The total number of generalized stable sets and kernels of graphs

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ABSTRACT: In [8] a graph - representation of the Fibonacci numbers F_n and Lucas numbers F_n^* was presented. It is interesting to know that they are the total numbers of all stable sets of undirected graphs P_n and C_n, respectively. In this paper we discuss a more general concept of stable sets and kernels of graphs. Our aim is to determine the total numbers of all k-stable sets and $(k, k-1)$ - kernels of graphs P_n and C_n. The results are given by the second-order linear recurrence relations containing generalized Fibonacci and Lucas numbers. Recent problem were investigated in [9], [10].

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1.Introduction

For general concepts, we refer the reader to [5]. By a graph G we mean a finite undirected connected graph without loops and multiple edges. $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The length of the shortest path joining vertices x and y in G will be denoted by $d_G(x,y)$. Recall that the length of the path is the number of edges in it. If $X \subseteq V(G)$ and $x \in V(G)$, then we put $d_G(x, X) = min\{d_G(x,x'); x' \in X\}$. The notation $G - X$ means the graph obtained from G by deleting the subset X. By P_n and C_n, for $n \geq 2$, we mean graphs with the vertex sets $V(P_n) = V(C_n) = \{x_1, ..., x_n\}$ and the edge sets $E(P_n) = \{[x_i, x_{i+1}]; i = 1, 2, ..., n-1\}$ and $E(C_n) = E(P_n) \cup \{[x_n, x_1]\}$, respectively. In addition, $C_1 = P_1$ where P_1 is a graph consists of only one vertex.

Let k be a fixed integer, $k \geq 2$.
A subset $J \subset V(G)$ is said to be a $(k, k-1)$- kernel of G if
(1) for each two distinct vertices $x, y \in J$, $d_G(x,y) \geq k$ and
(2) for each $x' \in V(G) \setminus J$, there exists $x \in J$ such that $d_G(x', x) \leq k-1$.
In addition, a subset containing only one vertex also is a $(k, k-1)$- kernel of G.

Note that for $k = 2$ the definition reduces to the definition of a kernel of the graph G.
In further investigations a subset $S \subseteq V(G)$ satisfying the condition in (1) will be called a k-stable set of G. It has been proved in [6] that every maximal k-stable set of the graph G is a $(k, k-1)$- kernel of G, for $k \geq 2$. The k-stable sets and $(k, k-1)$- kernels (also called n - independent dominating sets) and more generalized kernels called (k, l) - kernels, for $l \geq 1$, were studied in [2],[3],[4],[6],[7],[9],[10].

Let $X = \{1, 2, ..., n\}$, $n \geq 1$ and let $Y \subseteq X$ such that

(3) $|Y| = p$, for a fixed p, $0 \leq p \leq n$ and

(4) Y does not contain two consecutive integers, where $|Y|$ denotes the cardinality of Y.

We add that, if $n = 1$, then $Y = X$ (and for this case $p = n$) and note that $Y = \emptyset$ (i.e. $p = 0$) also is to be taken into consideration. By $f(n,p)$ we denote the number of all subsets Y having exactly p elements and

(5) $f(n,p) = \binom{n-p+1}{p}$.

The number $F_n = \sum_p f(n,p)$ is called the Fibonacci number, see [1]. In graph terminology, the number F_n, for $n \geq 1$ is equal to the total number of subsets $S \subseteq V(P_n)$ such that each two vertices of S are not adjacent. In other words, F_n is the total number of all 2-stable sets (short:stable sets) of the graph P_n. We mean $Y = \emptyset$ as a stable set of a graph so that $f(n, 0)$ has a graph interpretation. It may be interesting to note that Fibonacci numbers F_n are defined by the second-order recurrence relations:
$F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ with initial conditions $F_0 = 1$ and $F_1 = 2$, where $n = 0$ corresponds to $X = \emptyset$, see [1]. For a graph interpretation of the number F_0 we introduce the empty graph P_0 having a unique stable set $X = \emptyset$.

Let $X = \{1, 2, ..., n\}$, $n \geq 1$ and let $Y^* \subseteq X$ such that

(6) $|Y^*| = p$, for a fixed $p, 0 \leq p \leq n$ and

(7) Y^* does not contain either two consecutive integers or both 1 and n simultaneously.

The number of all subsets Y^* having exactly p elements is denoted by $f^*(n,p)$. Moreover, for $n \geq 3$,

(8) $f^*(n,p) = f(n-3, p-1) + f(n-1, p) = \frac{n}{n-p}\binom{n-p}{p}$, see [1].

Of course $f^*(n,p) = f(n,p)$ for $n = 0, 1, 2$.
The number $F_n^* = \sum_p f^*(n,p)$ is called the Lucas number, see [1], and in the graph interpretation the number F_n^* is equal to the total number of stable sets of the graph C_n, see [8].
For Lucas numbers there is the well- known formula $F_n^* = F_{n-1}^* + F_{n-2}^*$, for $n \geq 2$, with initial conditions $F_0^* = 1$ and $F_1^* = 2$, see [1]. Note that the initial conditions correspond to graphs $C_0 = P_0$ and C_1, mentioned earlier.

2. Generalizations

In this section we present some relevant features of generalized Fibonacci and Lucas numbers.

Let $k \geq 2$ be an integer and let $X = \{1, 2, ..., n\}, n \geq 1$. In addition, we put $n = 0$ for $X = \emptyset$.
We say that two distinct integers $i, j \in X$ are k - distance consecutive if $|i - j| < k$.
Let $Y \subseteq X$ such that

(9) $|Y| = p$ for a fixed p, $0 \leq p \leq n$ and

(10) $i, j \in Y$ if they are not k - distance consecutive.

By $f(k, n, p)$ we denote the number of all such subsets Y having exactly p elements and further let $F(k, n) = \sum_p f(k, n, p)$. It is easy to see that for $k = 2$ the condition in (10) is equivalent to the condition in (4). Therefore, $f(2, n, p) = f(n, p)$ and $F(2, n) = F_n$

Proposition 1. *Let k, n, p be integers, $k \geq 2, n \geq 0, 0 \leq p \leq n$.*

Then we have the formula
$f(k, n, p) = \binom{n-p-(p-1)(k-2)+1}{p}$.

Proof: For $k = 2$ we have $f(2, n, p) = f(n, p)$ and by (5) the result follows. Let $k \geq 3$. Our intent is to calculate the number of all subsets of X (considering also $X = \emptyset$) having exactly p elements and not containing two k - distance consecutive elements. Suppose that S is one of such subsets of X. For convenience, instead of the subset S we can consider a sequence $\alpha = (\alpha_1, ..., \alpha_n)$ whose elements α_i satisfy the following conditions:
(11)

$$\alpha_i = \begin{cases} 1 & \text{if} \quad i \in S \\ 0 & \text{otherwise} \end{cases}$$

and

(12) $\sum_{i=1}^{n} \alpha_i = p$ and

(13) if $\alpha_i, \alpha_j = 1$, then $|i - j| \geq k$.

To calculate the number of all such sequences we start with a sequence $\beta = (\beta_1, ..., \beta_{n-p})$, where $\beta_i = 0$, for each $i = 1, ..., n-p$. Next, we choose from β a subsequence $\beta' = (\beta_{k-1}, \beta_{2k-2}, ..., \beta_{(p-1)k-(p-1)}, \beta_{(p-1)k-(p-1)+1}, ..., \beta_{n-p})$ on $(n-p)-(p-1)(k-2)$ elements. This follows by observing that for the building of sequence α it suffices to extend the subsequence β' by p 1's in such a way that no two 1's are consecutive. The total number of all possible sequences α is equal to the number $\binom{n-p-(p-1)(k-2)+1}{p}$ which shows that $f(k, n, p) = \binom{n-p-(p-1)(k-2)+1}{p}$.
Let $Y^* \subseteq X$ such that

(14) $|Y^*| = p$, for a fixed p, $0 \le p \le n$ and

(15) $i, j \in Y^*$ if i, j are not k - distance consecutive and $|i - j| \le n - k$. Further we denote by $f^*(k, n, p)$ the number of all subsets Y^* on p elements and we put $F^*(k, n) = \sum_p f^*(k, n, p)$.

Remark. From the condition in (15) it follows that if $i, j \in Y^*$, then $k \le |i - j| \le n - k$. Therefore, if $n < 2k$, then $p = 0$ or $p = 1$.

Proposition 2. *Let $k \ge 2$ and $0 \le p \le n$. If $n \ge 2k$ and $p \ge 2$, then we have the formula*

$$f^*(k, n, p) = (k - 1)f(k, n - (2k - 1), p - 1) + f(k, n - (k - 1), p).$$

If $n \ge 0$, then $f^*(k, n, 1) = n$, $f^*(k, n, 0) = 1$.

Proof: For $p = 0, 1$ the result follows immediately. Using Remark, it remains to consider the case that $n \ge 2k$ and $p \ge 2$. Let $Y^* \subset X$. We recall that Y^* has exactly p elements, such that for each $i, j \in Y^*$, $i \ne j$, $|i - j| \ge k$ and $|i - j| \le n - k$. Let i be a fixed integer, $1 \le i \le k - 1$. To calculate the number of all subsets Y^*, we first calculate the number of all subsets containing the fixed integer i. Assume that $j \in Y^*$ and $j \ne i$. Then $j > i$ and the condition $k \le |i - j| \le n - k$ is equivalent to $k \le j - i \le n - k$. Therefore, $i + k \le j \le n - k + i$. This means that the others $(p - 1)$ integers (different from i) from Y^* must be chosen among $n - k + i - (i + k) + 1 = n - (2k - 1)$ integers from X. Then by Proposition 1 the number of all such possible choices is equal to $f(k, n - (2k - 1), p - 1)$. Since the integer i can be any of the integers $1, ..., k - 1$, the total number of all subsets Y^* containing the integer i, $1 \le i \le k - 1$, is equal to $(k - 1)f(k, n - (2k - 1), p - 1)$, $p \ge 2$.

Now, we calculate the number of sets Y^* not containing the integer i. Let $i \notin Y^*$, i.e., $i = 1, ..., k - 1 \notin Y^*$, and moreover for each $l, j \in X \setminus \{i : i = 1, ..., k - 1\}$, $|l - j| \le n - k$. Thus, we can conclude from this that to form subset Y^* on p integers we can choose these p integers from $n - (k - 1)$ integers of X. If $l, j \in Y^*$, then the condition $|l - j| \ge k$ must be fulfilled. It follows by Proposition 1 that the number of all subsets Y^* not containing the integer i is equal to $f(k, n - (k - 1), p)$.

Finally $f^*(n, k, p) = (k - 1)f(k, n - (2k - 1), p - 1) + f(k, n - (k - 1), p)$, this completes the proof.

From Proposition 2 and the condition in (8) there follows:

Corollary. For $k = 2$ and $n \ge 0$, we have the identity $f^*(2, n, p) = f^*(n, p)$. Thus $F^*(2, n) = F_n^*$

3. The total number of k-stable sets of $\mathbf{P_n}$ and $\mathbf{C_n}$

We can observe that the numbers $F(k,n)$ and $F^*(k,n)$ are total numbers of all k-stable sets of graphs P_n and C_n, respectively, for a fixed $k \geq 2$. From investigations in section 2 it follows that

$F(k,n) = \sum_p \binom{n-p-(p-1)(k-2)+1}{p}$ and

$F^*(k,n) = \sum_p f^*(k,n,p)$ where $f^*(k,n,p)$ is determined in Proposition 2. Now we present numbers $F(k,n,p)$ and $F^*(k,n,p)$ by second-order linear recurrence relations.

Theorem 1. *Let $k \geq 2$ and $n \geq 0$ be integers. Then the numbers $F(k,n)$ satisfy the following recurrence:*
$F(k,n) = F(k,n-1) + F(k,n-k)$, for $n \geq k$, with initial conditions $F(k,n) = n+1$ for $n = 0,1,...,k-1$.

Proof: Let k,n,p, be as mentioned in the statement of the theorem.
If $n = 0$, then $p = 0$ and $F(k,0) = 1$ since it was mentioned before that the empty set is meant as a k-stable set of the graph P_0.
For $n = 1,...,k-1$, each of vertices of $V(P_n)$ and also the empty set can be a k-stable set of P_n. This implies that $F(k,n) = n+1$ in this case.
Now suppose that $n \geq k$ and let S be an arbitrary k-stable set of P_n with the vertex set $V(P_n)$ numbered in the natural fashion.
Two cases can occur now:
Case 1. $x_n \notin S$.
If $\mathcal{S}_1$ is the family of all such sets S, then its cardinality $|\mathcal{S}_1|$ is equal to the total number of k-stable sets of the graph $P_n - x_n$ isomorphic to P_{n-1}. In other words, $|\mathcal{S}_1| = F(k,n-1)$.
Case 2. $x_n \in S$.
Then it is clear that $x_{n-i} \notin S$, for each $i = 1,...,k-1$. This implies, that $S = S^* \cup \{x_n\}$, where S^* is an arbitrary k-stable set of the graph $P_n - \bigcup_{i=1}^{k-1} x_{n-i}$ which is isomorphic to P_{n-k}. If we denote by $\mathcal{S}_2$ the family of all k-stable sets such that the condition in Case 2 is fulfilled, then $|\mathcal{S}_2| = F(k,n-k)$. In consequence, for the numbers $F(k,n)$ we have a second-order linear recurrence $F(k,n) = F(k,n-1) + F(k,n-k)$. This completes the proof.

Theorem 2. *If $k \geq 2$, then the numbers $F^*(k,n)$ satisfy the following recurrence*
$F^*(k,n) = (k-1)F(k,n-(2k-1)) + F(k,n-(k-1))$, for $n \geq 2k$, with initial conditions $F^*(k,n) = n+1$, for $n = 0,1,...,2k-1$.

Proof: If $n = 0$, then also $p = 0$ and this implies $F^*(k,0) = f^*(k,0,0) = 1$, by the definition of $F^*(k,n)$.

If $n = 1,..,2k-1$, then p can be 0 or 1. Hence $F^*(k,n) = \sum_{p=0}^{1} f^*(k,n,p) = f^*(k,n,0) + f^*(k,n,1) = n+1$ by Proposition 2.

If $n \geq 2k$, then $F^*(k,n) = \sum_{p} f^*(k,n,p) = f^*(k,n,0) + \sum_{p\geq 1}((k-1)f(k,n-(2k-1),p-1) + f(k,n-(k-1),p)) = 1 + (k-1)\sum_{p-1=r\geq 0} f(k,n-(2k-1),r) + \sum_{p\geq 1} f(k,n-(k-1),p)$.

Since $f(k,n-(k-1),0) = 1$, using Proposition 2 we can write

$F^*(k,n) = (k-1)\sum_{r\geq 0} f(k,n-(2k-1),r) + \sum_{p\geq 0} f(k,n-(k-1),p) =$

$(k-1)F(k,n-(2k-1)) + F(k,n-(k-1))$, as required.

Thus, the theorem is proved.

4. The total numbers of $(\mathbf{k},\mathbf{k}-1)$ - kernels of $\mathbf{P_n}$ and $\mathbf{C_n}$

We say that a k-stable set S of the graph G is maximal if for any $x \in V(G)\setminus S$, $S \cup \{x\}$ is not k-stable set of G. Additionally, if $|V(G)| = 1$, then $V(G)$ will mean a maximal k-stable set of G. It has been noted by the author of [6] that

Proposition 3[6].*Every maximal k-stable set of G is a $(k,k-1)$- kernel of G, for any $k \geq 2$.*

The following observation says that the total number of all $(k,k-1)$ - kernels of P_n is equal to the total number of all its maximal k-stable sets. Let us denote by $J(k,n)$ the number of all $(k,k-1)$- kernels of P_n. We determine it recursively.

Theorem 3.*Let $k \geq 2, n \geq 0$ be integers. Then*

$J(k,0) = 1$ and $J(k,n) = n$, for $n = 1,...,k$,

$J(k,n) = J(k,n-1) + J(k,n-k) - 1$, for $k+1 \leq n \leq 2k$ and

$J(k,n) = J(k,n-1) + J(k,n-k) - J(k,n-2k)$, for $n > 2k$.

Proof: The empty set is a $(k,k-1)$- kernel of the empty graph P_0, hence for $n = 0$ the result follows.

For $n = 1,2,...,k \geq 2$, a $(k,k-1)$- kernel $J \subset V(P_n)$ contains exactly one vertex. Moreover, each of vertices of $V(P_n)$ is a $(k,k-1)$- kernel. This implies that $J(k,n) = n$ in this case.

Suppose that $n > k$.

Let $S \subset V(P_n)$ be an arbitrary maximal k-stable set of P_n. Two cases can appear.

Case 1. Let $x_n \in S$.

In this case, for $i = n-1, n-2, ..., n-(k-1)$, $x_i \notin S$. Furthermore, if S^* is an arbitrary maximal k-stable set of $P_n - \bigcup_{i=0}^{k-1} x_{n-i}$, then $S^* \cup \{x_n\}$ is a k-stable set of P_n. We shall show that $S^* \cup \{x_n\}$ is maximal. By an easy observation it follows that among the vertices of S^* there must be x_j such that $n-k-(k-1) \leq j \leq n-k$. Otherwise, we could add the vertex x_{n-k} to S^*, this would contradict the maximality of S^*. Consequently, to prove that $S^* \cup \{x_n\}$ is maximal it suffices to estimate the distance between vertices x_j and x_n in P_n. By simple calculations we obtain that $d_{P_n}(x_n, x_j) \leq n-(n-2k+1) = 2k-1 < 2k$. This means that it is not possible to add to $S^* \cup \{x_n\}$ any vertex of the successive vertices $x_j, x_{j+1}, ..., x_{n-k}$. This shows that $S^* \cup \{x_n\}$ is maximal and $S = S^* \cup \{x_n\}$. This implies, that the total number of maximal k-stable sets, i.e., $(k, k-1)$- kernels of P_n containing the vertex x_n, is equal to $J(k, n-k)$.
Case 2. $x_n \notin S$.
Then all maximal k-stable sets of $P_n - x_n$ are k-stable sets of P_n.
Suppose that S^* is a maximal k-stable set of $P_n - x_n$. It should be noted that, if $x_{n-k} \in S^*$, then S^* could not be a $(k, k-1)$- kernel of P_n, since then $d_{P_n}(x_n, S^*) = k$. Observe that if $x_{n-k} \notin S^*$, then there must be a vertex x_j, $n-k < j \leq n-1$, which belongs to S^* by the maximality of S^* in the graph $P_n - x_n$. Thus, we can conclude that S^* is a maximal k-stable set of P_n. This means that to calculate the total number of maximal k-stable sets of P_n not containing the vertex x_n it sufficies to subtract the number of all subsets S^* which contain the vertex x_{n-k} from the number $J(k, n-1)$. Let r denotes the number of all maximal k-stable sets of $P_n - x_n$ containing the vertex x_{n-k}.
Consider two possibilities:
Subcase 2.1. $k+1 \leq n \leq 2k$.
Since $n-k < k$, there exists exactly one maximal k-stable set S^* of the graph $P_n - x_n$ containing the vertex x_{n-k}, namely $S^* = \{x_{n-k}\}$. This means that $r = 1$. So, in this case the number of all maximal k -stable sets is equal to $J(k, n-k) - 1$.
Subcase 2.2. $n > 2k$.
Consider the graph $P_n - \bigcup_{i=0}^{k-1} x_{n-i}$ isomorphic to P_{n-k}. Since r denotes the number of all maximal k-stable sets containing the vertex x_{n-k}, it follows from case 1 and preceding observations that $r = J(k, (n-k)-k) = J(k, n-2k)$.
All this together gives the result that

$J(k,n) = J(k, n-1) + J(k, n-k) - 1$, for $k+1 \leq n \leq 2k$ and

$J(k,n) = J(k, n-1) + J(k, n-k) - J(k, n-2k)$, for $n > 2k$.

Thus, the theorem is proved.

Let us denote by $J^*(k,n)$ the number of all $(k,k-1)$- kernels of C_n. As a consequence of Theorems 2 and 3, we have

Theorem 4. *Let $n \geq 0, k \geq 2$. Then*
$J^*(k,0) = 1$ *and* $J^*(k,n) = n$, *for* $n = 0,1,...,2k-1$ *and*
$J^*(k,n) \leq (k-1)J(k,n-(2k-1)) + J(k,n-(k-1))$, *for* $n \geq 2k$.

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