

A Study of the Genus of a Group

Hung-Lin Fu¹ and I-Fan Sun

Department of Applied Mathematics
Nation Chiao Tung University
Hsin Chu, Taiwan, R.O.C.

Abstract

In this paper, we improve the upper bounds for the genus of the group $\mathcal{A} = Z_{m_1} \times Z_{m_2} \times Z_{m_3}$ (in canonical form) with at least one even m_i , $i = 1, 2, 3$. As a special case, our results reproduce the known results in the cases $m_3 = 3$ or both m_2 and m_3 are equal to 3.

1 Introduction

We shall mainly use the notations from the book by Gross and Tucker [5]. A surface is a compact 2-manifold and it is well-known that every closed connected orientable surface can be obtained by adding some handles to a sphere in 3-space. We denote the sphere with g handles by S_g and the number g is called the (orientable) genus of the surface.

By an embedding of a graph G on a surface S ($G \hookrightarrow S$), we mean a drawing where the edges do not cross but meet only at vertices, and the components of $S - G$, which we call faces (or regions) of the embeddings. The "(orientable) genus" of a graph G , denoted by $\gamma(G)$, is the smallest number g such that the graph G embeds on the orientable surface S_g . If all of regions are 2-cells (where "2-cell" can be considered as an 2-dimensional open disk), then the embedding is said to be a 2-cell embedding. The well known formula for the sphere by Euler [4] and later extended to the surfaces

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of higher genus by White [11] show that the alternating sum of the number of vertices, edges and regions of a graph 2-cell embedded in a surface S_g is an invariant of the embedding surface and does not depend on the graph, i.e. $p - q + r = 2 - 2g$.

Let \mathcal{A} be a group and X be a generating set for \mathcal{A} . The Cayley color graph $C(\mathcal{A}, X)$ has its vertex set the elements of group \mathcal{A} and its edges set the Cartesian product $X \times \mathcal{A}$ such that the edge (x, a) has its endpoints the vertices a and ax , with its direction from a to ax . The designation of a plus direction and a color for every edge is an intrinsic part of a Cayley color graph $C(\mathcal{A}, X)$. If these designation are suppressed, the result is a graph $C(\mathcal{A}, X)^0$, called the "Cayley graph" for the group \mathcal{A} and the generating set X . The genus of a group \mathcal{A} is given by :

$$\gamma(\mathcal{A}) = \min\{\gamma(C(\mathcal{A}, X)^0)\}$$

where the minimum is taken over all generating sets X for \mathcal{A} .

Since any finite abelian group \mathcal{A} of rank r (the number of elements of the minimum generating set) has a unique canonical form

$$Z_{m_1} \times \cdots \times Z_{m_r}$$

such that m_{i+1} divides m_i for $i = 1, \dots, r-1$, and $m_r > 1$. It is convenient to study the genus of a group in canonical form. Here are some of the well-known results.

Theorem 1.1. (Jungerman and White, [6]) Let the abelian group \mathcal{A} be of the canonical form $Z_{m_1} \times \cdots \times Z_{m_r}$ such that $r > 1$ and $m_i \geq 4$ for all i . Then

$$\gamma(\mathcal{A}) \geq 1 + \frac{|\mathcal{A}|}{4}(r - 2).$$

Theorem 1.2. [9] Let the group $Z_{m_1} \times Z_{m_2}$ be of the canonical form. Then

$$\gamma(Z_{m_1} \times Z_{m_2}) = \begin{cases} 1 & , \text{ if } m_2 > 2 ; \text{ and} \\ 0 & , \text{ if } m_2 = 2 . \end{cases}$$

Theorem 1.3. [9] For $n \geq 2$, $\gamma((Z_2)^n) = 1 + 2^{n-3}(n - 4)$.

Theorem 1.4. (Jungerman and White, [6]) Let the abelian group \mathcal{A} have the canonical form $Z_{m_1} \times \cdots \times Z_{m_r}$, where $r > 1$ and every $m_i > 3$, and such that either every m_i is even or $r > 3$. Then $\gamma(\mathcal{A}) = 1 + \frac{|\mathcal{A}|}{4}(r - 2)$, whenever the number on the right-hand side of this equation is an integer. ■

Theorem 1.5. (Mohar et al.[8], Brin and Squier,[1]) The genus of the group $Z_3 \times Z_3 \times Z_3$ is 7. ■

2 The main result

First, we derive an upper bound which comes directly from the following theorem obtained by Burnside. Also, it is an easy consequence of the theory of coverings and branched covering theory of surfaces; see Riemann-Hurwitz equation in [5].

Theorem 2.1. [2] *If \mathcal{A} is finite and is minimally generated by $\{g_1, g_2, \dots, g_n\}$ and satisfies at least the relation $g_i^{m_i} = e = (\prod_{j=1}^n g_j)^k$, $1 \leq i \leq n$, then*

$$\gamma(\mathcal{A}) \leq 1 + \frac{|\mathcal{A}|}{2} \left(n - 1 - \frac{1}{k} - \sum_{j=1}^n \frac{1}{m_j} \right).$$

Corollary 2.2. *Let \mathcal{A} have the canonical form $Z_{m_1} \times Z_{m_2} \times Z_{m_3}$ such that $m_3 \geq 3$, then*

$$\gamma(\mathcal{A}) \leq 1 + \frac{|\mathcal{A}|}{2} \left(2 - \frac{2}{m_1} - \frac{1}{m_2} - \frac{1}{m_3} \right).$$

Proof. Obviously, $g_1 = (1, 0, 0)$, $g_2 = (0, 1, 0)$, and $g_3 = (0, 0, 1)$ form a minimal generating set such that $g_i^{m_i} = e$, $i = 1, 2, 3$. Furthermore, $(g_1 g_2 g_3)^{m_1} = e$. Therefore $\gamma(\mathcal{A}) \leq 1 + \frac{|\mathcal{A}|}{2} \left(2 - \frac{2}{m_1} - \frac{1}{m_2} - \frac{1}{m_3} \right)$. ■

Now we consider $\mathcal{A} = Z_{m_1} \times Z_{m_2} \times Z_{m_3}$ with m_1, m_2 even and m_3 odd, and \mathcal{A} in canonical form. Mainly, we shall use a different way of selecting a generating set to improve the upper bound obtained in Corollary 2.2.

Proposition 2.3. Let $A = Z_{m_1} \times Z_{m_2} \times Z_{m_3}$ be in canonical form where $m_1 = 2^{k_1}n_1$, $m_2 = 2^{k_2}n_2$, $m_3 = n_3$, $k_1, k_2 \geq 1$, and n_i is odd for $i = 1, 2, 3$. Then

$$\gamma(A) \leq 1 + \frac{|A|}{4} \left(1 + \frac{1}{n_1}\right).$$

Proof. Since $Z_{2^{k_1}n_1} \times Z_{2^{k_2}n_2} \times Z_{n_3}$ is isomorphic to $Z_{2^{k_1}n_3} \times Z_{2^{k_2}n_2} \times Z_{n_1}$, we shall consider the Cayley graph $C(Z_{2^{k_1}n_3} \times Z_{2^{k_2}n_2} \times Z_{n_1}, \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\})$. It is not difficult to see that we can embed the Cayley graph $H = C(Z_{2^{k_2}n_2} \times Z_{n_1}, \{(1, 0), (0, 1)\})$ quadrilaterally on torus. Figure 2.1 shows the embedding of H . By a "surgery technique" illustrated in [5], corresponding to the bipartition of $V(C_{2^{k_1}n_3})$ we obtain $2^{k_1-1}n_3$ copies of H which are embedded minimally on torus(S) and $2^{k_1-1}n_3$ copies of H on torus(S') with reverse orientation. Now we are ready to make two joins for each copy of H on S and then correspondingly on S' with reverse orientation. Since n_1 is odd, there is no way that we can find two sets of mutually disjoint regions(4-cycles) each contain $|V(H)|$ vertices(edges). Therefore, we obtain two sets of regions each covers all the vertices of H , but with as less common vertices as possible. They are designated by "O" and "X" respectively in Figure 2.1.

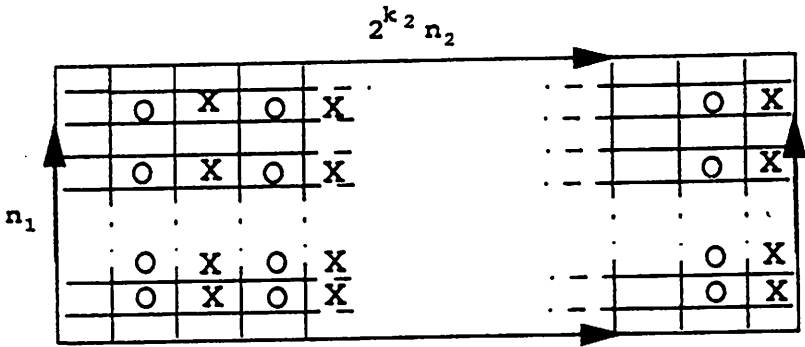


Figure 2.1.

We shall make a tube(handle) for each region to connect the copies. First, consider those regions with common edge. There are 6 edges to connect two regions with a common edge. The new regions are depicted in Figure 2.2. Thus we have a region which is an 8-cycle and the other regions are 4-cycles. Now we are ready to use Euler's formula to figure out

the genus of the surface obtained.

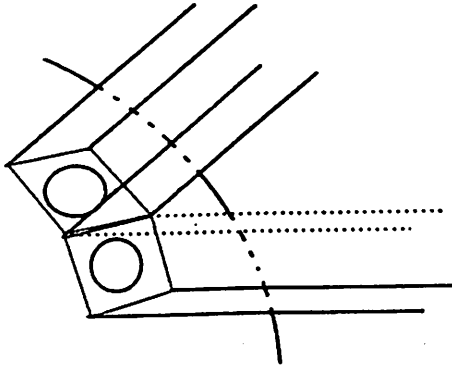


Figure 2.2.

Since, there are $2^{k_2}n_2n_1 - 2 \cdot [2^{k_2-2}n_2(n_1 + 1)]$ regions left on each copy of H after the surgery process, and we create $2^{k_2}n_2n_1$ new regions for each join. Hence

$$\begin{aligned} |F| &= 2^{k_1}n_3[2^{k_2}n_2n_1 - 2 \cdot 2^{k_2-2}n_2(n_1 + 1) + 2^{k_2}n_2n_1] \\ &= 2^{k_1+k_2-1}n_3n_2(3n_1 - 1). \end{aligned}$$

Since $|V| = 2^{k_1+k_2}n_3n_2n_1$ and $|E| = 3 \cdot 2^{k_1+k_2}n_3n_2n_1$, by Euler's formula, we have

$$2^{k_1+k_2}n_3n_2n_1 - 3 \cdot 2^{k_1+k_2}n_3n_2n_1 + 2^{k_1+k_2-1}n_3n_2(3n_1 - 1) = 2 - 2g.$$

Thus

$$\begin{aligned} g &= 1 + 2^{k_1+k_2}n_3n_2n_1 - 2^{k_1+k_2-2}n_3n_2(3n_1 - 1) \\ &= 1 + \frac{2^{k_1+k_2}n_3n_2n_1}{4} \left(4 - 3 + \frac{1}{n_1}\right) \\ &= 1 + \frac{|\mathcal{A}|}{4} \left(1 + \frac{1}{n_1}\right). \end{aligned}$$

This completes the proof. ■

Proposition 2.4. *Let the abelian group \mathcal{A} have the canonical form $Z_{m_1} \times Z_{m_2} \times Z_{m_3}$, where $m_1 = 2^k n_1$, $m_2 = n_2$, $m_3 = n_3$, $k \geq 1$ and n_i is odd for all i . Then*

$$\gamma(\mathcal{A}) \leq 1 + \frac{|\mathcal{A}|}{4} \left(1 + \frac{1}{n_1} + \frac{1}{n_2} - \frac{3}{n_1 n_2}\right).$$

Since the proof is similar to Proposition 2.3, instead of going through the details again we use an example to illustrate the theorem.

Let $\mathcal{A} = Z_{2 \times 15} \times Z_5 \times Z_5$. Clearly \mathcal{A} is isomorphic to the group $Z_{10} \times Z_5 \times Z_{15}$. And then we consider the quadrilateral embedding of $C(Z_5 \times Z_{15}, \{(1,0), (0,1)\})$ on the torus. For convenience, let H denote the Cayley graph $C(Z_5 \times Z_{15}, \{(1,0), (0,1)\})$. Figure 2.3 shows that the quadrilateral embedding on the torus and there exists two disjoint sets each containing $\frac{75+5+15-3}{4} = 23$ regions such that both sets containing all vertices.

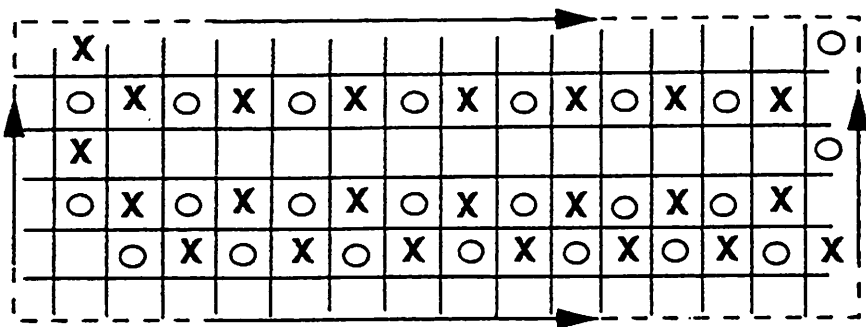


Figure 2.3. H is embedded on the torus and the number of the regions designated "O" and "X" are both 23.

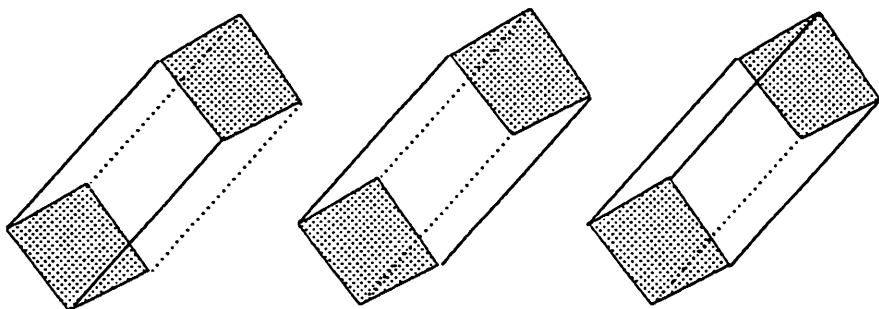


Figure 2.4.

Now partition the 10 copies of H such that the 5 copies of H embed on the torus(S) and the other 5 copies of H on the torus(S'). Make two joins of each copy in order to construct the embedding of the whole graph. Each

join contains 23 tubes and observe that some tubes are not carrying exactly 4 edges because of the fact that they may pass common vertices between them. See Figure 2.4. Note that no matter how many edges a tube carries, it always creates the same number of new regions, i.e. the number of new regions in each join is fixed and it is equal to $|V(H)| = 75$. Then we can compute the genus of this new surface.

Clearly, $|V| = 750$, $|E| = 2250$ and $|F| = 10(75 - 2 + 75) = 10 \times 104 = 1040$.

Again, by Euler's formula, we have

$$g = 231.$$

So $\gamma(Z_{30} \times Z_5 \times Z_5) \leq 231$.

In case that $m_3 = 3$, or m_2 and m_3 are equal to 3, then we can apply the idea in [1] to reproduce good lower bounds which are known, see [9].

Proposition 2.5. *Let $\mathcal{A} = Z_{2^{k_1 n_1}} \times Z_{2^{k_2 n_2}} \times Z_3$ be in canonical form and $k_1, k_2 \geq 1$, and n_i is odd for $i = 1, 2, 3$. Then*

$$\gamma(\mathcal{A}) \geq 1 + \frac{5}{24}|\mathcal{A}|.$$

Proof. Let X be a minimal generating set for \mathcal{A} . The Cayley graph $H = C(\mathcal{A}, X)$ contains at most $\frac{1}{3}|V|$ triangles since each triangle must consist of three order-3 edges. Then

$$\begin{aligned} 2|E| &\geq 3f_3 + 4(|F| - f_3) \\ &= 4|F| - f_3. \end{aligned}$$

Hence

$$\begin{aligned} f_3 &\geq 4|F| - 2|E| \\ &= 4(2 - 2g - |V| + |E|) - 2|E| \\ &= 8 - 8g + 2|V|. \end{aligned}$$

By the fact that $f_3 \leq \frac{|V|}{3}$, we conclude

$$g \geq 1 + \frac{5}{24}|\mathcal{A}|.$$

■

By a special embedding of $C(Z_3 \times Z_3, \{(1, 0), (0, 1)\})$. We can improve the upper bound obtained in Proposition 2.4 in case that $\mathcal{A} = Z_{2^{k_n}} \times Z_3 \times Z_3$, and obtain a similar lower bound as in Proposition 2.5.

Proposition 2.6. *Let the group $A = Z_{2^k n} \times Z_3 \times Z_3$ be in canonical form and $k \geq 1$. Then*

$$1 + \frac{1}{6}|A| \leq \gamma(Z_{2^k n} \times Z_3 \times Z_3) \leq 1 + \frac{5}{18}|A|.$$

Proof. First, we consider a special embedding for the Cayley graph $H = C(Z_3 \times Z_3, \{(1, 0), (0, 1)\})$ on the torus and this embedding was first used in [8] for showing that $\gamma(Z_3 \times Z_3 \times Z_3)$, illustrated as Figure 2.5.

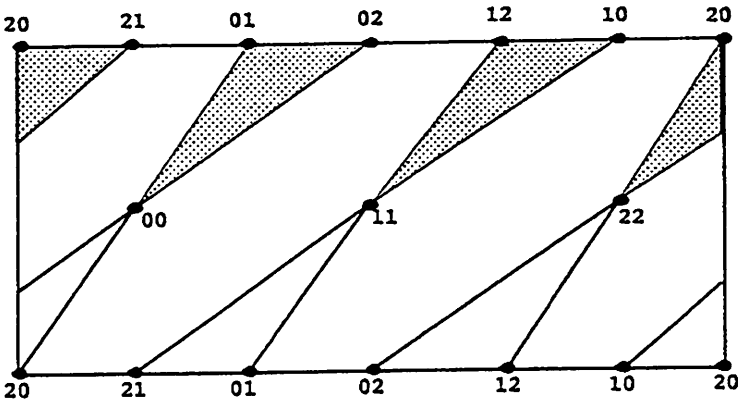


Figure 2.5. A toroidal embedding of $C_3 \times C_3$.

By observation we can select two disjoint sets of regions such that one set consists of 3 mutually vertex-disjoint triangles and the other set consists of two hexagons. The same construction will require to make the two joins for $2^k m$ copies of H . The construction of one join is attaching one end of a tube in the interiors of each region in the above selected set which consists of 3 mutually vertex-disjoint triangles for this copy. So the first join consists of 3 tubes each containing 3 new 4-cycles. The construction of the other join is attaching one end of a tube to the regions of the second selected set which contains two hexagons, and then the second join consists of 2 tubes one of which contains 6 4-cycles and the other one contains 3 hexagons. Now we are ready to compute the genus of the new surface.

Clearly, $|V| = 9 \cdot 2^k n$, $|E| = 27 \cdot 2^k n$ and $|F| = 2^k n \cdot (9 - 5 + 9) = 13 \cdot 2^k n$. From Euler's formula, we have

$$\begin{aligned} g &= \frac{1}{2}(2 - |V| + |E| - |F|) \\ &= \frac{1}{2}(2 - 9 \cdot 2^k n + 27 \cdot 2^k n - 13 \cdot 2^k n) \end{aligned}$$

$$\begin{aligned}
&= 1 + (-9 + 27 - 13)2^{k-1}n \\
&= 1 + \frac{5}{18}|A|.
\end{aligned}$$

Since the number of order-3 edges is $\frac{2}{3}|E|$, i.e. $f_3 \leq \frac{2}{3}|V|$, $1 + \frac{1}{6}|A| \leq \gamma(\mathcal{A})$ can be obtained by a similar counting.

This completes the proof. ■

By way of the idea used in Proposition 2.3 and 2.4, we also have the following result by using the technique in [9].

Proposition 2.7. *Let the group $A = Z_{m_1} \times Z_{m_2} \times Z_{m_3}$ be in canonical form where m_i is odd for $i = 1, 2, 3$. Then*

$$\gamma(\mathcal{A}) \leq 1 + \frac{|A|}{4} \left(1 + \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} - \frac{1}{m_1 m_2} - \frac{1}{m_1 m_3} - \frac{3}{m_2 m_3} + \frac{5}{m_1 m_2 m_3} \right).$$

Proof. In Theorem 1 of [9] Case (3b) take $G = C_{m_2} \times C_{m_3}$, $H = C_{m_1}$, $s = 2$, $e_1 = e_2 = \frac{1}{2}(m_1 - 1)$, $r_1 = r_2 = \frac{1}{4}(m_2 m_3 + m_2 + m_3 - 3)$, $e_0 = 1$. Then

$$\begin{aligned}
g &= 1 + |V(H)|(1 - 1) + \sum_{i=1}^s e_i r_i + \frac{1}{2} e_0 (|V(G)| + 1) \\
&= 1 + e_1 r_1 + e_2 r_2 + \frac{1}{2} e_0 (|V(G)| + 1) \\
&= 1 + 2 \left(\frac{m_1 - 1}{2} \right) \left(\frac{m_2 m_3 + m_2 + m_3 - 3}{4} \right) + \frac{m_2 m_3 + 1}{2} \\
&= 1 + \frac{(m_1 - 1)(m_2 m_3 + m_2 + m_3 - 3)}{4} + \frac{2(m_2 m_3 + 1)}{4} \\
&= 1 + \frac{1}{4} (m_1 m_2 m_3 + m_1 m_2 + m_1 m_3 - 3m_1 + m_2 m_3 - m_2 - m_3 + 5) \\
&= 1 + \frac{|A|}{4} \left(1 + \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} - \frac{1}{m_1 m_2} - \frac{1}{m_1 m_3} - \frac{3}{m_2 m_3} + \frac{5}{m_1 m_2 m_3} \right).
\end{aligned}$$

■

To conclude this note, we also observe that the asymptotics of the genus of $\mathcal{A} = Z_{m_1} \times Z_{m_2} \times Z_{m_3}$, $\gamma(\mathcal{A}) \rightarrow 1 + \frac{|A|}{4}$, provided that one of the following conditions holds:

- (1) all m_i are even,
- (2) two of the m_i 's are even and $m_1 \rightarrow \infty$,
- (3) one m_i is even and $m_2 \rightarrow \infty$,
- (4) all three m_i 's are odd and $m_3 \rightarrow \infty$.

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References

- [1] M. G. Brin and C. C. Squier. *On the genus of $Z_3 \times Z_3 \times Z_3$* , Europ. J. Combin. 9, 431-443. (1988)
- [2] W. Burnside. *Theory of Groups of Finite Order*, Second Edition, Cambridge University Press, Cambridge.
- [3] A. Cayley. *The theory of groups: Graphical representations*, Amer. J. Math. 1 (1878), 174-176.
- [4] L. Euler, *Demonstratio nonnullarum insignium proprietatum quibus solida hedris planis inclusa sunt praedita*, Novi Comm. Acad. Sci. Imp. Petropol. 4 (1758), 140-160.
- [5] J. L. Gross and T. W. Tucker. *Topological Graph Theory*, John Wiley & Sons, New York. (1987)
- [6] M. Jungerman and A. T. White. *On the genus of finite abelian groups*, European J. Combin. 1, 243-251.
- [7] S. MacLane and G. Birkhoff, *Algebra*, Macmillan, New York. (1967)
- [8] B. Mohar, T. Pisanski, M. Škovič and A. T. White. *The cartesian product of three triangles can be embedded into a surface of genus 7*, Discrete Math. 56 (1985), 87-89.
- [9] B. Mohar, T. Pisanski and A. T. White, *Embeddings of Cartesian Products of Nearly Bipartite Graphs*, Journal of Graph Theory, Vol. 14, No. 3, 301-310. (1990)
- [10] T. Pisanski and A. T. White, *Nonorientable embeddings of groups*, Eur. J. Combinat. 9, 445-461. (1988)
- [11] A. T. White, *Graphs, Groups and Surfaces*, revised edition North-Holland, Amsterdam. (1984)
- [12] A. T. White, *On the genus of a group*, Trans. Amer. Math. Soc. 173 (1972), 203-214.