A Result of Erdös-Sós Cojecture

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ABSTRACT. Erdös and Sós conjectured in 1963 that if G is a graph of order p and size q with $q > \frac{1}{2}p(k-1)$, then G contains every tree of size k. This is proved in this paper when the girth of the complement of G is greater than 4.

1 Introduction

We shall use standard graph theory notation. We consider only simple graph of order p and size q. The complement of G is denoted by $\bar{G}$, and girth of G by $g(G)$. Other notation and terminology not defined here can be found in [1].

Erdös and Gallai in[2] proved that if $q > \frac{1}{2}p(k-1)$ then G contains a path of lenth k. In 1963, Erdös and Sós made the following

Conjecture. *If $q > \frac{1}{2}p(k-1)$, then G contains every tree of size k.*

Let G_1, G_2 be a pair of graphs. If G_1 is isomorphic to a spanning subgraph of $\bar{G}_2$, we say that G_1 can be embedded in $\bar{G}_2$, or that G_1 and G_2 are packable, and write $G_1 \prec \bar{G}_2$. From which, $G_1 \prec \bar{G}_2$ implies that there is a bijection σ from $V(G_1)$ to $V(G_2)$ such that σ is a isomorphic mapping from G_1 to a spanning subgraph of $\bar{G}_2$, and we say that the mapping σ is a packing of G_1 and G_2. We denote by T_k a tree of size k, and by E_k an independent set of k vertices. In this paper, we show that If $q > \frac{1}{2}(k-1)p$, and $g(\bar{G}) > 4$, then G contains every tree of size k.

2 Lemmas

Lemma 1. [1] *Let G be a simple graph and T_k a tree of size k. If $\delta(G) \geq k$ then T_k can be embedded in G.*

In a version of packing, Lemma 1 can be rewrited as the follows.

Lemma 2. *If $G_1 = T_{p-k} \cup E_{k-1}$, and $\Delta(G_2) \leq k-1$ for $1 \leq k \leq p-1$, then $G_1 \prec \bar{G}_2$.*

Lemma 3. *Let $G_1 = T_{p-k} \cup E_{k-1}$. If G_2 contains no K_3 as an induced subgraph and $k > \frac{p}{2}$, then $G_1 \prec \bar{G}_2$.*

Proof: Since $k > \frac{p}{2}$ we have that $p - k < \frac{p}{2}$. If $\Delta(G_2) \leq \frac{p}{2}$ then $\Delta(G_2) \leq k-1$. By Lemma 2, we get $G_1 \prec \bar{G}_2$. Assume that $\Delta(G_2) > \frac{p}{2}$. Choose a vertex $u \in V(G_2)$ such that $d_{G_2}(u) = \Delta(G_2) = m$. Since G_2 contains no K_3 as an induced subgraph, $N_{G_2}(u)$ is an independent set of G_2, and so it is a clique of $\bar{G}_2$. On the other hand, we have that $m \geq p - k + 1$ since $m > \frac{p}{2}$ and $\frac{p}{2} > p - k$. So G_1 and G_2 are packable.

Lemma 4. [3] *Let H_i be a spanning subgraph of G_i, $i = 1, 2$. If G_1 and G_2 are packable, then H_1 and H_2 are also packable.*

Lemma 5. [3] *Let G_1 and G_2 be two graphs with the same order $p \geq 7$. If G_1 is a tree, then G_1 and G_2 are packable if and only if $\{G_1, G_2\}$ is not any one of the following pairs:*

(1) *$\{S_p, G\}$, where S_p denotes a star of order p and G contains no isolated vertex;*

(2) *$\{S'_p, C\}$, where S'_p is a tree obtained from S_{p-1} by adding an inserting vertex in some edge of S_{p-1} and C is a 2-regular graph;*

(3) *$\{S''_p, mk_3\}$, where S''_p is a tree obtained from S_{p-2} by adding two inserting vertices in some edge of S_{p-2};*

(4) *$\{T, S\}$, where T is a tree and S is a graph obtained from S_p by adding an edge which connects two vertices with degree 1.*

3 Main result and proof

In the version of packing theory our main result can be rewrite as follows.

Theorem. *Let $G_1 = T_{p-k} \cup E_{k-1}$. If $q(G_2) < \frac{1}{2}pk$ $(k = 1, \ldots, p-1)$ and $g(G_2) > 4$, then $G_1 \prec G_2$.*

Proof: We prove by induction on p. By Lemma 3, we need only to consider the case $k \leq \frac{p}{2}$.

It is easy to see that the theorem is true when $p < 6$. So we assume that $p \geq 6$. From the argument above, we only need to consider the case T_{p-k} is not a star.

Since T_{p-k} is not a star, we can choose a vertex $v_1 \in G_1$ such that

(1) $d_{G_1}(v_1) \geq 2$;

(2) $N_{G_1}(v_1)$ contains only one vertex which is not an end vertex;

(3) Subject to (1) and (2), $d(v_1)$ is as small as posible.

Let $v \in N_{G_1}(v_1)$ be an end vertex. Put

$$N_{G_1}(v_1) = \{v, g_1, \ldots, g_{t-1}\}, \quad E_{k-1} = \{g'_1, \ldots, g'_{k-1}\},$$

$$V(G_1) - (N_{G_1}(v_1) \cup E_{k-1}) = \{g'_k, \ldots, g'_n\}.$$

Denote by h the number of vertices in $\{g'_1, \ldots, g'_n\}$ with degree 1. By the choice of v_1, we have that $h \geq t-1$. Assume, without loss of generality, that $d_{G_1}(g_i) = 1$ for $i = k, \ldots, k+h-1$.

Let $u \in V(G_2)$ be a vertex with maximum degree. Set $d_{G_2}(u) = \Delta(G_2) = m$. By Lemma 2, we only need to consider the case $m \geq k$. Put

$$N_{G_2}(u) = \{u_1, \ldots, u_m\}, \quad N_{G_2}(u_1) = \{w_1, \ldots, w_r\},$$

$$V(G_2) - (N_{G_2}(u) \cup N_{G_2}(u_1)) = \{w'_1, \ldots, w'_b\}.$$

Since $g(G_2) > 4$, we have that $N_{G_2}(u) \cap N_{G_2}(u_1) = \emptyset$.

Let $H_1 = G_1 - \{v\}$, $H_2 = G_2 - \{u\}$. Then $q(H_1) = (p-1) - k$, and $q(H_2) = q(G_2) - m < \frac{1}{2}pk - k < \frac{1}{2}(p-1)k$. By induction, there is a packing τ of H_1 and H_2. If $\tau(v_1) \notin N_{G_2}(u)$, then we can extend τ to a required packing σ of G_1 and G_2 by letting $\sigma(v) = u$ and $\sigma(w) = \tau(w)$ for any $w \in V(H_1)$. So we just consider the case $\tau(v_1) \in N_{G_2}(u)$. We assume more that

$$\tau(v_1) = u_1, \ \tau(g_i) = x_i (i = 1, \ldots, t-1), \ \tau(g'_j) = x'_j (j = 1, \ldots, s) \quad (1)$$

since τ is a packing of H_1 and H_2, we have that $x_i \in \{w'_1, \ldots, w'_b\} \cup \{u_2, \ldots, u_m\}$.

If there is a vertex x'_i $(1 \le i \le k-1) \notin N_{G_2}(u_1)$, then we can extend τ to a required packing σ of G_1 and G_2 by letting $\sigma(v) = x'_i$, $\sigma(g'_j) = u$, and $\sigma(w) = \tau(w)$ for any $w \in V(H_1) - \{g'_i\}$. Now we assume that

$$x'_i \in N_{G_2}(u_1), i = 1, \ldots, k-1, x'_i = w_i, i = 1, \ldots, k-1 \qquad (2)$$

Two cases must be considered.

Case 1. There is an integer $i \in \{k, \ldots, k+h-1\}$ such that $x'_i \notin N_{G_2}(u_1)$. Let x'_j $(j > k+h-1)$ be such a vertex of G_2 such that $\tau^{-1}(x'_i)\tau^{-1}(x'_j) = g'_i g'_j \in E(G_1)$.

If $x'_j \notin N_{G_2}(u)$, then we can extend τ to a required packing σ of G_1 and G_2 by letting $\sigma(v) = x'_i$, $\sigma(g'_i) = u$, and $\sigma(w) = \tau(w)$ for any $w \in V(H_1) - \{g'_i\}$.

If $x'_j \in N_{G_2}(u)$, since $g(G_2) > 4$ we can extend τ to a required packing σ of G_1 and G_2 by letting $\sigma(v) = x'_i$, $\sigma(g'_i) = w_1$, $\sigma(g'_1) = u$, and $\sigma(w) = \tau(w)$ for any $w \in V(H_1) - \{g'_1, g'_i\}$.

Case 2. $x'_i \in N_{G_2}(u_1)$, $i = k, \ldots, k+h-1$. Assume, without loss of generality, that $x'_i = w_i$ for $i = k, \ldots, k+h-1$, and together with (2), we have that

$$x'_i = w_i, \quad i = 1, \ldots, k+h-1 \qquad (3)$$

since $g(G_2) > 4$, w'_i is adjacent to at most one vertex in $\{w_1, \ldots, w_r\}$ for $i = 1, \ldots, b$. Put

$$M = \{w'_i \mid i = 1, \ldots, b\} \cap \{x_i \mid i = 1, \ldots, t-1\} = \{w'_1, \ldots, w'_x\}$$

we see that $x \le t-1$, and $k+h-1 > t-1 \ge x$ since $h \ge t-1$ and $k \ge 2$.

If $M = \emptyset$, then $x_i \in \{u_2, \ldots, u_m\}$ for $i = 1, \ldots, t-1$, and so we can extend τ to a required packing σ of G_1 and G_2 by letting $\sigma(v) = u$, $\sigma(g'_1) = u_1$, $\sigma(v_1) = w_1$, and $\sigma(w) = \tau(w)$ for any $w \in V(H_1) - \{g'_1, v_1\}$ since $g(G_2) > 4$.

If $M \ne \emptyset$, there is $w_i \in \{w_j \mid j = 1, \ldots, k+h-1\}$ such that $N(w_i) \cap M = \emptyset$ since $k+h-1 > x$.

Subcase 1. $i \in \{1, \ldots, k-1\}$. Since $\tau(g'_i) = x'_i = w_i$, and $N_{G_2}(w_i) \cap \{u_2, \ldots, u_m\} = \emptyset$, we can extend τ to a required packing σ of G_1 and G_2 by letting $\sigma(v) = u$, $\sigma(g'_i) = u_1$, $\sigma(v_1) = w_i$, and $\sigma(w) = \tau(w)$ for any $w \in V(H_1) - \{g'_i, v_1\}$.

Subcase 2. $i \in \{k, \ldots, k+h-1\}$. Let j be an integer such that $g'_i g'_j \in E(G_1)$. If $x'_j \notin N_{G_2}(u_1)$, then we can extend τ to a required packing σ of G_1 and G_2 by letting $\sigma(v) = u$, $\sigma(g'_i) = u_1$, $\sigma(v_1) = w_i$, and $\sigma(w) = \tau(w)$ for any $w \in V(H_1) - \{g'_i, v_1\}$. If $x'_j \in N_{G_2}(u_1)$, then we can extend τ to a required packing σ of G_1 and G_2 by letting $\sigma(v) = u$, $\sigma(g'_i) = w_1$, $\sigma(v_1) = w_i$, $\sigma(g'_1) = u_1$, and $\sigma(w) = \tau(w)$ for any $w \in V(H_1) - \{g'_i, v_1, g'_1\}$. The proof of the theorem is completed.

References

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