod 2)$

$2(\sum_{i=0}^{k-1} i) \equiv 0(\mod k)$. In other words if $n \equiv 0(\mod 2k)$ then E_k-cordial labelling of S_n with $f(v_c) = 0$ satisfies

$$v_f(0) = \frac{n}{k+1}, v_f(1) = v_2(2) = ... = v_f(k-1) = \frac{n}{k}$$

Again assume that we add a new vertex to obtain $S_n, n \equiv 1(\mod 2), k \equiv 0(\mod 2)$. Whatever label we give to this new edge both the new vertex and center will take the same label and this violate the condition for E_k-cordiality.

Sufficiency: Let $e_1, e_2, ..., e_n$ be the set of all edges of S_n. Then label these edges as follows:

$$0, 1, 2, ..., k-1, 0, 1, 2, ..., k-1, 0, 1, 2, ..., k-1, ..., 0, 1, 2, ..., i, \text{ where } i \leq k-1.$$

It can easily be verified that as long as the condition of the theorem holds the edge labels induce vertex labelling $\bar{f}$ which satisfies $|v_f(i) - v_f(j)| \leq 1, i, j = 0, 1, ..., k-1$. Thus the labelling is E_k-cordial. □

E_k-cordial labelling of other classes of trees such as paths, caterpillars, symmetric trees etc., will be given in a future work.

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