Transverse families of matchings in the plane

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ABSTRACT. We investigate the following problem: given a set $S \subset \mathbb{R}^2$ in general position and a positive integer k, find a family of matchings $\{M_1, M_2, \ldots, M_k\}$ determined by S so that if $i \neq j$ then each segment in M_i crosses each segment in M_j. We give improved linear lower bounds on the size of the matchings in such a family.

1. Introduction

Let $S \subset \mathbb{R}^2$ be a finite set in general position (no three points collinear). A *matching* on S is a collection M of line segments determined by points of S such that no point of S is endpoint to more than one segment in M. A matching is *fully crossing* if each pair of its segments intersect. A family of matchings $\{M_1, M_2, \ldots, M_k\}$ on S is *transverse* if $\cup_{i=1}^{k} M_i$ is a matching and whenever $i \neq j$ every segment in M_i intersects every segment in M_j.

We will say that the collection $\{A_1, A_2, \ldots, A_m\}$ of subsets of S is *convexly arranged* if for each j the convex hull $\mathrm{co}((\cup_{i=1}^{m} A_i) \setminus A_j)$ contains no point of A_j. (Throughout this paper we will use $\mathrm{co}(X)$ to denote the convex hull of set X.) Thus $\{A_1, A_2, \ldots, A_m\}$ is convexly arranged if and only if each A_i can be separated by a line from the union of the other sets. Equivalently, $\{A_1, A_2, \ldots, A_m\}$ is convexly arranged if and only if whenever $p_1, p_2, \ldots, p_m$ are points with $p_i \in A_i$ for each i then each p_i is a vertex of $\mathrm{co}(\{p_1, p_2, \ldots, p_m\})$.

The results of this paper are related to those of [1] and [2]. In [1] the authors prove that any set S of n points admits a fully-crossing matching of size at least $\sqrt{n/12}$ and ask whether this can be improved to a linear bound. We modify the question by seeking a transverse family of k matchings on S, each of some minimum size. In [2] the authors generalize a well-known theorem of Erdös and Szekeres by proving that for each k there is a constant

$c_k > 0$ such that each set S of n points admits a convexly arranged collection of k subsets, each of size $\lfloor c_k n \rfloor$. The constants c_k appear to diminish exponentially; however, it is proved in [2] that $c_4 \geq 1/22$. Clearly a convexly arranged collection of $2k$ subsets of size m gives rise to a transverse family of of k matchings, each of size m. Thus, transverse families of linear-sized matchings always exist.

In this paper we establish better lower bounds on the size of these matchings. Specifically, we show that given any set S of n points in general position and any $i \geq 0$ there is a transverse family of 2^{i+1} matchings on S, each of size at least $\lfloor \frac{n}{20 \cdot 16^i} \rfloor$. The $i = 0$ case (Theorem 1) provides a modest improvement on the bound for c_4 from [2], and while the matchings we produce for $i > 0$ do not necessarily arise from convexly arranged collections of subsets, their sizes improve considerably on bounds obtainable from previous results. All of our proofs are constructive; the desired transverse families of matchings being obtainable in $O(n \log n)$ time.

By way of notation, if $\vec{l}$ is an oriented line then we denote by $H_L(\vec{l})$ and $H_R(\vec{l})$ the open halfplanes to the left and right, respectively, of $\vec{l}$. We will use the following well-known fact several times.

LEMMA 1. *Let $S \subset \mathbb{R}^2$ be a finite set of points in general position. Let $\vec{l}$ be an oriented line missing S and let S_L and S_R denote the sets $S \cap H_L(\vec{l})$ and $S \cap H_R(\vec{l})$, respectively. Given nonnegative integers $m_L \leq |S_L|$ and $m_R \leq |S_R|$ there is an oriented line $\vec{l'}$ missing S such that $\left|S_L \cap H_L(\vec{l'})\right| = m_L$ and $\left|S_R \cap H_R(\vec{l'})\right| = m_R$.* □

2. Results

Our first goal will be to construct (for any set S of n points in general position) a convexly arranged collection of four not-too-small subsets $\{A_1, A_2, A_3, A_4\}$. It will be useful to consider a property for collections of sets that is weaker than that of being convexly arranged: we will say that the family $\{A_1, A_2, \ldots, A_m\}$ of subsets of S is *independently arranged* if no line meets three of the convex hulls $\mathrm{co}(A_i)$ $(1 \leq i \leq m)$. It is easy to see that every convexly arranged family is independently arranged. By a 4CA collection we will mean a convexly arranged collection of four subsets of S, and by a 4INCA collection we will mean a collection of four subsets of S that is independently arranged but not convexly arranged. Note that if $\{A_1, A_2, A_3, A_4\}$ is a 4INCA collection then one of the four sets is contained entirely in the convex hull of the union of the other three. We will say that this set is the *center set* of the 4INCA collection. Note that $\{A_1, A_2, A_3, A_4\}$ is a 4INCA collection with center A_4 if and only if for $1 \leq i \leq 3$ the set A_i can be separated by a line from the union of the other sets, and if A_4 can be separated by a line from the union of any two other sets, but not from $A_1 \cup A_2 \cup A_3$.

LEMMA 2. *Let $S \subset \mathbb{R}^2$ be a set of n points in general position. Then either*

(a) *there is a 4INCA collection $\{A_1, A_2, A_3, A_4\}$ of subsets of S with center set A_4 such that the cardinality of A_1 and A_2 are at least $\lfloor \frac{n}{20} \rfloor$ each, the cardinality of A_3 is at least $\lfloor \frac{n}{5} \rfloor$, and the cardinality of A_4 is at least $\lfloor \frac{n}{10} \rfloor$, or*

(b) *there is a 4CA collection $\{B_1, B_2, B_3, B_4\}$ of subsets of S, each of cardinality at least $\lfloor \frac{n}{20} \rfloor$.*

Proof: Using Lemma 1 and applying a shear transformation, we may assume that each of the four open quadrants of $\mathbb{R}^2$ contains at least $\lfloor \frac{n}{4} \rfloor$ points of S. Let these open quadrants be denoted U_1, U_2, U_3, and U_4 (with the quadrants numbered in the usual counterclockwise fashion with U_1 being the set of points (x, y) with both x and y positive) and define $S_i = S \cap U_i$ $(i = 1, 2, 3, 4)$. Apply Lemma 1 twice to the set $S_2 \cup S_4$ and the positively oriented x-axis to obtain lines $\vec{l_1}$ and $\vec{l_2}$ such that

$$\left|S_2 \cap H_L(\vec{l_1})\right| = \left|S_2 \cap H_R(\vec{l_2})\right| = \left|S_4 \cap H_L(\vec{l_1})\right| = \left|S_4 \cap H_R(\vec{l_2})\right| = \left\lfloor \frac{n}{20} \right\rfloor$$

(with $\vec{l_1}$ and $\vec{l_2}$ oriented compatibly so that $S_2 \cap H_L(\vec{l_1}) \cap H_R(\vec{l_2}) = \emptyset$). Then the region $K = H_R(\vec{l_1}) \cap H_L(\vec{l_2})$ must contain at least $\lfloor \frac{3n}{20} \rfloor$ points in each of S_2 and S_4. If $|S_1 \cap K| \geq \lfloor \frac{n}{10} \rfloor$ then condition **(a)** is satisfied with

$$\begin{aligned} A_1 &= S_2 \cap H_R(\vec{l_2})\,, \\ A_2 &= S_4 \cap H_R(\vec{l_2})\,, \\ A_3 &= S_3\,, \text{ and} \\ A_4 &= S_1 \cap K \end{aligned}$$

(see Figure 1). Similarly, condition **(a)** is satisfied if $|S_3 \cap K| \geq \lfloor \frac{n}{10} \rfloor$. Assume, then, that both $|S_1 \cap K|$ and $|S_3 \cap K|$ are less than $\lfloor \frac{n}{10} \rfloor$. In this case, both $\left|S_1 \cap H_R(\vec{l_2})\right|$ and $\left|S_3 \cap H_L(\vec{l_1})\right|$ must be at least $\lfloor \frac{3n}{20} \rfloor$. Let S' be a subset of S consisting of $\lfloor \frac{3n}{20} \rfloor$ points in each of $S_1 \cap H_R(\vec{l_2})$, $S_3 \cap H_L(\vec{l_1})$, $S_2 \cap K$, and $S_4 \cap K$. As before, define $S'_i = S' \cap U_i$ $(i = 1, 2, 3, 4)$.

Now repeat the previous procedure on S', this time applying Lemma 1 twice to $S'_1 \cup S'_3$ to yield two lines $\vec{l_3}$ and $\vec{l_4}$ so that

$$\left|S'_1 \cap H_L(\vec{l_3})\right| = \left|S'_1 \cap H_R(\vec{l_4})\right| = \left|S'_3 \cap H_L(\vec{l_3})\right| = \left|S'_3 \cap H_R(\vec{l_4})\right| = \left\lfloor \frac{n}{20} \right\rfloor .$$

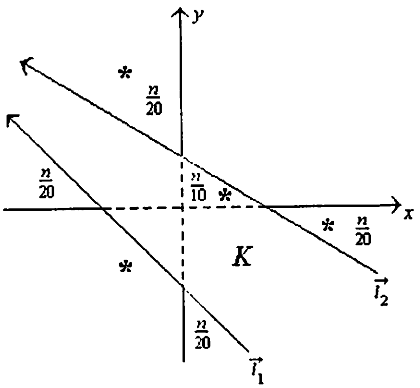

Figure 1.

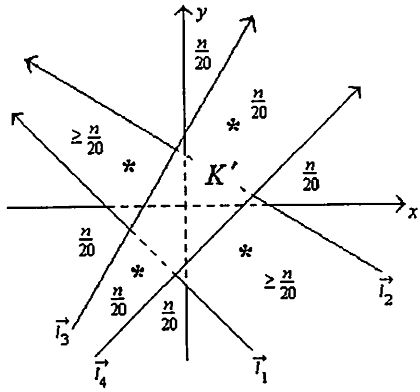

Figure 2.

Then the region $K' = H_R(\vec{l_3}) \cap H_L(\vec{l_4})$ must contain at least $\lfloor \frac{n}{20} \rfloor$ points in each of S'_1 and S'_3, none of which lie in K. Again, condition **(a)** is satisfied if either $|K' \cap U_2| \geq \lfloor \frac{n}{10} \rfloor$ or $|K' \cap U_4| \geq \lfloor \frac{n}{10} \rfloor$. Assuming, then, that this is not the case, we conclude that $K \cap H_L(\vec{l_3})$ and $K \cap H_R(\vec{l_4})$ each contain at least $\lfloor \frac{n}{20} \rfloor$ points of S'. It is now clear (see Figure 2) that condition **(b)** is satisfied with

$$\begin{aligned} B_1 &= S'_1 \cap K' \cap H_R(\vec{l_2}) , \\ B_2 &= S'_2 \cap K \cap H_L(\vec{l_3}) , \\ B_3 &= S'_3 \cap K' \cap H_L(\vec{l_1}) , \text{ and} \\ B_4 &= S'_4 \cap K \cap H_R(\vec{l_4}) . \end{aligned}$$

□

THEOREM 1. *Let $S \subset \mathbb{R}^2$ be a set of n points in general position. Then there is a 4CA collection of subsets of S, each of cardinality at least* $\lfloor \frac{n}{20} \rfloor$.

Proof: By Lemma 2 it is enough to show that given a 4INCA collection $\{A_1, A_2, A_3, A_4\}$ with center set A_4 and with $|A_1| = |A_2| = k$, $|A_3| = 4k$, and $|A_4| = 2k$, we can construct a 4CA collection of sets each of cardinality at least k. So, let such a 4INCA collection be given. By Lemma 1 there exists a line $\vec{l}$ such that $A_{3L} = A_3 \cap H_L(\vec{l})$ and $A_{3R} = A_3 \cap H_R(\vec{l})$ each have cardinality $2k$ while $A_{4L} = A_4 \cap H_L(\vec{l})$ and $A_{4R} = A_4 \cap H_R(\vec{l})$ each have cardinality k. Assume that (as in Figure 3) $A_1 \subset H_L(\vec{l})$ (so that of necessity $A_2 \subset H_R(\vec{l})$). There is a line $\vec{l'}$ separating A_3 from $A_1 \cup A_2 \cup A_4$. Translate $\vec{l'}$ until the halfplane determined by it and containing A_4 first

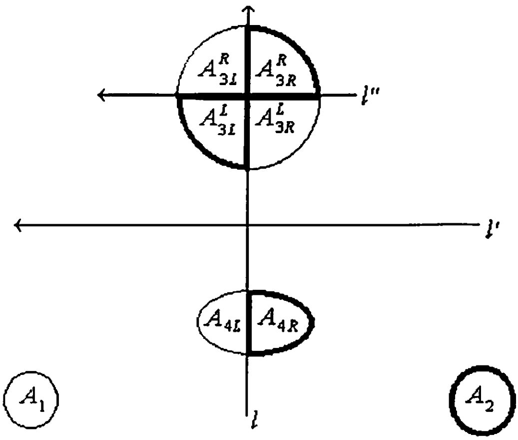

Figure 3.

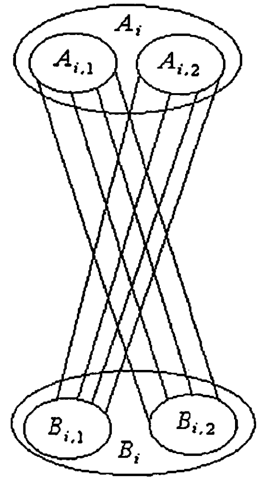

Figure 4.

contains k points of either A_{3L} or A_{3R}. Let the translated line be denoted $\vec{l''}$ and suppose that the halfplane containing A_4 is $H_L(\vec{l''})$. Defining $A_{3L}^L = A_{3L} \cap H_L(\vec{l''})$, $A_{3L}^R = A_{3L} \cap H_R(\vec{l''})$, $A_{3R}^L = A_{3R} \cap H_L(\vec{l''})$, and $A_{3R}^R = A_{3R} \cap H_R(\vec{l''})$, we may assume that $|A_{3L}^L| = |A_{3L}^R| = k$, and $|A_{3R}^R| \geq k$ (see Figure 3). Then $\{A_2, A_{3L}^L, A_{3R}^R, A_{4R}\}$ is a 4CA collection with each set of cardinality at least k. □

Theorem 1 allows the formation of a transverse family of two matchings on S, each of cardinality at least $\lfloor \frac{n}{20} \rfloor$. The strategy for finding larger transverse families of matchings is simple: suppose $\{M_1, M_2, \ldots, M_k\}$ is a transverse family of matchings on S. Then for each i there are sets A_i and B_i (separated by a line) such that each segment in M_i has an endpoint in each of A_i and B_i. Suppose we now find a 4CA collection $\{A_{i,1}, A_{i,2}, B_{i,1}, B_{i,2}\}$ such that $A_{i,1} \cup A_{i,2} \subset A_i$ and $B_{i,1} \cup B_{i,2} \subset B_i$. Then we can define new matchings $M_{i,1}$ and $M_{i,2}$ (as in Figure 4) so that the family $\{M_{i,j}\}_{1 \leq i \leq k, 1 \leq j \leq 2}$ is transverse. The key to making this scheme work is the following lemma. Its proof is a simple adaptation of the demonstration by Bárány and Valtr (see [2]) of their bound for c_4.

LEMMA 3. *Let A and B be sets each of cardinality $16k$ separated by a line $\vec{l_0}$. Then there exists a 4CA collection $\{A_1, A_2, B_1, B_2\}$ of sets, each of cardinality at least k, so that $A_1 \cup A_2 \subset A$ and $B_1 \cup B_2 \subset B$.*

Proof: We may assume $\vec{l_0}$ is the positively oriented y-axis, A lies to the left of $\vec{l_0}$, B lies to the right of $\vec{l_0}$, and no vertical line contains more than one point of $A \cup B$. We will also assume that all vertical lines are oriented

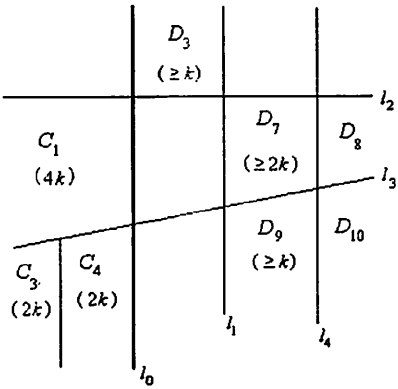

Figure 5.

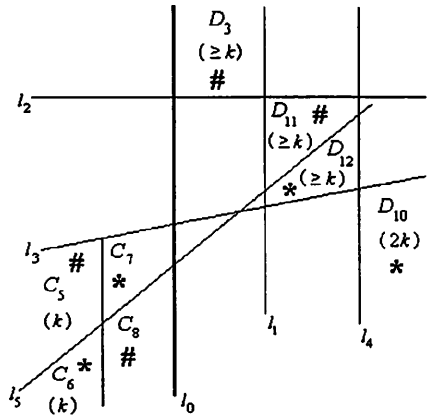

Figure 6.

consistent with $\vec{l_0}$ and that non-vertical lines are oriented so as to cross $\vec{l_0}$ from left to right.

Find a vertical line $\vec{l_1}$ separating B into two sets, D_1 (left of $\vec{l_1}$) and D_2 (right of $\vec{l_1}$) such that $|D_1| = 2k$ and $|D_2| = 14k$. Apply Lemma 1 to the sets A and D_2 to obtain a line $\vec{l_2}$ splitting both sets into equal halves. Assume (without loss of generality) that at least k points of D_1 lie above $\vec{l_2}$ and refer to this set of points as D_3. Let D_4 denote the set of $7k$ points of D_2 lying below $\vec{l_2}$ and let C denote the set of $8k$ points of A lying below $\vec{l_2}$. Now apply Lemma 1 to the sets C and D_4 to obtain a line $\vec{l_3}$ separating C into sets C_1 (above $\vec{l_3}$) and C_2 (below $\vec{l_3}$) and separating D_4 into sets D_5 (above $\vec{l_3}$) and D_6 (below $\vec{l_3}$) so that $|C_1| = |C_2| = 4k$, $|D_5| = 4k$, and $|D_6| = 3k$. Using a vertical line, separate C_2 into sets C_3 (on the left) and C_4 (on the right), each of cardinality $2k$. Then find a vertical line $\vec{l_4}$ separating D_5 into sets D_7 (left of $\vec{l_4}$) and D_8 (right of $\vec{l_4}$) and separating D_6 into sets D_9 (left of $\vec{l_4}$) and D_{10} (right of $\vec{l_4}$) so that $|D_7| \geq 2k$, $|D_9| \geq k$, and so that one of D_8 or D_{10} contains $2k$ points (see Figure 5). There are now two cases to consider.

Case 1: $|D_{10}| = 2k$. In this case, apply Lemma 1 to the sets C_3 and D_7 to find a line $\vec{l_5}$ separating C_3 into sets C_5 (above $\vec{l_5}$) and C_6 (below $\vec{l_5}$) and separating D_7 into sets D_{11} (above $\vec{l_5}$) and D_{12} (below $\vec{l_5}$) so that all four new sets have cardinality at least k. The line $\vec{l_5}$ also separates C_4 into two sets C_7 (above $\vec{l_5}$) and C_8 (below $\vec{l_5}$), one of which must contain at least k points. If $|C_7| \geq k$ then $\{C_6, C_7, D_{10}, D_{12}\}$ is a 4CA collection. If $|C_8| \geq k$ then $\{C_5, C_8, D_3, D_{11}\}$ is a 4CA collection. (See Figure 6.)

Case 2: $|D_8| = 2k$. In this case, apply Lemma 1 to the sets C_1 and

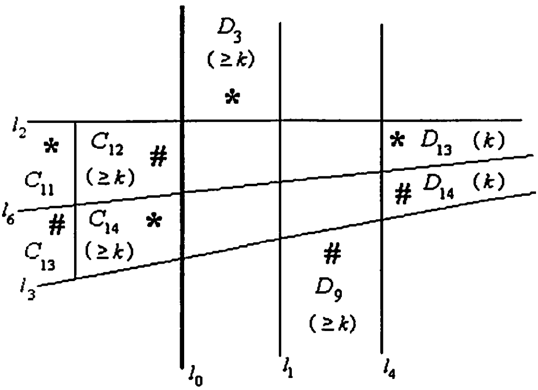

Figure 7.

D_8 to find a line $\vec{l_6}$ separating C_1 into sets C_9 (above $\vec{l_6}$) and C_{10} (below $\vec{l_6}$) and separating D_8 into sets D_{13} (above $\vec{l_6}$) and D_{14} (below $\vec{l_6}$) so that $|C_9| = |C_{10}| = 2k$ and $|D_{13}| = |D_{14}| = k$. Now find a vertical line separating C_9 into sets C_{11} (on the left) and C_{12} (on the right) and separating C_{10} into sets C_{13} (on the left) and C_{14} (on the right) so that both C_{12} and C_{14} contain k or more points while one of C_{11} or C_{13} contains k points. If $|C_{11}| = k$ then $\{C_{11}, C_{14}, D_3, D_{13}\}$ is a 4CA collection, while if $|C_{13}| = k$ then $\{C_{12}, C_{13}, D_9, D_{14}\}$ is a 4CA collection. (See Figure 7.) □

Following the strategy outlined previously (Theorem 1 followed by repeated application of Lemma 3) yields our desired result on transverse families of matchings. Note that this theorem implies a $O(nk^{-4})$ lower bound for the cardinality of matchings in a transverse family of k matchings on a set of n points.

THEOREM 2. *Let $S \subset \mathbb{R}^2$ be a set of n points in general position. For every $i \geq 0$ there is a transverse family of 2^{i+1} matchings on S with each matching of cardinality at least* $\lfloor \frac{n}{20 \cdot 16^i} \rfloor$. □

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