On exact bicoverings of 12 points

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Abstract

The minimum number of incomplete blocks required to cover, exactly λ times, all t-element subsets from a set V of cardinality v ($v > t$) is denoted by $g(\lambda, t; v)$. The value of $g(2,2;v)$ is known for $v = 3, 4, \ldots, 11$. It was previously known that $13 \le g(2,2;12) \le 16$. We prove that $g(2,2;12) \ge 14$.

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1 Introduction

A *pairwise balanced design* of *index* λ and *order* v (PBD($v;\lambda$)) is a pair $(V,\mathcal{B})$, where V is a set of cardinality v (the *points*) and $\mathcal{B}$ is a family of subsets of V (the *blocks*) with the property that every pair of elements of V occurs in exactly λ blocks of $\mathcal{B}$. We are concerned with the case $\lambda = 2$ and the PBD is then referred to as an (exact) *bicovering* of V.

This paper focuses on the minimisation of $|\mathcal{B}|$ for given v in the case $\lambda = 2$, with the additional constraint that each $B \in \mathcal{B}$ satisfies $|B| < v$, i.e. $\mathcal{B}$ contains only *incomplete* blocks. This constraint excludes the trivial answer $|\mathcal{B}| = 2$. Following Woodall [6], the notation $g(\lambda, t; v)$ is generally used to denote the minimum number of incomplete blocks required to cover, exactly λ times, all t-element subsets from a set V with $|V| = v > t$. Woodall writes μ instead of t and, for this reason, the problem is sometimes referred to as the λ-μ problem. For the case $\lambda = t = 2$ the existing state of knowledge is complete for $v = 3, 4, \ldots, 11$ and is summarised in Table 1, the results being taken from [5].

v	3	4	5	6	7	8	9	10	11
$g(2,2;v)$	6	4	6	7	7	9	11	11	11

Table 1.

It is known [3] that $g(2,2;v) \geq v$. Equality occurs if and only if there exists a symmetric *balanced incomplete block design* (BIBD) with parameters $(v, v, k, k, 2)$, and this design then provides a minimal bicovering. Moreover, except for $v = 7$ where there is an alternative minimal bicovering, all minimal bicoverings of cardinality v are of this form. (See [1] for an explanatory discussion of symmetric BIBDs.)

For $v = 12$ it is only known that $13 \leq g(2,2;12) \leq 16$. The upper bound follows from the existence of a symmetric BIBD(16,16,6,6,2) by deleting points, and the lower bound follows from the non-existence of a symmetric BIBD(12,12,k,k,2). In this paper we prove that $g(2,2;12) \neq 13$. In obtaining this result, we also obtain some information about the structure of any possible bicoverings which correspond to $g(2,2;12) = 14$ or 15.

In proving our results, we make use of the concept of a *Steiner triple system* of order v (STS(v)). This comprises a pair $(V,\mathcal{B})$, where V is a set of cardinality v (the points) and $\mathcal{B}$ is a set of subsets of V (the blocks or *triples*) with the property that every 2-element subset of V occurs in exactly one triple. Such a system is said to be *resolvable* if the triples can be grouped into *resolution* or *parallel classes*, the triples of each parallel class collectively covering all v points precisely once. There is, up to isomorphism, a unique STS(9). This is resolvable into four parallel classes $\mathcal{P}_i$, for $i = 1, 2, 3, 4$ as shown below.

$$\begin{aligned} V &= \{1,2,3,4,5,6,7,8,9\}, \\ \mathcal{P}_1 &= \{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}, \\ \mathcal{P}_2 &= \{\{1,4,7\},\{2,5,8\},\{3,6,9\}\}, \\ \mathcal{P}_3 &= \{\{1,5,9\},\{2,6,7\},\{3,4,8\}\}, \\ \mathcal{P}_4 &= \{\{1,6,8\},\{2,4,9\},\{3,5,7\}\}. \end{aligned}$$

Figure 1: the STS(9).

Use will also be made of results from [2] which concern the values of $g^{(k)}(v)$, the minimum number of blocks required to cover, exactly once, each pair of elements from a set V of cardinality v, subject to the restriction that the maximum block size is precisely k ($k < v$). A complete tabulation of the values of $g^{(k)}(v)$ for $v \leq 13$ is given in [2], together with an enumeration of all corresponding non-isomorphic solutions to this problem.

2 Proof that $g(2,2;12) \geq 14$

Throughout this section we denote $g(2,2;12)$ simply by g. We make extensive use of two further parameters associated with a minimal exact bicovering, namely the length l of the longest block and the cardinality d of the largest intersection of distinct blocks. We establish that $g \geq 14$ by proving that $g = 13$ entails $l \geq 6$, followed by $d \neq 2$, $d \leq 4$, $d \neq 4$ and, finally, $d \neq 3$. We take our set of 12 points to be $\{1,2,\ldots,12\}$ but we write 10, 11 and 12 as t, e and w respectively. We often omit brackets and commas, for example writing the triple $\{1,2,3\}$ as 123. A block written as $B = \{1,2,3,4,\ldots\}$ or as $B = 1234\ldots$ indicates that the points 1, 2, 3 and 4 definitely lie in B, and that B may or may not contain additional points. A block containing precisely n points will be referred to as an n-block.

Lemma 2.1 *If* $13 \leq g \leq 15$ *then* $l \geq 6$.
Proof. Suppose $l \leq 5$. Denote the number of blocks of length i in the bicovering by n_i. Counting pairs of elements gives

$$n_2 + 3n_3 + 6n_4 + 10n_5 = 132.$$

Counting blocks gives

$$n_2 + n_3 + n_4 + n_5 = g.$$

Hence

$$9n_2 + 7n_3 + 4n_4 = 10g - 132. \tag{1}$$

If $g = 13$ then (1) has no solutions. If $g = 14$, the only solution is $n_2 = 0, n_3 = 0, n_4 = 2$, giving also $n_5 = 12$. But then there is a point x occurring only in blocks of size five and such an x cannot occur in 22 $\{x, y\}$ pairs. If $g = 15$, the possible solutions are:

(A) $n_2 = 2. n_3 = 0, n_4 = 0, n_5 = 13$ and

(B) $n_2 = 0. n_3 = 2, n_4 = 1, n_5 = 12$.

But in each of these two cases there is a point x occurring only in blocks of size five, again giving a contradiction. □

Lemma 2.2 *If $l \geq 6$ and $d = 2$ then $g \geq 16$.*
Proof. Suppose $l \geq 6$ and $d = 2$. Consider a block $B = 123456\ldots$ of the bicovering having at least six points. The pairs from $\{1, 2, 3, 4, 5, 6\}$ must then occur a second time in distinct blocks. Hence $g \geq \binom{6}{2} + 1 = 16$. □

Lemma 2.3 *If $d \geq 5$ then $g \geq 16$.*
Proof. Suppose $d \geq 5$. Then there exist blocks of the bicovering, $B_1 = 12345\ldots$ and $B_2 = 12345\ldots$, both of cardinality at least five.

(A) Suppose there are two distinct points, say e, w, such that $e, w \notin B_1 \cup B_2$. The ten pairs $1e, 1e, 2e, 2e, \ldots, 5e, 5e$ must lie in ten distinct blocks and likewise the ten pairs $1w, 1w, 2w, 2w, \ldots, 5w, 5w$. It is possible that two of the latter collection lie in a common block with two of the former. Even so, we have $g \geq 2 + 10 + (10 - 2) = 20$.

(B) Suppose there is a unique point, say w, such that $w \notin B_1 \cup B_2$. If $|B_1 \cap B_2| \geq 7$ we may assume $B_1 = 1234567\ldots$ and $B_2 = 1234567\ldots$, and consideration of the pairs $1w, 1w, 2w, 2w, \ldots, 7w, 7w$ gives $g \geq 2 + 14 = 16$. In the case $|B_1 \cap B_2| = 6$ we may assume $B_1 = 123456789\ldots$ and $B_2 = 123456\ldots$. Consideration of the pairs $1w, 1w, 2w, 2w, \ldots, 6w, 6w$ and $17, 27, 37, 47, 57, 67$ gives $g \geq 2 + 12 + (6 - 2) = 18$. Finally in case (B), if $|B_1 \cap B_2| = 5$ we may assume $B_1 = 12345678\ldots$ and $B_2 = 12345\ldots$. Consideration of the pairs $1w, 1w, 2w, 2w, \ldots, 5w, 5w$; $16, 26, 36, 46, 56$ and $17, 27, 37, 47, 57$ gives $g \geq 2 + 10 + (5 - 2) + (5 - 3) = 17$.

(C) Suppose $B_1 \cup B_2 = 12\ldots w$. In this case, $|B_1 \cap B_2| = 11$ is not possible given that the blocks are incomplete. If $|B_1 \cap B_2| = 10$ then we may take $B_1 = 12\ldots te$ and $B_2 = 12\ldots tw$; consideration of the pairs $1e, 2e, \ldots, te$ and $1w, 2w, \ldots, tw$ gives $g \geq 2 + 10 + (10 - 2) = 20$. If $|B_1 \cap B_2| = 9$ then we may take $B_1 = 12\ldots 9te$ and $B_2 = 12\ldots 9w$; consideration of the pairs $1t, 2t, \ldots, 9t$ and $1e, 2e, \ldots, 9e$ gives $g \geq 2 + 9 + (9 - 1) = 19$. If $|B_1 \cap B_2| = 8$ then we may take $B_1 = 12\ldots 89t\ldots$

and $B_2 = 12\ldots8\ldots$; consideration of the pairs $19, 29, \ldots, 89$ and $1t, 2t, \ldots, 8t$ gives $g \geq 2 + 8 + (8-1) = 17$. If $|B_1 \cap B_2| = 7$ then we may take $B_1 = 12\ldots789t\ldots$ and $B_2 = 12\ldots7\ldots$; consideration of the pairs $18, 28, \ldots, 78$; $19, 29, \ldots, 79$ and $1t, 2t, \ldots, 7t$ gives $g \geq 2 + 7 + (7-1) + (7-2) = 20$. If $|B_1 \cap B_2| = 6$ then we may take $B_1 = 12\ldots6789\ldots$ and $B_2 = 12\ldots6\ldots$; consideration of the pairs $17, 27, \ldots, 67$; $18, 28, \ldots, 68$ and $19, 29, \ldots, 69$ gives $g \geq 2 + 6 + (6-1) + (6-2) = 17$. If $|B_1 \cap B_2| = 5$ then we may take $B_1 = 123456789\ldots$ and $B_2 = 12345\ldots$; consideration of the pairs $16, 26, 36, 46, 56$; $17, 27, 37, 47, 57$; $18, 28, 38, 48, 58$ and $19, 29, 39, 49, 59$ gives $g \geq 2 + 5 + (5-1) + (5-2) + (5-3) = 16$. □

Lemma 2.4 *If $d = 4$ then $g \geq 14$.*

Proof. Suppose $d = 4$. Then there exist blocks of the bicovering, $B_1 = 1234\ldots$ and $B_2 = 1234\ldots$, both of cardinality at least four.

(A) Suppose there are two distinct points, say e, w, such that $e, w \notin B_1 \cup B_2$. Consideration of the pairs $1e, 1e, 2e, 2e, 3e, 3e, 4e, 4e$ and $1w, 1w, 2w, 2w, 3w, 3w, 4w, 4w$ gives $g \geq 2 + 8 + (8-2) = 16$.

(B) Suppose there is a unique point, say w, such that $w \notin B_1 \cup B_2$. Then we may assume that $B_1 = 12345678\ldots$ and $B_2 = 1234\ldots$. Consider the pairs $1w, 1w, 2w, 2w, 3w, 3w, 4w, 4w$. These must lie in eight blocks distinct from one another and from B_1 and B_2. Denote these eight blocks by $C_1, C_2, \ldots, C_8$. Now consider the pairs $15, 25, 35, 45$. At most two of these can lie in $C_1, C_2, \ldots, C_8$. So the remaining blocks, say $D_1, D_2, \ldots$, contain at least two occurrences of the point 5. Similarly, $D_1, D_2, \ldots$ contain at least two occurrences of each of the points 6, 7 and 8. Now consider packing the points 5, 6, 7 and 8 into $D_1, D_2, \ldots$. Without loss of generality, there are three possibilities:

(1) $5, 6, 7, 8 \in D_1$, or

(2) $5, 6, 7 \in D_1$ but $8 \notin D_1$, or

(3) each of $D_1, D_2, \ldots$ contains at most a pair from $\{5, 6, 7, 8\}$.

In case (B1) there must be blocks D_2, D_3, D_4 and D_5 containing respectively the points 5, 6, 7 and 8. Hence, in case (B1), $g \geq 2+8+5 = 15$. In case (B2) there must be blocks D_2, D_3 and D_4 containing respectively the points 5, 6 and 7. Hence, in case (B2), $g \geq 2+8+4 = 14$. In case (B3) there must be blocks D_1, D_2, D_3 and D_4 each containing at most a pair from $\{5, 6, 7, 8\}$ so that every one of these four points appears twice. Hence, in case (B3), $g \geq 2 + 8 + 4 = 14$.

(C) Suppose $B_1 \cup B_2 = 12\ldots w$. We split this case into subcases depending on the value of $|B_1|$. Clearly we may assume $|B_1| \geq 8$.

(1) $|B_1| = 11$. We take $B_1 = 12\ldots e$. Consider the intersections of the remaining blocks of the bicovering with B_1. These yield an exact single covering of the pairs from B_1. Because $d = 4$ and $|B_1 \cap B_2| = 4$, this single covering has largest block length four. It was shown in [2] that $g^{(4)}(11) = 13$ and so such a single covering has at least 13 blocks. Reinstating B_1, we have $g \geq 13 + 1 = 14$.

(2) $|B_1| = 10$. We take $B_1 = 12\ldots t$ and then $B_2 = 1234ew$. Repeating the argument of (C1) we see that $g \geq 14$ unless the exact single covering of $\{1, 2, \ldots, t\}$ is the unique single covering of ten points by twelve blocks having maximum size four given in [2]. This single covering is formed by adding a point to each of the blocks of a parallel class of an STS(9). To examine this possibility we may therefore, without loss of generality, take the blocks of the bicovering to be:

$B_1 =$	$12\ldots t$			
$B_2 =$	$1234ew$	$158\ldots$	$16t\ldots$	$179\ldots$
	$5674\ldots$	$269\ldots$	$278\ldots$	$25t\ldots$
	$89t4\ldots$	$37t\ldots$	$359\ldots$	$368\ldots$

where undeclared entries are from $\{e, w\}$. If there are any further blocks then $g \geq 14$ and we are finished with this subcase. So suppose that there are no further blocks and consider the point e. This occurs in B_2 and must occur also in *one* of $5674\ldots$ and $89t4\ldots$ in order to cover two $4e$ pairs. So we may assume a block $B_3 = 5674e\ldots$. Now consider pairs of the form xe for $x \in \{1, 2, \ldots, t\} \backslash \{4\}$. There are 18 such pairs to be covered. However, in order to cover each of $8e, 9e, te$ twice, we must adjoin e to 6 triples of the single covering. But then e occurs in 24 xe pairs for $x \in \{1, 2, \ldots, t\} \backslash \{4\}$, a contradiction. Thus, if $|B_1| = 10$, we must have $g \geq 14$.

(3) $|B_1| = 9$. Repeating the argument of (C2) and noting from [2] that $g^{(4)}(9) = 12$, we have $g \geq 14$ unless the 13 blocks of the bicovering are derived from the unique exact single covering of $\{1, 2, \ldots, 9\}$ in twelve blocks having maximum block size four given in [2] (see also [4]). In this case the 13 blocks of the bicovering may be taken as:

$B_1 =$	$12\ldots 9$			
$B_2 =$	$1234tew$	$258\ldots$	$269\ldots$	$27\ldots$
	$1567\ldots$	$368\ldots$	$379\ldots$	$35\ldots$
	$189\ldots$	$478\ldots$	$459\ldots$	$46\ldots$

where undeclared entries are from $\{t, e, w\}$. Now consider the point t. This must appear in *one* of $1567\ldots$ and $189\ldots$ in order to cover two $1t$ pairs.

Suppose there is a block $1567t\ldots$ and consider pairs of the form xt for $x \in \{2,3,4,5,6,7,8,9\}$. There are 16 such pairs to be covered. However, in order to cover each of $8t$ and $9t$ twice, we must adjoin t to four triples of the single covering. But then t occurs in 18 xt pairs for $x \in \{2,3,4,5,6,7,8,9\}$, a contradiction.

So now suppose there is a block $189t\ldots$. Then t must be adjoined to two more triples of the single covering. But then, however we add t to pairs of the single covering, it is impossible to achieve 16 xt pairs for $x \in \{2,3,4,5,6,7,8,9\}$, again a contradiction.

Thus, if $|B_1| = 9$, we must have $g \geq 14$.

(4) $|B_1| = 8$. We take $B_1 = 12345678$ and $B_2 = 12349tew$. Without loss of generality, there are three possibilities:

(a) there exists a block $B_3 = 5678\ldots$, or

(b) there exists a block $B_3 = 567\ldots$ and $8 \notin B_3$, or

(c) all blocks apart from B_1 and B_2 contain at most two of 5, 6, 7 and 8, and at most two of $9, t, e$ and w.

In case (C4a) consider the pairs 15, 25, 35, 45; 16, 26, 36, 46; 17, 27, 37, 47 and 18, 28, 38, 48. The block B_3 cannot contain any of these pairs because, if it did, then $|B_3 \cap B_1| \geq 5 > d$. But then we must have $g \geq 3 + 16 = 19$.

In case (C4b) suppose first that $1, 2, 3, 4 \notin B_3$ and consider the pairs 15, 25, 35, 45; 16, 26, 36, 46 and 17, 27, 37, 47. None of these pairs can appear in a common block (apart from B_1) and so we have distinct blocks

$$
\begin{aligned}
B_1 &= 12345678\\
B_2 &= 12349tew\\
B_3 &= 567\ldots \quad (1,2,3,4,8 \notin B_3)\\
& 15\ldots \quad 25\ldots \quad 35\ldots \quad 45\ldots\\
& 16\ldots \quad 26\ldots \quad 36\ldots \quad 46\ldots\\
& 17\ldots \quad 27\ldots \quad 37\ldots \quad 47\ldots
\end{aligned}
$$

Now consider pairs $x8$ for $x \in \{1,2,3,4,5,6,7\}$. There are 14 such pairs to be covered but the blocks listed can cover at most: seven such pairs from B_1, plus two such pairs from $15\ldots, 25\ldots, 35\ldots, 45\ldots$, plus two such pairs from $16\ldots, 26\ldots, 36\ldots, 46\ldots$ and plus two such pairs from $17\ldots, 27\ldots, 37\ldots, 47\ldots$. This leaves at least one more such pair to be covered. Thus $g \geq 3 + 12 + 1 = 16$.

If, on the other hand, say $1 \in B_3$, then we have $2, 3, 4 \notin B_3$. We cannot have all four of $\{9, t, e, w\}$ in B_3 since this would give $|B_3 \cap B_2| = 5 > d$, so suppose $9 \notin B_3$. We must therefore have distinct blocks

$$\begin{array}{lll} B_1 = 12345678 & & \\ B_2 = 12349tew & & \\ B_3 = 1567\ldots & (2,3,4,8,9 \notin B_3) & \\ 25\ldots & 35\ldots & 45\ldots \\ 26\ldots & 36\ldots & 46\ldots \\ 27\ldots & 37\ldots & 47\ldots \end{array}$$

Now consider pairs $x9$ for $x \in \{1,2,3,4,5,6,7\}$. There are 14 such pairs to be covered. But the blocks listed can cover at most: four such pairs from B_2, plus two such pairs from $25\ldots, 26\ldots, 27\ldots$, plus two such pairs from $35\ldots, 36\ldots, 37\ldots$ and plus two such pairs from $45\ldots, 46\ldots, 47\ldots$. This leaves at least four more such pairs to be covered. Since every pair from $\{1,2,3,4,5,6,7\}$ already appears twice in the twelve blocks listed, there must be at least four more distinct blocks to cover the four missing $x9$ pairs for $x \in \{1,2,3,4,5,6,7\}$. Thus $g \geq 12 + 4 = 16$.

In case (C4c) there must be six blocks distinct from B_1 and B_2 with the structure:

$$56\ldots, 57\ldots, 58\ldots, 67\ldots, 68\ldots, 78\ldots$$

Now consider the pairs $15, 25, 35$ and 45. No two of these pairs can appear together in a single block (apart from B_1) and so there must be a block additional to those given above which contains the point 5. Similarly, there are three further distinct blocks containing respectively the points 6, 7 and 8. This accounts for a minimum of twelve blocks.

Suppose that $g \leq 14$. Then there are at most two blocks extra to the twelve already identified and such blocks cannot contain any pair from $\{5,6,7,8\}$. Thus, without loss of generality, we may assume that the only blocks containing the points 5 or 6 are those already identified, namely $B_1, 56\ldots, 57\ldots, 58\ldots, 67\ldots, 68\ldots$, $5\ldots$ and $6\ldots$. But the point 5 must occur twice with each of $9, t, e, w$ and so the blocks $56\ldots, 57\ldots, 58\ldots$ and $5\ldots$ must each contain a pair from $\{9, t, e, w\}$. Similarly the blocks $56\ldots, 67\ldots$, $68\ldots$ and $6\ldots$ must each contain a pair from $\{9, t, e, w\}$. But there are only six distinct pairs from $\{9, t, e, w\}$ and so at least one pair must be repeated in the seven distinct blocks $56\ldots$, 57..., 58..., 67..., 68..., 5... and 6.... But this pair also appears once in B_2 and hence three times altogether, a contradiction. It follows that, in case (C4c), $g \geq 15$. □

Lemma 2.5 *If $d = 3$ then $g \geq 14$.*

Proof. Suppose that the longest block $B_1 = 12\ldots l$ intersects the other blocks in m_2 pairs and m_3 triples. Then $m_2 + m_3 \leq g - 1$ and $m_2 + 3m_3 = \binom{l}{2}$. We examine the implication of these relationships for different possible values of l.

(A) $l = 11$ gives $m_2 + 3m_3 = 55$ and the minimum value of $m_2 + m_3$ is then $1 + 18 = 19$, giving $g \geq 20$.

(B) $l = 10$ gives $m_2 + 3m_3 = 45$ and the minimum value of $m_2 + m_3$ is then $0 + 15 = 15$, giving $g \geq 16$.

(C) $l = 9$ gives $m_2 + 3m_3 = 36$. Solutions of this immediately give $g \geq 15$, apart from the case $m_2 = 0, m_3 = 12$. This remaining possibility corresponds to the twelve triples of an STS(9) on the nine points of the longest block. The associated bicovering has at least 13 distinct blocks which we may take as:

$$\begin{array}{lllll} B_1 = & 12\ldots9 & & & \\ & 123\ldots & 147\ldots & 159\ldots & 168\ldots \\ & 456\ldots & 258\ldots & 267\ldots & 249\ldots \\ & 789\ldots & 369\ldots & 348\ldots & 357\ldots \end{array}$$

where undeclared entries are from $\{t, e, w\}$. Suppose that this is a complete list of the blocks of the bicovering and consider the pair $1t$. Without loss of generality, we may assume that this appears as $123t\ldots$ and $147t\ldots$. To cover the pair $2t$ twice there are then three alternatives, namely $267t\ldots$ or $249t\ldots$ or $258t\ldots$. For the first of these three alternatives, it is only then possible to adjoin t to $456\ldots, 369\ldots$ and $348\ldots$, and thus the pair $5t$ can only be covered once, a contradiction. A similar argument applies to the second alternative. In the case of the third alternative we have blocks $123t\ldots, 147t\ldots, 258t\ldots$ and, by a similar argument reapplied to the pairs $3t$, we can assume that we also have the block $369t\ldots$. There are 18 xt pairs to cover for $x \in \{1, 2, \ldots, 9\}$ and so t must appear in six blocks. It follows that we must therefore also have the blocks $456t\ldots$ and $789t\ldots$, i.e. t appears in blocks corresponding to two of the four parallel classes of the STS(9). But then the same argument can be applied to e and w. Consequently, at least one of the pairs te, tw and ew must appear with all three triples of at least one parallel class, a contradiction. Thus $g \geq 14$.

(D) $l = 8$ gives $m_2 + 3m_3 = 28$. Solutions of this immediately give $g \geq 15$, apart from two cases, namely

(1) $m_2 = 1, m_3 = 9$, and

(2) $m_2 = 4, m_3 = 8$.

Consider first case (D1) and assume that the unique pair is 12. Then the point 1 must occur in triples with the points 3, 4, 5, 6, 7 and 8, and likewise the point 2. Without loss of generality, six of the nine triples must be 134, 156, 178, 245, 267 and 283. But then the missing pairs are 35, 36, 37, 46, 47, 48, 57, 58 and 68, and these cannot be partitioned into three triples. We therefore turn our attention to case (D2). It was shown in [2] that $g^{(3)}(8) = 12$ and that the unique corresponding design may be obtained by taking the twelve triples of an STS(9) and deleting a point. We may therefore take 13 blocks of the bicovering to be:

$B_1 =$	12...8			
	123...	147...	15...	168...
	456...	258...	267...	24...
	78...	36...	348...	357...

where undeclared entries are from $\{9, t, e, w\}$. Suppose that this is a complete list of the blocks of the bicovering. The point 9 occurs in 16 $x9$ pairs for $x \in \{1, 2, \ldots, 8\}$. If 9 occurs with a_2 pairs and a_3 triples from $\{1, 2, \ldots, 8\}$, we therefore have $2a_2 + 3a_3 = 16$, giving $a_2 = 2$ and $a_3 = 4$ as the only feasible solution. A similar argument applies to the points t, e and w. Thus each of the points $9, t, e$ and w must be adjoined to two of the pairs and four of the triples from $\{1, 2, \ldots, 8\}$ given above. Without loss of generality, we may assume that we have 789... and 369....

Suppose that the point 9 also appears with the triple 267 as a block 2679.... Then we cannot have 4569..., or 1479..., or 1689..., or 3579..., and so 9 must appear in all of 1239.... 2589... and 3489.... But now the pair 29 appears three times, a contradiction. A similar argument applies if we attempt to adjoin the point 9 to any of the triples 348, 168 or 357. Thus the point 9 must be adjoined to triples and pairs corresponding to two complete parallel classes of the STS(9). The same argument applies to t, e and w, and so at least one of the pairs from $\{9, t, e, w\}$ must appear more than twice. We conclude that $g \geq 14$.

(E) $l = 7$ gives $m_2 + 3m_3 = 21$. Solutions of this immediately give $g \geq 14$ apart from three cases, namely

(1) $m_2 = 0, m_3 = 7$,

(2) $m_2 = 3, m_3 = 6$, and

(3) $m_2 = 6, m_3 = 5$.

We may take as two blocks of the bicovering $B_1 = 1234567$ and $B_2 = 123\ldots$. Suppose that $|B_2| \leq 5$, so that we can assume $t, e, w \notin B_1 \cup B_2$. Consideration of the pairs $1t, 1t, 2t, 2t, 3t, 3t$: $1e, 1e, 2e, 2e, 3e, 3e$ and $1w, 1w, 2w, 2w, 3w, 3w$ then gives $g \geq 2+6+(6-2)+(6-4) = 14$. We can therefore assume that every block intersecting B_1 in a triple extends to a 6- or a 7-block of the bicovering.

Now considering case (E1), we see that the bicovering must have seven 6- or 7-blocks each of which contain three points from $\{1,2,3,4,5,6,7\}$ and at least three points from $\{8,9,t,e,w\}$. These blocks must therefore cover at least $7 \times 3 = 21$ pairs from $\{8,9,t,e,w\}$. However, there are only $\binom{5}{2} \times 2 = 20$ pairs to be covered, and so case (E1) yields a contradiction.

In case (E2), we see in a similar fashion that the existence of a 7-block containing four points from $\{8,9,t,e,w\}$, together with five further 6- or 7-blocks each containing three or four points from $\{8,9,t,e,w\}$ again produces a contradiction. There remains, however, the possibility of exactly six 6-blocks of the form $xxxyyy$ with x denoting elements from $\{1,2,3,4,5,6,7\}$ and y denoting elements from $\{8,9,t,e,w\}$. Collectively these blocks cover $6 \times 3 = 18$ yy pairs, leaving two more blocks, say C_1 and C_2, to contain the remaining two yy pairs. Now consider the xy pairs; the six 6-blocks cover $6 \times 3 \times 3 = 54$ of these $7 \times 5 \times 2 = 70$ pairs. At most eight more xy pairs can come from the blocks C_1 and C_2, leaving a deficit of at least eight xy pairs. In fact, the deficit will be greater unless both C_1 and C_2 contain an xx pair. Consideration of C_1 and C_2 together with the blocks required to cover the deficit of xy pairs shows that at least seven further blocks are required, giving $g \geq 1+6+2+7 = 16$.

In case (E3), it is again easy to see that there cannot be two 7-blocks of the form $xxxyyyy$ with x denoting elements from $\{1,2,3,4,5,6,7\}$ and y denoting elements from $\{8,9,t,e,w\}$ because the 6- and 7-blocks would then contain at least $2 \times 6 + 3 \times 3 = 21$ yy pairs. So first suppose that there is precisely one 7-block of this form and hence four 6-blocks of the form $xxxyyy$. These blocks cover $6 + 4 \times 3 = 18$ yy pairs, leaving two yy pairs uncovered which must therefore lie in two further blocks, say C_1 and C_2. The 7-block and the four 6-blocks together cover $12 + 4 \times 9 = 48$ of the 70 xy pairs. At most eight more xy pairs can come from the blocks C_1 and C_2, leaving a deficit of at least 14 xy pairs. Again, the deficit will be greater unless both C_1 and C_2 contain an xx pair. Consideration of C_1 and C_2 together with the blocks required to cover the deficit of xy pairs shows that at least ten further blocks are required, giving $g \geq 1+5+2+10 = 18$.

We may therefore reduce case (E3) to consideration of the subcase in which there are five 6-blocks of the form $xxxyyy$ with x denoting elements from $\{1,2,3,4,5,6,7\}$ and y denoting elements from $\{8,9,t,e,w\}$. These cover $5 \times 3 = 15$ yy pairs, leaving five yy pairs uncovered. These five yy pairs may either occur in five separate blocks C_1, C_2, C_3, C_4, C_5 or in three blocks D_1, D_2, D_3, where D_1 contains three points from $\{8,9,t,e,w\}$. The five 6-blocks cover $5 \times 9 = 45$ of the 70 xy pairs. At most 20 xy pairs can come from the blocks C_1, C_2, C_3, C_4, C_5, leaving in this case a deficit of at least five xy pairs. The deficit will be greater unless C_1, C_2, C_3, C_4 and C_5 each contain an xx pair. Consideration of these blocks together with the blocks required to cover the deficit of xy pairs shows that at least four further blocks are required, giving $g \geq 1+5+5+4 = 15$. At most 14 xy pairs can come from the blocks D_1, D_2, D_3, leaving in this case a deficit of at least eleven xy pairs. By a similar argument to before, this requires at least eight further blocks, giving $g \geq 1+5+3+8 = 17$.

(F) $l = 6$ gives $m_2 + 3m_3 = 15$. If $m_3 = 0$ then $m_2 = 15$ and so $g \geq 16$. So suppose $m_3 > 0$. Then we have blocks $B_1 = 123456$ and $B_2 = 123\ldots$, where $|B_2| \leq 6$. Consequently, we may assume that $t, e, w \notin B_2$. Now consideration of the pairs $1t, 1t, 2t, 2t, 3t, 3t$; $1e, 1e, 2e, 2e, 3e, 3e$ and $1w, 1w, 2w, 2w, 3w, 3w$ gives $g \geq 2+6+(6-2)+(6-4) = 14$.

□

We conclude this section by combining the results of Lemmas 2.1 - 2.5.

Theorem 2.1 $g(2,2;12) \geq 14$. □

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