Covering designs with minimum overlap

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ABSTRACT. Let H be a graph, and let k be a positive integer. A graph G is *H-coverable with overlap k* if there is a covering of all the edges of G by copies of H such that no edge of G is covered more than k times. The number $ol(H, G)$ is the minimum k for which G is H-coverable with overlap k.

It is established (Theorem 2.1) that if n is sufficiently large then

$$ol(H, K_n) \leq 2.$$

For H being a path, a matching or a star it is enough to assume $|H| \leq n$ (Theorem 3.1).

The same result is obtained (Main Theorem) for any graph H having at most four vertices, or else at most four edges with a single exception $ol(K_4, K_5) = 3$.

1 Introduction

All graphs in this paper are finite, undirected and simple, unless otherwise noted. We will use the notation of [21].

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1.1 Recent approach in covering theory

The theory of graph decomposition has been accompanied since the beginning by a parallel theory of covering (see [9], [22], [29] and [33]). In the early stages of the development of covering theory the main motivation was to minimize the number of covering graphs. In recent years covering with certain structures were investigated (see [1], [3], [9], [29], and [36]). One of the natural approaches is to minimize the number of times any edge is covered, which was anticipated in Statistical Designs (see [24]). Some recent attempts along this line were made in [3].

This paper is devoted entirely to that minimum overlap problem. The following example illustrates the two approaches.

Example: It is well known that K_5 has no exact decomposition into copies of K_3. The minimum number of copies of K_3 needed to cover K_5 is four. Moreover (see [15]) in every covering with four copies one edge of K_5 has to be covered three times. Now the covering $(1,2,4) \pmod 5$ shows that using five copies of K_3 one can cover the edges of K_5 such that no edge is covered more than *twice*.

1.2 Definitions and notation

The vertices of K_n are denoted by the elements of Z_n whenever a particular decomposition is needed.

We denote by $P(H,n)$, the *packing number*, namely, the maximum number of pairwise edge disjoint copies of H, in the complete graph K_n, and by $C(H,n)$, the *covering number*, namely, the minimum number of copies of H whose union is K_n.

Beside those definitions from the minimum covering problem approach we need some new definitions.

Let H be a graph, and let k be a positive integer. A graph G is *H-coverable with overlap k* if there is a covering of the edges of G by copies of H such that no edge of G is covered more than k times. Denote by $ol(H,G)$ the minimum k for which G is H-coverable with overlap k . Denote also by $CO(H,G)$ the minimum number of copies of H in a cover of G that realizes $ol(H,G)$. In case $G = K_n$ we write $ol(H,n)$ and $CO(H,n)$.

Clearly, $ol(H,G) = 1$ means that G has a H-decomposition usually denoted $H \mid G$.

1.3 Purpose and content of the paper

For some pairs H and G the graph G is not H coverable with overlap k, for any k. But if there is a k such that G is H coverable with overlap k then, $ol(H,G) \leq e(G) - e(H) + 1$. For some pairs H and G equality is achieved and moreover $ol(H, G)$ can be quite large. For instance, if D_n is a triangle with $n-2$ edges attached to one of its vertices then, $CO(D_3, D_n) = ol(D_3, D_n) = e(D_n) - e(D_3) + 1 = n - 2$. In contrast if $n \geq |V(H)|$ then K_n is always H-coverable with overlap k for some k and it will turn out that $ol(H, n) \leq 2$ in most cases.

The above considerations justifies restriction of our interest to the case $G = K_n$.

In the next section we prove, among others, the asymptotic result that for any fixed graph H and sufficiently large n, $ol(H, n) \leq 2$.

In sections 3 and 4 we show that for some graphs H one can obtain exact results and reduce the condition on n to the trivial one, namely,

$$n \geq |V(H)|, \tag{0}$$

and still conclude, $ol(H, n) \leq 2$.

This is done in chapter 3 for paths, matchings, stars and some particular forests.

In Chapter 4 it is done for all graphs having at most four vertices or else at most four edges, with a single exception, namely, $ol(K_4, 5) = 3$.

Subsequently we shall assume that the trivial condition (0) holds.

2 An asymptotic result

We start with a general theorem.

Theorem 2.1. *Given a graph H there exists $M(H)$ such that if $n \geq M(H)$ then,*

$$ol(H, K_n) \leq 2.$$

In the proof bellow we shall use the following two known theorems.

Theorem HS. (Hajnal-Szemeredi [19]). *Let G be a graph on n vertices and $k \geq 2$ an integer such that $\delta(G) \geq \frac{(k-1)n}{k}$ and $k \mid n$. Then G has a K_k factor.*

Theorem W. (Wilson [43]). *Let H be a graph on q edges and let d be the greatest common divisor of its degree sequence. There exists a positive integer $N(H) = N$ such that if:* (a) $n \geq N$; (b) $q \mid \binom{n}{2}$; (c) $d \mid n-1$, *then* $H \mid K_n$.

Proof of Theorem 2.1: It is enough to prove the theorem in the case H is a complete graph. By Theorem W there exists K_H, the smallest complete graph such that $H \mid K_N$. Now if the theorem is true for complete graphs then by the exact decomposition of K_H into copies of H both parts of the required result follow.

Let $\overline{m}$ be an integer such that $\overline{m} \geq N(k+1)$ and $k(k+1) \mid \overline{m}(\overline{m}-1)$. Suppose $n \geq \overline{m}$ and denote the largest integer $\overline{m}$ not exceeding n, by m. Then, $n = m + t$ and $0 \leq t \leq k(k+1)$. If $t = 0$ we are done. So we may assume $t > 0$.

We have to cover the edges of $K_n = K_m \cup K_{m,t} \cup K_t$. The edges of K_m are covered by the K_{k+l}-decomposition of K_m insured by Theorem W. For to cover the remaining edges let v be a vertex of K_m and consider $K_m \setminus \{v\}$. That graph satisfies the assumptions of Theorem HS, provided m is large enough, namely, $m \geq (t-1)(k-1)k + 2k$. Hence, it has a K_k-factor. The graph obtained by deleting the edges of that factor still satisfies the assumptions of Theorem HS and this can be repeated t times so that we get t edge-disjoint K_k-factors, $L_1, \dots, L_t$ of $K_m \setminus \{v\}$. If

$$\lfloor \frac{m-1}{k-1} \rfloor \geq \max\{l + t(k-1), \binom{t}{2} + t\},$$

then the remaining graph still satisfies the assumptions of Theorem HS for k, so that we have also a K_{k-1}factor L edge-disjoint of the former t K_k-factors and it contains at least $\binom{t}{2} + t$ edge-disjoint copies of K_{k-1}.

The above exhibited K_k-factors together with the additional K_{k-1}-factor L, enable us to cover as required the edges of $K_{m,t} \cup K_t$ as follows:

Let $\{v_1, \dots, v_t\}$ be the vertices of K_t. Then for $i = 1, 2, \dots, t$ join the vertex v_i to each vertex of L_i, thus each K_k of L_i is completed to a K_{k+1} and all edges of $K_{m,t}$ are covered except those stemming out of v.

Now the t non-covered edges of $K_{m,t}$ will be covered by completing t copies of K_{k-1} in L to K_{k+1} by joining respectively the vertices of a K_{k-1} to end-vertices of the edges adjacent to v. Similarly the remaining $\binom{t}{2}$ copies of K_{k-1} in L will be completed to K_{k+1} using respectively the end-vertices of the edges of K_t. In this way all edges of K_n are covered and the only edges covered twice are the edges of $L_1, \dots, L_t$ and L.

Observe that the number of copies of K_{k+1} used in the above cover is bounded above namely,

$$CO(H, K_n) \leq \frac{n(n-1)}{2e(H)} + c(H)n,$$

where $c(H)$ is a constant depending only on H. □

3 Results and proofs in the case of paths, matchings, stars and other particular forests

In this section we shall assume that condition (0) holds, namely, whenever $ol(H, G)$ appears, then $|H| \leq |G|$.

The bound M in Theorem 2.1 as shown in the beginning of the previous section, can be very large. Hence we shall be interested in establishing some classes of graphs H for which this bound is reduced to the trivial condition (0). In Theorem 3.1 this goal is achieved for H being a star a matching or a path. In theorem 3.2 we consider some other forests.

Theorem 3.1. *If $H \in \{tK_2, P_{t+1}, K_{1,t}\}$, then, $ol(H, n) \leq 2$.*

In the proof of the theorem we use the following three lemmas.

Lemma 3.2. *If $G \in \{P_n, C_n\}$, and $H = tK_2$, then*

$$ol(H,G) = \begin{cases} 1 & t \mid e(G) \\ 2 & otherwise \end{cases}$$

Proof of Lemma 3.2: We shall prove the lemma for $G = P_n$. If $G = C_n$ the proof is similar. Let $n - 1 = lt + r$, $0 \leq r < t$. Denote the vertices of P_n by the numbers of the set $\{1, 2, \ldots, n\}$. The path formed by the first lt edges of P_n has the following decomposition: $\{(1+i, 2+i), (1+l+i, 2+l+i), (1+2l+i, 2+2l+i), \ldots, (1+(t-1)l+i, 2+(t-1)l+i)\}$, $0 \leq i \leq l-1$. Now the r edges left can be decomposed into two matchings of size at most $\frac{t}{2}$ such that each of them can be completed into tK_2 by adding edges from one of the copies used earlier and no edge is covered more than twice. If $r = 0$ this is a decomposition. □

Lemma 3.3. *If $G \in \{P_n, C_n\}$, and $H = P_{t+1}$, then*

$$ol(H,G) = \begin{cases} 1 & t \mid e(G) \\ 2 & otherwise \end{cases}$$

Proof: We shall prove the lemma for $G = P_n$. If $G = C_n$ the proof is similar. Let $n - 1 = lt + r$, $0 \leq r < t$. Divide the tl path into l paths P_t. The remainder P_{r+1} is attached to the last $t - r$ edges of the last path in the former decomposition. Then only $t - r$ edges are covered twice. □

Lemma 3.4. ([3]) *Let $t > 1$ be an integer. Let G be a graph such that every edge of G has an endpoint whose degree is at least t. Then $ol(K_{1,t}, G) \leq 2$. Consequently, if $\delta(G) \geq t$ then $ol(K_{1,t}, G) \leq 2$.*

Proof of Theorem 3.1: It is well known (see [21]) that K_n can be decomposed into Hamilton cycles, C_n or paths P_n according to whether n is odd or even. Hence the proof for paths and matchings follows from that fact and Lemmas 3.2 and 3.3. The proof in the case of stars is an immediate consequence of Lemma 3.4 since, $\delta(K_n) = n - 1 \geq t$, for $n \geq t + 1$ required by condition (0). □

Remark 1: The decomposition conditions for K_n into $tK_2, P_{t+1}, K_{1,t}$, are known (see [2] , [41], [10] [22] and [42]). If they are fulfilled one has $ol(H,n) = 1$, in all other cases $ol(H,n) = 2$.

For many graphs H, as enumerated in Theorem R below, one can conclude $ol(H,n) = 2$ from the following self evident lemma.

Lemma 3.5. *Let H be a graph and n a positive integer. If, $C(H,n) - P(H,n) = 1$ and the non covered edges in the packing can be covered with exactly one extra copy of H, then, $ol(H,n) = 2$.*

The following result summarizes the values of $ol(H,n)$ for small trees and forests.

Theorem 3.6. *Let H be a forest with m vertices and q edges, such that $m \leq 7$ if H is a tree, and $m \leq 6$ otherwise. Then,*

$$ol(H,n) = \begin{cases} 1 & \text{if } q \mid \binom{n}{2} \text{ and } n \geq 2q \text{ and } H \text{ is a star} \\ 1 & \text{if } q \mid \binom{n}{2} \text{ and } n \geq m \text{ and } H \text{ is not a star} \\ 2 & \text{otherwise.} \end{cases}$$

□

In all cases that the assumptions of Theorem 3.6 hold, the result, for $n \geq 11$, follows from [23], [33] - [37] and Theorem R.

For $n \leq 10$, the result follows from particular constructions.

Theorem R. (Roditty [33] - [37]).

$$P(F,n) = \lfloor \frac{n(n-1)}{2e(T)} \rfloor \text{ and } C(F,n) = \lceil \frac{n(n-1)}{2e(T)} \rceil, \text{ for } n \geq n_0,$$

where F is any tree of order at most seven or any forest of order at most six, and n_0 is a constant which does not exceed 11. Furthermore, in all cases where, $n \geq n_0$ the conditions of Lemma 3.5 hold.

The exact values of n_0 have been determined for the relevant forests in [33] - [37].

4 Proofs in the case G has at most four vertices or at most four edges

In this section we shall prove the following Main Theorem.

Main Theorem: *If H is any graph having at most four vertices or else at most four edges, and if condition (0) is satisfied then,*

$$ol(H,n) \leq 2,$$

with a single exception $ol(K_4,5) = 3$.

The relevant graphs mentioned in the Main Theorem are listed in Table 1 below. The first column enumerates those graphs. The graphs in the table denoted H_5 and D, are the star $K_{1,3}$ with an edge adjacent to one of its end vertices, respectively the triangle K_3 with an edge adjacent to one of its vertices.

In the rest of the columns there is information which is used later. The number in the square brackets indicates the appropriate reference. As it is seen there are 21 graphs to deal with. Since for some graphs the proof is long, we shall divide the proof of the theorem into several propositions.

No.	The graph H	Decomposition	$P(H,n)$	$C(H,n)$	$ol(H,n) \le 2$ follows by
1	K_2	trivial	trivial	trivial	trivial
2	P_3	$n \equiv 0,1 \pmod 4$[12]	$\lfloor \frac{n(n-1)}{4} \rfloor$[33]	$\lceil \frac{n(n-1)}{4} \rceil$[33]	Theorem 3.1
3	$P_3 \cup K_2$	$n \equiv 0,1 \pmod 3$, $n \ge 6$[4]	$\lfloor \frac{n(n-1)}{6} \rfloor$[34]	$\lceil \frac{n(n-1)}{6} \rceil$[34]	Thereom 3.6
4	$2K_2$	$n \equiv 0,1 \pmod 4$[2]	$\lfloor \frac{n(n-1)}{4} \rfloor$[33]	$\lceil \frac{n(n-1)}{4} \rceil$[33]	Theorem 3,1
5	$K_{1,3}$	$n \equiv 0,1 \pmod 3$, $n \ge 6$[10],[42]	$\lfloor \frac{n(n-1)}{6} \rfloor$[33]	$\lceil \frac{n(n-1)}{6} \rceil$[33]	Thereom 3.1
6	K_3	$n \equiv 1,3 \pmod 6$[40]	$\lfloor \frac{n}{3} \lfloor \frac{n-1}{2} \rfloor \rfloor$, $n \not\equiv 5 \pmod 6$[38] $\lfloor \frac{n}{3} \lfloor \frac{n-1}{2} \rfloor \rfloor - 1$, $n \equiv 5 \pmod 6$[38]	$\lceil \frac{n}{3} \lceil \frac{n-1}{2} \rceil \rceil$, $n \ge 3$[15]	Pro. 4.3
7	P_4	$n \equiv 0,1 \pmod 3$[6]	$\lfloor \frac{n(n-1)}{6} \rfloor$[33]	$\lceil \frac{n(n-1)}{6} \rceil$[33]	Theorem 3.1
8	$3K_2$	$n \equiv 0,1 \pmod 3$ $n \ge 6$[7],[44]	$\lfloor \frac{n(n-1)}{6} \rfloor$[37]	$\lceil \frac{n(n-1)}{6} \rceil$[37]	Theorem 3.1
9	C_4	$n \equiv 1 \pmod 8$[25]	$\lfloor \frac{n}{4} \lfloor \frac{n-1}{2} \rfloor \rfloor$, $n \not\equiv 5,7 \pmod 8$[39] $\lfloor \frac{n}{4} \lfloor \frac{n-1}{2} \rfloor \rfloor - 1$, otherwise[39]	$\lceil \frac{n}{4} \lceil \frac{n-1}{2} \rceil \rceil$, $n \not\equiv 3 \pmod 8$[39] $\lceil \frac{n}{4} \lceil \frac{n-1}{2} \rceil \rceil + 1$, otherwise[39]	Pro. 4.4
10	P_5	$n \equiv 0,1 \pmod 8$[32]	$\lfloor \frac{n(n-1)}{8} \rfloor$[34]	$\lceil \frac{n(n-1)}{8} \rceil$[34]	Theorem 3.1
11	$K_{1,4}$	$n \equiv 0,1 \pmod 8$ (see [10],[42])	$\lfloor \frac{n(n-1)}{8} \rfloor$[34]	$\lceil \frac{n(n-1)}{8} \rceil$[34]	Theorem 3.1
12	H_5	$n \equiv 0,1 \pmod 8$[32]	$\lfloor \frac{n(n-1)}{8} \rfloor$[34]	$\lceil \frac{n(n-1)}{8} \rceil$[34]	Theorem 3.6
13	D	$n \equiv 0,1 \pmod 8$[5]	$\lfloor \frac{n(n-1)}{8} \rfloor$[33]	$\lceil \frac{n(n-1)}{8} \rceil$[33]	Pro. 4,7
14	$K_{1,3} \cup K_2$	$n \equiv 0,1 \pmod 8$[44]	$\lfloor \frac{n(n-1)}{8} \rfloor$[37]	$\lceil \frac{n(n-1)}{8} \rceil$[37]	Theorem 3.6
15	$P_4 \cup K_2$	$n \equiv 0,1 \pmod 8$[44]	$\lfloor \frac{n(n-1)}{8} \rfloor$[37]	$\lceil \frac{n(n-1)}{8} \rceil$[37]	Theorem 3.6
16	$2P_3$	$n \equiv 0,1 \pmod 8$[44]	$\lfloor \frac{n(n-1)}{8} \rfloor$[37]	$\lceil \frac{n(n-1)}{8} \rceil$[37]	Theorem 3.6
17	$K_3 \cup K_2$	$n \equiv 0,1 \pmod 8$[44]	not known	not known	Pro. 4.7
18	$P_3 \cup 2K_2$	$n \equiv 0,1 \pmod 8$[30]	not known	not known	Pro. 4.8
19	$4K_2$	$n \equiv 0,1 \pmod 8$[2]	$\lfloor \frac{n(n-1)}{8} \rfloor$[37]	$\lceil \frac{n(n-1)}{8} \rceil$[37]	Theorem 3.1
20	$Q = K_4 \setminus e$	$n \equiv 0,1 \pmod 5$, $n \ne 5$[5]	$\lfloor \frac{n(n-1)}{10} \rfloor$[33]	$\lceil \frac{n(n-1)}{10} \rceil$[33]	Pro. 4.6
21	K_4	$n \equiv 1,4 \pmod{12}$[20]	$\lfloor \frac{n}{4} \lfloor \frac{n-1}{3} \rfloor \rfloor - 1$, $n \equiv 7,10 \pmod{12}$[8] $\lfloor \frac{n}{4} \lfloor \frac{n-1}{3} \rfloor \rfloor$, otherwise[8]	$\lceil \frac{n}{4} \lceil \frac{n-1}{3} \rceil \rceil$, $n \ne 5,7,9$ $, 10, 19$[27],[28]	Pro. 4.9

Table 1

Proposition 4.1. *The Main Theorem holds for the graphs numbered in Table 1 as: 1,2,4,5,7,8,10,11,12,14,19.*

Proof: Those graphs are paths, matchings or stars. Hence, the proposition is a corollary of Theorem 3.1. □

Proposition 4.2. *The Main Theorem holds for the graphs numbered in Table 1 as: 3,15, 16.*

Proof: Those graphs satisfy the assumption of Theorem 3.6. □

For each of the remaining graphs the proofs are more specific and therefore for each of them we present an appropriate proposition.

Proposition 4.3. *The main theorem holds for the graph numbered 6, namely,*

$$ol(K_3, n) = \begin{cases} 1 & n \equiv 1,3 \pmod 6 \\ 2 & \textit{otherwise.} \end{cases}$$

Proof: Denote the triangle K_3 by (a, b, c).

In Table 2 we prove $ol(K_3, n) = 2$ for $4 \leq n \leq 6$.

n	The covering	edges covered twice
4	$(0,1,2) \pmod 4$	all edges
5	$(0,1,3) \pmod 5$	(0,1)(1,2)(2,3) (3,4)
6	$(0,1,3) \pmod 6$	(0,3)(1,4)(2,5)

Table 2

Let $n = 6m + k$, $0 \leq k \leq 5$. Since the existence of Steiner triple systems [40], we have to deal only with $k = 0, 2, 4, 5$. If $k = 0, 2, 4$ then the covering appearing in [15] is such that no edge is covered more than twice. For $k = 5$ and $n \geq 11$ the covering in [15] shows that only one edge say, e is covered three times and all the others once. It is easy then to use only two triangles from those covering e and cover the edges, different from e, of the third triangle, by two other suitable triangles. □

Remark 2: It follows from Proposition 4.3 and the covering number $C(K_3, n)$ proved in [15] that

$$CO(K_3, n) = \begin{cases} C(K_3, n) + 1 & \text{if } n \equiv 5 \pmod 6 \\ C(K_3, n) & \text{otherwise.} \end{cases}$$

Proposition 4.4. *The main theorem holds for the graph numbered 9, namely,*

$$ol(C_4, n) = \begin{cases} 1 & n \equiv 1 \pmod 8 \\ 2 & \textit{otherwise.} \end{cases}$$

The following Lemma establishes the C_4-coverability with overlap 2, of complete bipartite graphs and is extremely useful in proving Proposition 4.4 and also Proposition 4.6 below.

Lemma 4.5. *Let $m, n > 1$ be integers. Then,*

$$ol(C_4, K_{m,n}) = \begin{cases} 1 & m \text{ and } n \text{ are both even} \\ 2 & \text{otherwise.} \end{cases}$$

Proof: The case m and n even is an easy consequence of the fact that $C_4 \mid K_{2,2t}$.

Denote the complete bipartite graph $K_{m,n}$ by $K(A, B)$, $|A| = m$, $|B| = n$. Now if only one of m, n is odd, say, $m = 2a+1$, let u, v be some vertices in A. Then, $C_4 \mid K(A \setminus v, B)$ and the remaining star $K(v, B)$ can be covered by adding to it the star $K(u, B)$, the edges of which are covered twice.

Finally if m, n are both odd and at least three, then one can use induction. First observe that $ol(C_4, K_{3,3}) = 2$. Indeed, there are 3 cycles in the covering and three independent edges are covered twice. Now as before denote the graph $K_{m,n}$ by $K(A, B)$. Then at least one of the sets A, B has order at least 5. Let A be that set. Consider two vertices u, v in A. We have, $ol(C_4, K(A \setminus \{u, v\}, B) = 2$ and $ol(C_4, K(\{u, v\}, B) = 2$. Since, $K(A, B) = K(A \setminus \{u, v\}, B) \cup K(\{u, v\}, B)$ we are done. □

Proof of Proposition 4.4: The proof is by considering the different residue classes (mod 8), n belongs to.

If $n \equiv 1 \pmod 8$ we have the result due to Kotzig [25], namely,

$$C_4 \mid K_{8m+1}. \tag{1}$$

In view of condition (0) it remains to prove the proposition for $n = 8m+t$, $m \geq 0$, $t \in \{4, 5, 6, 7, 8, 10, 11\}$.

The coverings for $m = 0$ and $t \in \{4, 5, 6, 7\}$ are exhibited in Table 3 below.

n	The covering
4	(0,1,2,3), (0,1,3,2)
5	(0,1,2,3), (0,1,4,2), (0,2,3,4), (1,2,4,3)
6	(0,1,2,3), (0,1,3,2), (4,0,5,1), (4,3,5,2), (4,2,1,5)
7	(0,1,2,3), (0,1,3,2), (5,3,6,2), (5,1,6,0), (0,4,5,6), (4,6,1,2), (4,2,5,3), (4,1,5,0)

Table 3

Using those coverings and Lemma 4.5 one obtains the coverings for $t \in \{8, 10, 11\}$, $m = 0$ by observing that,

$$\begin{aligned} K_8 &= K_4 \cup K_{4,4} \cup K_4, \\ K_{10} &= K_4 \cup K_{4,6} \cup K_6, \\ K_{11} &= K_5 \cup K_{5,6} \cup K_6. \end{aligned}$$

An induction argument using

$$K_{8m+8} = K_8 \cup K_{8m+8} \cup K_{8m},$$

shows that $ol(C_4, K_n) = 2$ for $n \equiv 0 \pmod 8$.

Finally, we finish the general case by considering

$$K_{8m+t} = K_t \cup K_{8m,t} \cup K_{8m},$$

for $t \in \{4, 5, 6, 7, 10, 11\}$. □

Proposition 4.6. *The main theorem holds for the graph numbered 20, namely,*

$$ol(Q, n) = \begin{cases} 1 & n \equiv 0, 1 \pmod 5,\ n \neq 5 \\ 2 & \textit{otherwise}. \end{cases}$$

Proof: The case $ol(Q, n) = 1$, for $n \equiv 0, 1 \pmod 5$, is the decomposition result due to Bermond and Schönheim [5], namely,

$$Q \mid K_n, \text{ for } n \equiv 0, 1 \pmod 5,\ n \neq 5. \tag{2}$$

Denote the graph Q by $(a, \widehat{b, c}, d)$.

So that we have to prove $ol(Q, n) = 2$ for $n = 5m + k$, $k = 2, 3, 4$.

The only cases that does not follow from the proof below are $n = 4, 5$. Their covering is found in Table 4.

n	The covering	edges covered twice
4	$(0, \widehat{1, 2}, 3), (3, \widehat{0, 1}, 2)$	(0,1), (2,3)
5	$(3, \widehat{0, 1}, 2), (0, \widehat{1, 3}, 4), (4, \widehat{3, 2}, 1)$	(0,1), (1,2), (2,3), (3,4), (1,4)

Table 4

Now consider the general case: $5m + k$, $k \in \{2, 3, 4\}$.

k = 2:

Observe that $K_{5m+2} = K_2 \cup K_{2,5m} \cup K_{5m}$ and let $m \geq 1$, with $V(K_2) = \{x, y\}$. We separate our proof into two cases:

(a) $m = 2t$.

Then $n = 10t + 2$. In this case, $C_4 \mid K_{2,10t}$, namely, $K_{2,10t} = 5tK_{2,2}$ and we take an edge from a complete matching of K_{10t} for each of the $K_{2,2}$ to form Q. In this way no edge is covered more than twice since by (2) $Q \mid K_{10t}$. One covers the edge (x, y) by using any copy of $K_{2,2}$ again giving with (x, y) a copy of Q.

(b) $m = 2t + 1$.

Then $n = (10t + 5) + 2$. In this case we put aside one vertex of K_{10t+5}, say, u. Then $K_{2,10t+4} = (5t + 2)K_{2,2}$ and as above we take an edge of a complete matching from K_{10t+4} for each of the $K_{2,2}$ to form Q. No edge is covered more than twice. Now let w be some vertex of K_{10t+4}. We form the following copy of Q: $(w, \widehat{y, u, x})$, where the edges $(x, w), (y, w)$ are covered twice. This works even if $t = 0$, since in the covering of K_5 there is a maximal matching $(1, 3), (2, 4)$ covered only once and choose $u = 0$.

k = 3:

Suppose $m \geq 1$. Observe that,

$$K_{5m+3} = K_3 \cup K_{3,5m} \cup K_{5m}, \tag{3}$$

and also

$$K_{5m+3} = K_2 \cup K_{2,5m+1} \cup K_{5m+1}. \tag{4}$$

We shall use (3) if $5m$ is even and (4) if it is odd, hence, $5m + 1$ is even.

The case (4) is exactly as (a) in $k = 2$. For (3) the covering of the last two terms is based upon the C_4-covering of $K_{3,5m}$ and upon (2).

Denote the vertices of K_3.in (3) by $\{x, y, z\}$, and the vertices of K_{5m} by the elements of Z_{5m}. Now in the C_4-covering of $K_{3,5m}$ the C_4's formed by edges stemmed out from x, y can be completed into copies of Q by using a complete matching of K_{5m}. The other copies of C_4 formed by the stars stemming out from z, x, are similarly completed by a second matching, edge disjoint from the first one. There are two such matchings since from (2) there are no overlaps. Now the edges of the triangle (x, y, z) are covered by taking (x, y), (x, z) instead of one in each of the former matchings, while (y, z) can be added to C_4 already covered but only once passing through y and z.

k = 4:

Suppose $m \geq 1$. Observe that,

$$K_{5m+4} = K_4 \cup K_{4,5m} \cup K_{5m}, \tag{5}$$

and also

$$K_{5m+4} = K_3 \cup K_{3,5m+1} \cup K_{5m+1}. \tag{6}$$

We shall use (5) if $5m$ is even and (6) otherwise. For (6) the construction is exactly as in the case (3) with $Q \mid K_{5m+1}$. For (5) the graph K_4 can be covered as mentioned, while the copies of C_4 can be completed by using two independent matchings of K_{5m}. □

Proposition 4.7. *The main theorem holds for the graphs numbered 13,17, namely, let H be one of the graphs either D or $F = K_3 \cup K_2$. Then,*

$$ol(H,n) = \begin{cases} 1 & n \equiv 0,1 \pmod 8 \\ 2 & \textit{otherwise.} \end{cases}$$

Proof: The case $ol(H,n) = 1$ for $n \equiv 0,1 \pmod 8$ is the decomposition result and appeared in [5] if $H = D$, and in [4] if $H = F$.

Denote the graph $K_3 \cup K_2$ by $(a,b,c)(x,y)$. The graph D is denoted $(a,b,c;d)$ and d is adjacent to c.

Our proof for the 2-overlap case is based upon the cyclic construction of the decomposition in the case $n = 8m+1$. For the graph D we have the following base blocks:

$$(1, 2m-2i, 4m-i+1; 6m-2i+1), \; i = 0,1,\ldots,m-1.$$

For the graph F we have the following base blocks:

$$(1, 2m-2i, 4m-i+1)(4m-i, 6m-2i), \; i = 0,1,\ldots,m-1.$$

We shall construct the required coverings for $n = 8m+j$, $j = 2,3,\ldots,7$ using $j-1$ additional vertices labeled as: $\infty_1, \infty_2, \ldots, \infty_{j-1}$.

The base blocks for the coverings are those from above and the following additional $j-1$ blocks. For D:

$$(1, 2m-2i, 4m-i+1; \infty_{i+1}), \; i = 0,1,\ldots,j-2.$$

For F:

$$(1, 2m-2i, 4m-i+1)(4m-i, \infty_{i+1}), \; i = 0,1,\ldots,j-2.$$

At this stage all edges in $V = \{0,1,\ldots,8m\}$ and all edges connecting V to $W = \{\infty_1, \infty_2, \ldots, \infty_{j-1}\}$, are covered and only the triangles resulting from the last $j-1$ base blocks are covered twice. We now have to cover the edges of W at most twice. In Table 5 we represent the required covering.

j	D	F
2	no edges	no edges
3	$(1,2,\infty_1;\infty_2)$	$(1,2,3m+2)(\infty_1,\infty_2)$
4	$(\infty_1,\infty_2,\infty_3;4m)$	$(\infty_1,\infty_2,\infty_3;4m,6m)$
5	$(\infty_1,\infty_2,\infty_3;4m)$ $(\infty_1,\infty_2,\infty_3;4m-1)$ $(\infty_1,\infty_2,\infty_3;4m-2)$ $(\infty_1,\infty_2,\infty_3;4m-3)$	$(\infty_1,\infty_2,\infty_3;4m,6m)$ $(\infty_1,\infty_2,\infty_3;4m-1,6m-2)$ $(\infty_1,\infty_2,\infty_3;4m-2,6m-4)$ $(\infty_1,\infty_2,\infty_3;4m-3,6m-6)$
6	$(\infty_1,\infty_2,\infty_4;\infty_5)$ (mod 5)	$(\infty_1,\infty_2,\infty_4)(\infty_3,\infty_5)$ $(\infty_2,\infty_4,\infty_5)(\infty_1,\infty_3)$ $(\infty_1,\infty_4,\infty_5)(\infty_2,\infty_3)$ $(\infty_1,\infty_2,\infty_5)(\infty_3,\infty_4)$
7	$(\infty_1,\infty_2,\infty_4;\infty_3)$ (mod 6)	$(\infty_1,\infty_2,\infty_4)(\infty_5,\infty_6)$ (mod 6)

Table 5

The above constructions give a solution for $n \geq 48$. The rest of the cases are represented in Table 6.

n	D	F
4	$(3,1,2;4)(3,4,1;2)$	
5	$(1,2,4;0)$ (mod 5)	$(1,2,4)(3,0),(2,4,0)(1,3)$ $(1,4,0)(2,3),(1,2,0)(3,4)$
6	$(1,2,4;3)$ (mod 6)	$(1,2,4)(0,5)$ (mod 6)
7	$(1,2,4;0)$ (mod 7)	$(1,2,4)(3,6)$ (mod 7)
10	$(1,2,4;8),(1,2,6;9)$ (mod 10)	$(1,2,4)(3,7),(1,2,6)(4,7)$ (mod 10)
11	$(1,2,4;8),(1,2,6;0)$ (mod 11)	$(1,2,4)(3,7),(1,2,6)(3,8)$ (mod 11)
12	$(1,4,7;9),(1,2,6;8)$ (mod 12)	$(1,2,7)(3,5),(1,2,6)(3,5)$ (mod 12)
13	$(1,2,5;7),(1,3,8;11)$ (mod 13)	$(1,2,5)(4,6),(1,3,8)(4,7)$ (mod 13)
14	$(1,4,8;10),(1,2,7;9)$ (mod 14)	$(1,4,8)(3,5),(1,2,7)(3,5)$ (mod 14)
15	$(1,2,8;12),(1,3,6;10)$ (mod 15)	$(1,2,8)(3,7),(1,3,6)(4,8)$ (mod 15)
19	$(1,2,5;11),(1,2,5;12)$ (mod 19) $(1,9,18;4)$ (mod 19)	$(1,2,5)(3,6),(1,2,5)(3,10)$ (mod 19) $(1,9,18)(3,8)$ (mod 19)
20	$(1,2,7;10),(1,3,10;13)$ (mod 20) $(1,5,13;3)$ (mod 20)	$(1,2,7)(3,6),(1,3,10)(4,7)$ (mod 20) $(1,5,13)(2,12)$ (mod 20)
21	$(1,11,20;4),(1,9,16;0)$ (mod 21) $(1,2,5;11)$ (mod 21)	$(1,11,20)(2,7),(1,9,16)(2,7)$ (mod 21) $(1,2,5)(3,9)$ (mod 21)
22	$(1,2,12;15),(1,6,14;15)$ (mod 22) $(1,3,7;14)$ (mod 22)	$(1,2,12)(3,6),(1,6,14)(4,7)$ (mod 22) $(1,3,7)(2,9)$ (mod 22)
23	$(1,11,22;0),(1,9,18;0)$ (mod 23) $(1,4,8;13)$ (mod 23)	$(1,11,22)(2,3),(1,9,18)(2,7)$ (mod 23) $(1,4,8)(2,7)$ (mod 23)

Mention here that one has no choice but to be technical in proving the existence of the coverings for small values of n.

The method used is to prescribe directed lengths $1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ on the edges of a suitable number of edge disjoint copies of H, each length at least once and at most twice and rotating. The work done here is to see that this is possible in the considered cases.

28	(1, 2, 15; 23), (1, 12, 24; 4) (mod 28) (1, 3, 7; 16)(1, 4, 11; 20) (mod 28)	(1, 2, 15)(3, 11), (1, 12, 24)(3, 11) (mod 28) (1, 3, 7)(2, 11)(1, 4, 11)(4, 13) (mod 28)
29	(1, 3, 16; 24), (1, 7, 18; 27) (mod 29) (1, 2, 5; 15)(1, 6, 13; 21) (mod 29)	(1, 3, 16)(2, 10), (1, 7, 18)(2, 11) (mod 29) (1, 2, 5)(3, 13)(1, 6, 13)(2, 10) (mod 29)
30	(1, 2, 4; 19), (1, 5, 10; 24) (mod 30) (1, 7, 14; 26)(1, 9, 19; 0) (mod 30)	(1, 2, 4)(3, 18), (1, 5, 10)(3, 17) (mod 30) (1, 7, 14)(3, 15)(1, 9, 19)(3, 14) (mod 30)
31	(1, 2, 4; 19), (1, 5, 10; 24) (mod 31) (1, 7, 14; 26)(1, 9, 19; 0) (mod 31)	(1, 2, 4)(3, 18), (1, 5, 10)(3, 17) (mod 31) (1, 7, 14)(3, 15)(1, 9, 19)(3, 14) (mod 31)
37	(1, 19, 36; 7), (1, 17, 32; 4) (mod 37) (1, 15, 28; 2)(1, 2, 5; 13) (mod 37) (1, 7, 14; 23) (mod 37)	(1, 19, 36)(2, 10), (1, 17, 32)(2, 11) (mod 37) (1, 15, 28)(2, 10)(1, 2, 5)(2, 10) (mod 37) (1, 7, 14)(2, 11) (mod 37)
38	(1, 2, 20; 23), (1, 6, 22; 31) (mod 38) (1, 10, 24; 34)(1, 3, 7; 18) (mod 38) (1, 8, 16; 28) (mod 38)	(1, 2, 20)(3, 6), (1, 6, 22)(2, 11) (mod 38) (1, 10, 24)(2, 12)(1, 3, 7)(2, 13) (mod 38) (1, 8, 16)(2, 14) (mod 38)
39	(1, 18, 36; 4), (1, 16, 32; 2) (mod 39) (1, 14, 28; 38)(1, 2, 4; 11) (mod 39) (1, 6, 11; 30) (mod 39)	(1, 18, 36)(2, 9), (1, 16, 32)(2, 11) (mod 39) (1, 14, 28)(2, 12)(1, 2, 4)(3, 10) (mod 39) (1, 6, 11)(2, 21) (mod 39)
47	(1, 23, 46; 31), (1, 21, 42; 29) (mod 47) (1, 19, 36; 25)(1, 17, 34; 25) (mod 47) (1, 2, 5; 13)(1, 6, 13; 14) (mod 47)	(1, 23, 46)(2, 17), (1, 21, 42)(2, 15) (mod 47) (1, 19, 36)(2, 13)(1, 17, 34)(2, 11) (mod 47) (1, 2, 5)(3, 11)(1, 6, 13)(2, 3) (mod 47)

Table 6

We close our proof by saying that we did not use the known fact (see [11]) that for $n = 8k + 4$ and $n = 8k + 5$ the edges of K_n can be covered exactly twice by D respectively by F.

Example: In Table 7 we demonstrate the technique appearind in Proposition 4.7 for the case $H = F$ and $n = 56 + j$, $j = 1, 2, \ldots, 7$.

j	the covering
1	(1, 14, 29)(28, 42), (1, 12, 28)(27, 40), (1, 10, 27)(26, 38), (1, 8, 26)(25, 36), (1, 6, 25)(24, 34), (1, 4, 24)(23, 32), (1, 2, 23)(22, 30)
2	(1, 14, 29)(28, 42), (1, 12, 28)(27, 40), (1, 10, 27)(26, 38), (1, 8, 26)(25, 36), (1, 6, 25)(24, 34), (1, 4, 24)(23, 32), (1, 2, 23)(22, 30), (1, 14, 29)(28, ∞_1)
3	(1, 14, 29)(28, 42), (1, 12, 28)(27, 40), (1, 10, 27)(26, 38), (1, 8, 26)(25, 36), (1, 6, 25)(24, 34), (1, 4, 24)(23, 32), (1, 2, 23)(22, 30), (1, 14, 29)(28, ∞_1)(1, 12, 28)(27, ∞_2) (1, 2, 23)(∞_1, ∞_2)

4	$(1,14,29)(28,42), (1,12,28)(27,40), (1,10,27)(26,38),$ $(1,8,26)(25,36), (1,6,25)(24,34), (1,4,24)(23,32),$ $(1,2,23)(22,30), (1,14,29)(28,\infty_1)(1,12,28)(27,\infty_2)$ $(1,10,27)(26,\infty_3)(\infty_1,\infty_2,\infty_3)(28,42)$
5	$(1,14,29)(28,42), (1,12,28)(27,40), (1,10,27)(26,38),$ $(1,8,26)(25,36), (1,6,25)(24,34), (1,4,24)(23,32),$ $(1,2,23)(22,30), (1,14,29)(28,\infty_1)(1,10,27)(26,\infty_3)$ $(1,12,28)(27,\infty_2)(1,8,26)(25,\infty_4)(\infty_1,\infty_2,\infty_3)(28,42)$ $(\infty_1,\infty_2,\infty_4)(27,40)(\infty_1,\infty_3,\infty_4)(26,38)(\infty_2,\infty_3,\infty_4)(25,36)$
6	$(1,14,29)(28,42), (1,12,28)(27,40), (1,10,27)(26,38),$ $(1,8,26)(25,36), (1,6,25)(24,34), (1,4,24)(23,32),$ $(1,2,23)(22,30), (1,14,29)(28,\infty_1)(1,10,27)(26,\infty_3)$ $(1,12,28)(27,\infty_2)(1,8,26)(25,\infty_4)(1,6,25)(24,\infty_5)$ $(\infty_1,\infty_2,\infty_4)(\infty_3,\infty_5)(\infty_2,\infty_4,\infty_5)(\infty_1,\infty_3)$ $(\infty_1,\infty_4,\infty_5)(\infty_2,\infty_3)(\infty_1,\infty_2,\infty_5)(\infty_3,\infty_4)$
7	$(1,14,29)(28,42), (1,12,28)(27,40), (1,10,27)(26,38),$ $(1,8,26)(25,36), (1,6,25)(24,34), (1,4,24)(23,32),$ $(1,2,23)(22,30), (1,14,29)(28,\infty_1)(1,12,28)(27,\infty_2)$ $(1,10,27)(26,\infty_3)(1,8,26)(25,\infty_4)(1,6,25)(24,\infty_5)$ $(1,4,24)(23,\infty_6)(\infty_1,\infty_2,\infty_4)(\infty_5,\infty_6) \pmod 6$

Table 7

Proposition 4.8. *The main theorem holds for the graph numbered 18, namely,*

$$ol(P_3 \cup 2K_2, n) = \begin{cases} 1 & n \equiv 0,1 \pmod 8 \\ 2 & \textit{otherwise.} \end{cases}$$

Proof: The case $ol(P_3 \cup 2K_2, K_n) = 1$ is a decomposition result appears in [30]. To see that $ol(P_3 \cup 2K_2, K_n) = 2$ in the other cases observe that, $(P_3 \cup 2K_2) \mid P_9$. Hence, by Theorem 3.1 and since for $n \geq 9$, $P_9 \mid K_n$, we conclude the required result. The small cases $n = 7, 8$ are covered as follows:

For $n = 7$ the covering is: $(0,1,5)(2,4)(3,6) \pmod 7$. For $n = 8$ the covering is: $(0,1,6)(2,5)(3,7) \pmod 8$. □

Proposition 4.9. *The main theorem holds for the graph numbered 21, namely,*

$$ol(K_4, n) = \begin{cases} 1 & n \equiv 1,4 \pmod{12} \\ 3 & n = 5 \\ 2 & \textit{otherwise.} \end{cases}$$

Proof: The fact that $ol(K_4, n) = 1$ for $n \equiv 1, 4 \pmod{12}$ is a well-known decomposition result appears in [20], namely,

$$K_4 \mid K_n, \text{ for } n \equiv 1, 4 \pmod{12}. \tag{7}$$

For $n = 4$ it is trivial. For $n = 5$ one can easily see that $ol(K_4, 5) = 3$. Let $V(K_4) = \{a, b, c, d\}$. We shall denote it by $[a, b, c, d]$. In Table 8 we show that $ol(K_4, n) = 2$, for $6 \leq n \leq 12$.

n	The Covering
6	$[0,1,2,4] \pmod 6$
7	$[0,1,3,6] \pmod 7$
8	$[0,1,3,7] \pmod 8$
9	$[0,1,3,7] \pmod 9$
10	$[0,1,3,6] \pmod{10}$
11	$[0,1,3,7] \pmod{11}$
12	$[0,1,3,7] \pmod{12}$

Table 8

Next we follow the proof of Theorem 2.1 with its parameters t, h. We shall use them according to the residues $\pmod{12}$. For $n \equiv 1, 4 \pmod{12}$ we have (7). Put $n \equiv j \pmod{12}$. In Table 9 we find the appropriate values of t, h.

j	t	h
0	8	1
2	1	0
3	2	0
5	1	1
6	2	1
7	3	1
8	4	1
9	5	1
10	6	1
11	7	1

Table 9

Substituting those values in the conditions of Theorem 2.1 for M, we find that only for the following values of n the result can not be derived from Theorem 2.1, $5 \leq n \leq 12$, which was treated above. Also the following values of n has to be taken care:

$$n \in \{19, 20, 21, 22, 23, 24, 33, 34, 35, 36, 46, 47, 48, 59, 60, 72\}.$$

Before treating those values we recall a result due to Ray-Chaudhuri and Wilson ([32] and [31]) which states that if $n \equiv 3 \pmod 6$ then K_n has a K_3-resolvable design.

We give a common covering for all those values except for $n = 20, 24$. Observe that $n = 12k + 4 + j$, where, $1 \le k \le 5$ and $j \in \{3, 5, 6, 7, 8\}$. So we write,

$$K_{12k+4+j} = K_{12k+4} \cup K_{j,12k+4} \cup K_j.$$

By (7) we have the K_4-decomposition of K_{12k+4}. Let u be a vertex in K_{12k+4} and put, $K_{12k+3} = K_{12k+4} \setminus \{u\}$.

Applying the K_3-resolvable design to K_{12k+3} and taking each time a vertex from K_j with some appropriate K_3-factor we cover $K_{j,12k+3}$ leaving uncovered the graph $K_{j+1} = K_j \cup \{u\}$. For each of the values of j we use the appropriate covering presented in Table 8.

We are left with two values of n for which we give a different treatment.

n = 20:

Let,

$$K_{20} = K_{16} \cup K_4 \cup K_{4,16} \qquad (8)$$

By (7) we have the K_4-decomposition of K_{16}. Let u be some vertex in K_{16} and put $K_{15} = K_{16} \setminus \{u\}$. Then K_{15} has K_3-resolvable design results in 7 K_3-factors. So that each vertex of the vertices of K_4 together with the appropriate K_3-factor creates a complete covering of the bipartite graph $K_{4,15}$. To finish the covering let v be another vertex in K_{16}.Then $K_4 \cup \{u, v\} = K_6$. Let the vertices of that particular K_6 be denoted $\{1, 2, 3, 4, u, v\}$. Use the following K_4's in the cover (note that (v, i), $i = 1, 2, 3, 4, u$ are already used once) $[v, 1, 2, 3]$, $[1, 3, 4, u]$, $[2, 3, 4, u]$.

n = 24:

Let, $V(K_{24}) = A \cup B \cup C$, where, $|A| = |B| = |C| = 8$. Now the induced graph on the union of each pair of those sets creates K_{16} for which we use the K_4-decomposition assured by (7). The edges whose end-vertices are contained in either A, B or C are covered twice. All other edges are covered exactly once.

This completes the proof of the Main Theorem. □

5 Concluding remarks and open problems

1) The content of this paper raises further remarks and open problems. In this section we present those which we consider most appropriate.

One might wonder (as is essentially mentioned in [24]) about the parallel part of covering with overlap λ. Again the divisibility conditions suffices by the general form of Wilson's Theorem. In fact with slight modifications in the proof of Theorem 2.1, one can prove the following stronger form. However the essence is completely given in Theorem 2.1 and the only change is that one need to allocate λK_k-factors (respectively K_{k-1}-factors) to saturate each vertex (respectively edge), instead of allocating just one K_k-factor (K_{k-1}-factor).

Theorem 5.1. *Given a graph H and an integer λ. There exists $M(H,\lambda)$ such that if $n \geq M(H,\lambda)$ then, there exists a covering Π of the edges of K_n with copies of H such that every edge of K_n is covered either λ or $\lambda+1$ times. Further, the number of copies of H in Π is at most $\frac{\lambda n(n-1)}{2e(H)} + c(H,\lambda)n$, where, $c(H,\lambda)$ is a constant which depends only on H and λ.*

2) Etzion ([14]) raised the following question, related to Remark 2, and aimed to strengthen Theorem 2.1.

Is it true that:

$CO(H,n) - C(H,n) \leq c(H)$, where $c(H)$ is a constant depending only on H?

3) In [3] we find the following conjecture, extending Theorem 3.6.

Let T be any tree. Then,

$$ol(T,n) \leq 2, \text{ for all } n \geq |T|.$$

Below we state the 2-overlap analogue of the Oberwolfach Problem and the Gyarfas-Lehel Conjecture ([18]).

4) *Let, $C_{n_1}, C_{n_2}, \ldots, C_{n_s}$, be a set of cycles of sizes $n_1 < n_2 < \cdots < n_s$. Is it true that if $n = n_s$ then the edges of K_n can be covered by all the cycles from that set with each cycle appears at least once but the edges of K_n are covered at most twice?*

5) *Let, $T_{n_1}, T_{n_2}, \ldots, T_{n_s}$, be a set of trees of sizes $n_1 < n_2 < \cdots < n_s$. Is it true that if $n = n_s$ then the edges of K_n can be covered by all the trees from that set with each tree appears at least once but the edges of K_n are covered at most twice?*

Note that in both problems 4 and 5 if n is much larger than n_s then the answer is affirmative by a simple application of Theorem 2.1.

6) *One could ask for which H the required covering can be cyclic as in Tables 6 or 7.*

Note added in proof. Question 2 is recently solved in the affirmative by Caro and Yuster.

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