TWO CLASSES OF GRACEFUL GRAPHS

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Abstract. In this paper, we prove the gracefulness of a new class of graphs denoted by $K_n \otimes S_{2^{n-1}-\binom{n}{2}}$. We also prove that the graphs consisting of $2m+1$ internally disjoint paths of length $2r$ each, connecting two fixed vertices, are also graceful.

1. Introduction

A graph $G = (V, E)$ is numbered if each vertex v is assigned a non–negative integer $\varphi(v)$ and each edge uv is assigned the value $|\varphi(u) - \varphi(v)|$. The numbering is called graceful if, further, the vertices are labelled with distinct integers from $\{0,1,2,...,q\}$ and the edges with the integers from 1 to q, where q is the number of edges of G. A graph which admits a graceful numbering is said to be graceful. For the literature on graceful graphs we refer to [1,2] and the relevant reference given in them. In this paper we prove the gracefulness of a new class of graphs denoted by $K_n \otimes S_{2^{n-1}-\binom{n}{2}}$.

Definition 1.1 Let S_m be a star with m edges. Let $K_n \otimes S_m$ be a graph obtained by identifying any vertex of K_n with any vertex of S_m other than the centre of S_m.

Definition 1.2 Let u and v be two fixed vertices. We connect u and v by means of "b" internally disjoint paths of length "a" each. The resulting graph is denoted by $P_{a,b}$.

Example. The graph of $P_{4,3}$ is displayed in Fig. 1.

In this paper we also establish that the graphs $P_{2r,2m+1}$ are graceful.

2. GRACEFULNESS OF $K_n \otimes S_{2^{n-1}-\binom{n}{2}}$.

Theorem 2.1 For all values of n, the graphs $G_n = K_n \otimes S_{2^{n-1}-\binom{n}{2}}$ are graceful.

Proof. Clearly G_n has $n + 2^{n-1} - \binom{n}{2}$ vertices and 2^{n-1} edges. Let u_i, $i = 1, 2, \ldots, n$ be the vertices of K_n and let v_i, $i = 0, 1, 2, \ldots, 2^{n-1} - \binom{n}{2}$ be the vertices of $S_{2^{n-1}-\binom{n}{2}}$ where v_0 is the centre of the star and $u_n = v_{2^{n-1}-\binom{n}{2}}$.

Define φ on the vertices of G_n by the following rule.

$\varphi(u_i) = 2^{i-1}$, $i = 1, 2, \ldots, n$.

$\varphi(v_0) = 0$

and assign the $2^{n-1} - \binom{n}{2} - 1$ numbers from the set $\{1, 2, \ldots, 2^{n-1}\} - [\{2^j - 2^i \;/\; i = 0, 1, 2, \ldots, n-2; j = i+1, \ldots, n-1\} \cup \{2^{n-1}\}]$ to the vertices $v_1, v_2, \ldots, v_{2^{n-1}-\binom{n}{2}-1}$ of $S_{2^{n-1}-\binom{n}{2}}$ in any way so that each vertex receives exactly one number.

It is straightforward to verify that φ is a one–to-one map from the vertex set of G_n into $\{0, 1, \ldots, q\}$.

The labels of the edges of K_n are $2^j - 2^i$, $i = 0, 1, \ldots, n-2$ and $j = i+1, i+2, \ldots, n-1$ and the labels of the edges of the star are $\{1, 2, \ldots, 2^{n-1}\} - \{2^j - 2^i \;/\; i = 0, 1, 2, \ldots, n-2; j = i+1, \ldots, n-1\}$. Hence $\{ |\varphi(u) - \varphi(v)| \;/\; uv \in E\} = \{1, 2, \ldots, 2^{n-1}\}$ so that G_n is graceful for all n.

The graceful numbering of G_5 is displayed in Fig. 2.

Corollary 2.2 Every graph is a subgraph of a graceful graph.

3. GRACEFULNESS OF $P_{2r,\,2m+1}$

Theorem 3.1 $P_{2r,2m+1}$ are graceful for all values of r and m.

Proof. Let $v_0^i, v_1^i, v_2^i, \ldots, v_{2r}^i$ be the vertices of the i-th copy of the path of length $2r$, where $i = 1, 2, \ldots, 2m+1$, $v_0^i = u$ and $v_{2r}^i = v$ for all i. We observe that the number of vertices of this graph is $4mr + 2r - 2m+1$ and the number of edges of this graph is $4mr + 2r$.

Define φ on the vertex set of $P_{2r,2m+1}$ as follows.

$$\varphi(v_{2j+1}^i) = 4mr + 2r - j(2m+1) - i + 1, \quad i = 1, 2, \ldots, 2m+1$$

$$j = 0, 1, 2, \ldots, r-1$$

$$\varphi(v_{2j}^i) = 4m + 3 + (j-1)(2m+1) - 2i, \quad i = 1, 2, \ldots, 2m+1$$

$$j = 1, 2, \ldots, r-1$$

$\varphi(u) = 0$ and $\varphi(v) = (2m+1)r$.

It is not too hard to show that this assignment provides a graceful numbering of $P_{2r,\,2m+1}$. In Fig.3 we illustrate this graceful numbering for $P_{6,5}$.

We conclude with the following conjecture.

Conjecture: $P_{a,b}$ is graceful except when $a = 2r + 1$ and $b = 4s + 2$

References

[1] S.W.Golomb, How to number a graph, *Graph Theory and Computing* (Ed. R.C.Read), Academic Press, New York, 1972, 23–27.

[2] J.A.Gallian, A Survey: Recent results, conjectures and open problems in labeling graphs, *J. Graph Theory*, 13(1989), 491–504.

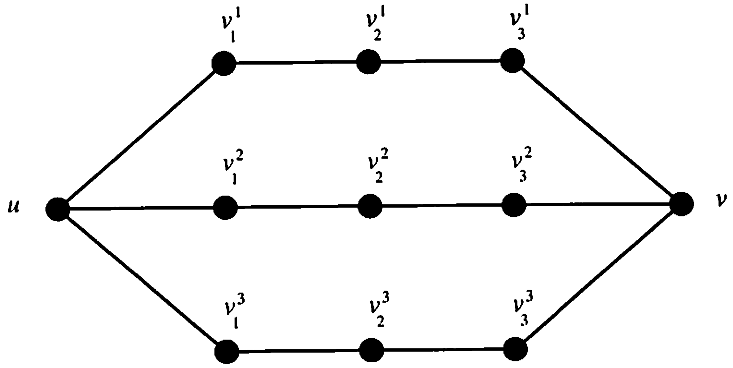

Fig. 1 Graph of $P_{4,3}$

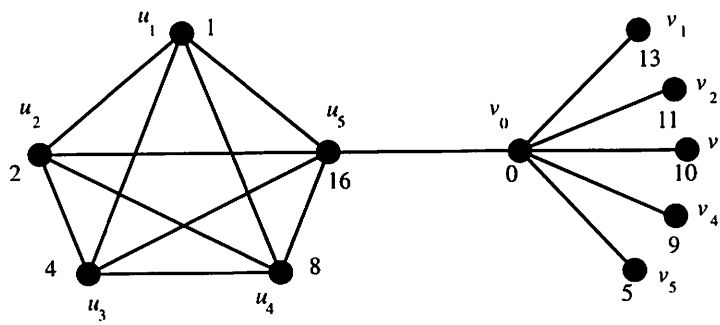

Fig. 2 Graceful numbering of G_5

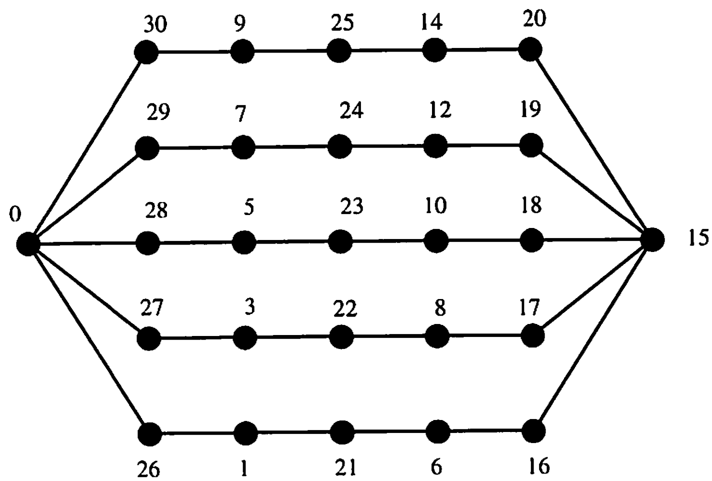

Fig. 3 Graceful numbering of $P_{6,5}$