Various partition function identities

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Introduction: Let $p(n)$, $q(n)$, $q_0(n)$ denote respectively the number of unrestricted partitions of n, partitions of n into distinct parts, partitions of n into distinct odd parts. (If $f(n)$ is any partition function, we define $f(\alpha) = 0$ if α is not a non-negative integer.) If the integer $t \geq 2$, let $c_t(n)$ denote the number of t-core partitions of n. In this note, we derive a number of apparently new identities concerning these partition functions.

Preliminaries: Let x, z denote complex variables with $|x| < 1$, $z \neq 0$. Let $(\frac{a}{p})$ denote the Legendre symbol. Let $t(k) = k(k+1)/2$ (the k^{th} triangular number). Then we have:

$$\sum_{n\geq 0} p(n)x^n = \prod_{n\geq 1}(1-x^n)^{-1} \tag{1}$$

$$\sum_{n\geq 0} q(n)x^n = \prod_{n\geq 1}(1+x^n) = \prod_{n\geq 1}(1-x^{2n-1})^{-1} \tag{2}$$

$$\sum_{n\geq 0} q_0(n)x^n = \prod_{n\geq 1}(1+x^{2n-1}) \tag{3}$$

$$\sum_{n\geq 0} c_t(n)x^n = \prod_{n\geq 1}(1-x^{tn})^t/(1-x^n) \tag{4}$$

$$c_2(n) = \begin{cases} 1 & \text{if n = j(j+1)/2} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$c_3(n) = \sum_{d|(3n+1)} \left(\frac{d}{3}\right) \tag{6}$$

$$\prod_{n\geq 1}(1-x^n) = 1 + \sum_{n\geq 1}(-1)^n\left(x^{\frac{n(3n-1)}{2}} + x^{\frac{n(3n+1)}{2}}\right) \tag{7}$$

$$\prod_{n\geq 1}(1-x^{2n})(1+x^{2n-1}z)(1+x^{2n-1}z^{-1}) = 1 + \sum_{n\geq 1} x^{n^2}(z^n + z^{-n}) \tag{8}$$

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$$\prod_{n\geq 1}(1-x^n)^3 = \sum_{j\geq 0}(-1)^j(2j+1)x^{j(j+1)/2} \tag{9}$$

$$\prod_{n\geq 1}\frac{1-x^{2n}}{1-x^{2n-1}} = \sum_{n\geq 0}x^{n(n+1)/2} \tag{10}$$

Remarks: Identity (6) has been proven independently by Granville & Ono (see [1]) and by Robbins (see [2]). Identities (8) and (9) are due to Jacobi, (8) being the celebrated triple product identity.

The Main Results

Theorem 1:

$$\sum_{j=0}^{n} q_0(n-j)q_0(j) = p(\frac{n}{2}) + 2\sum_{j\geq 1}p(\frac{n-j^2}{2})$$

Proof: Setting $z=1$ in (8), one obtains:

$$\prod_{n\geq 1}(1-x^{2n})(1+x^{2n-1})^2 = 1+\sum_{n\geq 1}2x^{n^2}$$

hence

$$\prod_{n\geq 1}(1+x^{2n-1})^2 = (\prod_{n\geq 1}(1-x^{2n})^{-1})(1+\sum_{n\geq 1}2x^{n^2})$$

By virtue of (1) and (3), we have:

$$(\sum_{n\geq 0}q_0(n)x^n)^2 = (\sum_{n\geq 0}p(\frac{n}{2})x^n)(\sum_{n\geq 0}b(n)x^n)$$

where

$$b(n) = \begin{cases} 1 & \text{if } n=0 \\ 2 & \text{if } n=j^2>0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$\sum_{n\geq 0}(\sum_{j=0}^{n}q_0(n-j)q_0(j))x^n = \sum_{n\geq 0}(p(\frac{n}{2})+2\sum_{j\geq 1}p(\frac{n-j^2}{2}))x^n$$

The conclusion follows by matching coefficients of like powers of x.

Theorem 2:

$$(-1)^{\frac{n}{3}}q_0(\frac{n}{3})+\sum_{k\geq 1}(-1)^{\frac{n-t(k)}{3}}q_0(\frac{n-t(k)}{3})=\begin{cases}2(-1)^m & \text{if } n=m^2, 3|m\\ (-1)^{m+1} & \text{if } n=m^2, 3\not|m\\ 0 & \text{otherwise}\end{cases}$$

Proof: Replace x by $-x$ in (8) to obtain:

$$\prod_{n\geq 1}(1-x^{2n})(1-x^{2n-1}z)(1-x^{2n-1}z^{-1})=1+\sum_{n\geq 1}(-1)^n x^{n^2}(z^n+z^{-n})$$

that is,

$$\prod_{n\geq 1}(1-x^{2n})(1-(z+z^{-1})x^{2n-1}+x^{4n-2})=1+\sum_{n\geq 1}(-1)^n x^{n^2}(z^n+z^{-n})$$

Let $z=e^{\frac{2\pi i}{3}}$, so that $z^n+z^{-n}=2\cos\frac{2\pi n}{3}$. This yields:

$$\prod_{n\geq 1}(1-x^{2n})(1+x^{2n-1}+x^{4n-2})=1+\sum_{n\geq 1}(-1)^n\cos\frac{2\pi n}{3}x^{n^2}$$

hence

$$(\prod_{n\geq 1}(1-x^{6n-3}))(\prod_{n\geq 1}\frac{1-x^{2n}}{1-x^{2n-1}})=1+\sum_{n\geq 1}(-1)^n\cos\frac{2\pi n}{3}x^{n^2}$$

$$(\sum_{n\geq 0}(-1)^n q_0(n)x^{3n})(\sum_{n\geq 0}x^{t(n)})=1+\sum_{n\geq 1}(-1)^n\cos\frac{2\pi n}{3}x^{n^2}$$

$$(\sum_{n\geq 0}(-1)^{\frac{n}{3}}q_0(\frac{n}{3}))(\sum_{n\geq 0}x^{t(n)})=1+\sum n\geq 1(-1)^n\cos\frac{2\pi n}{3}x^{n^2}=\sum_{n\geq 0}a_n x^n$$

where

$$a(n)=\begin{cases}2(-1)^m & \text{if } n=m^2, 3|m\\ (-1)^{m+1} & \text{if } n=m^2, 3\not|m\\ 0 & \text{otherwise}\end{cases}$$

Thus we have

$$\sum_{n\geq 0}(\sum_{k\geq 0}(-1)^{\frac{n-t(k)}{3}}q_0(\frac{n-t(k)}{3}))x^n=\sum_{n\geq 0}a(n)x^n$$

The conclusion now follows by matching coefficients of like powers of x.

Theorem 3:

$$q(n)+\sum_{k\geq 1}(-1)^k\{q(n-k(3k-1))+q(n-k(3k+1))\} = \begin{cases} 1 & \text{if } n=\frac{j(j+1)}{2} \\ 0 & \text{otherwise} \end{cases}$$

Proof: Identities (10), (2), and (7) imply

$$\sum_{n\geq 0} x^{\frac{n(n+1)}{2}} = (\prod_{n\geq 1}(1-x^{2n-1})^{-1})\prod_{n\geq 1}(1-x^{2n})$$

$$= (\sum_{n\geq 0} q(n)x^n)(1+\sum_{n\geq 1}(x^{n(3n-1)}+x^{n(3n+1)}))$$

$$= \sum_{n\geq 1}(q(n)+\sum_{k\geq 1}(q(n-k(3k-1))+q(n-k(3k+1))x^n$$

The conclusion follows from matching coefficients of like powers of x.

Theorem 4:

$$q(n) = \sum_{j\geq 0} p(\frac{n-t(j)}{2})$$

Proof: Invoking (4) with $t=2$, we have:

$$\sum_{n\geq 0} c_2(n)x^n = \prod_{n\geq 1}\frac{(1-x^{2n})^2}{1-x^n}$$

Therefore, invoking (2), we get

$$(\prod_{n\geq 1}(1-x^{2n})^{-1})(\sum_{n\geq 0} c_2(n)x^n) = \prod_{n\geq 1}\frac{1-x^{2n}}{1-x^n} = \prod_{n\geq 1}(1+x^n) = \sum_{n\geq 0} q(n)x^n$$

Thus by (1) we have

$$\sum_{n\geq 0} q(n)x^n = (\sum_{n\geq 0} p(\frac{n}{2})x^n)(\sum_{n\geq 0} c_2(n)x^n) = \sum_{n\geq 0}(\sum_{k=0}^{n} p(\frac{n-k}{2})c_2(k))x^n$$

The conclusion follows from (5), matching coefficients of like powers of x.

Theorem 5:

$$c_3(n) = \sum_{j\geq 0}(-1)^j(2j+1)p(n-3t(j))$$

Proof: Invoking (4) with $t=3$, and using (1) and (9) we have:

$$\sum_{n\geq 0} c_3(n)x^n = \prod_{n\geq 1}\frac{(1-x^{3n})^3}{1-x^n} = (\prod_{n\geq 1}(1-x^n)^{-1})(\prod_{n\geq 1}(1-x^{3n})^3)$$

$$= (\sum_{n\geq 0} p(n)x^n)(\sum_{n\geq 0}(-1)^n(2n+1)x^{3t(n)}) = \sum_{n\geq 0}(\sum_{j\geq 0}(-1)^j(2j+1)p(n-3t(j)))x^n$$

The conclusion now follows by matching coefficients of like powers of x.

Remark: Since $c_3(n)$ is easy to compute using (6), we rewrite Theorem 5 as:

Theorem 5*:

$$p(n) = c_3(n) + \sum_{j\geq 1}(-1)^{j-1}(2j+1)p(n-3t(j))$$

In similar fashion, that is, by invoking (4) with $t = 4$, one may prove:

Theorem 6:

$$\sum_{j=0}^{[\frac{n}{4}]} c_4(n-4j)p(j) = \sum_{j\geq 0}(-1)^j(2j+1)p(n-2j(j+1))$$

References:

1. A. Granville & K. Ono, *Defect zero p-blocks for finite simple groups*, Trans. Amer. Math. Soc. v.348 (1996) 331-347

2. N. Robbins, *On t-core partitions*, Fibonacci Quart. 38 (2000) 35-44