Adding and Deleting Crossings in Drawings of Graphs*

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ABSTRACT. Given a good drawing of a graph on some orientable surface, there exists a good drawing of the same graph with one more or one less crossing on an orientable surface which can be exactly determined. Our methods use a new combinatorial representation for drawings. These results lead to bounds related to the Thrackle Conjecture.

1 Introduction

A *good drawing* of a graph on an orientable surface, informally, will mean a drawing in which no two adjacent edges cross, no two edges cross more than once, no more than two edges cross at a point, no edge crosses itself, no edge intersects a vertex that is not one of its endpoints, and the regions so created are 2-cell. Two edges are *parallel* in a good drawing D of a graph G if they are nonadjacent and do not cross in D. All drawings in this paper are good and all surfaces are compact orientable 2-manifolds.

As observed by Dyck [1] and by Heffter [2] and formalized by Edmonds [3], a 2-cell embedding of a graph $G = (V, E)$ in an orientable surface can be described by a p-tuple of cyclic permutations of the open neighborhoods of the vertices $v_1, v_2, \ldots, v_p$ in V. The *Rotational Embedding Scheme*, discussed in detail by Youngs [4], states that there is a one-to-one

*Authors' work supported through the United States Office of Naval Research.

correspondence between the 2-cell embeddings of a graph G (on all possible orientable surfaces) and the p-tuples of cyclic permutations.

Using the definition of a good drawing and Rotational Embedding Schemes, we develop a combinatorial representation of a good drawing, which is the basis for the cross addition and cross deletion results.

2 Combinatorial Representation for a Good Drawing

In this section we present a formal structure for representing drawings of graphs on orientable surfaces. This structure has been used to examine certain drawings on orientable surfaces other than the plane [5].

For the remainder of the paper, a graph $G = (V, E)$ will be a graph with p vertices and q edges and $V = \{1, 2, \ldots, p\}$. Given a graph G, for every pair of distinct, nonadjacent edges associate exactly one new vertex called a *cross vertex* associated with that edge pair. Let $X(G)$ be the set of all cross vertices for a graph G. A *neighborhood set* of a graph G, $N_\pi(G)$, is a unique p-tuple $(\pi(1), \pi(2), \ldots, \pi(p))$, where for $i = 1, 2, \ldots, p$, $\pi(i)\colon N(i) \to N(i)$ is a cyclic permutation and $N(i)$ is the open neighborhood of the vertex i. Note that there exist

$$\prod_{v \in V} (\deg v - 1)!$$

distinct neighborhood sets of G, each one corresponding to a 2-cell embedding of G on some surface.

For Y a subset of $X(G)$, we define a *cross graph*, $G_x(Y)$, or G_x if the set Y is understood, for a graph G. For the purposes of this definition, label the edge between vertices i and j, with ij, where $i < j$. For each edge ij, let $Y(i, j)$ be the set of cross vertices from Y associated with edge ij, and let IJ be the order of this set. For each member, x, of $Y(i, j)$ associate uniquely an ordinal from the set $\{1, 2, \ldots, IJ\}$, and call this ordinal $ij(x)$. We label each cross vertex x with a unique ordered 4-set,

$$(ij, ij(x), uv, uv(x)),$$

where x is associated with edges ij and uv and ij is lexicographically smaller than uv. (That is, $i \le u$ and if $i = u$, then $j < v$.)

A cross graph G^* of G has vertex set $V(G^*) = V \cup Y$, and edge set defined by the following adjacencies:

(ij, r, st, w) is adjacent to (kl, y, mn, z) iff

$$\begin{aligned} ij = kl &\text{ and } y = r \pm 1 \text{ OR} \\ st = mn &\text{ and } z = w \pm 1 \text{ OR} \\ ij = mn &\text{ and } z = r \pm 1 \text{ OR} \\ st = kl &\text{ and } y = w \pm 1. \end{aligned}$$

AND
For i, j in V, i and j are adjacent iff they are adjacent in G and there are no cross vertices in Y associated with edge ij.
AND
Vertex i in V is adjacent to (mn, z, st, w) iff exactly one of the following holds:

$$i = m \text{ and } z = 1 \text{ OR}$$
$$i = n \text{ and } z = MN \text{ OR}$$
$$i = s \text{ and } w = 1 \text{ OR}$$
$$i = t \text{ and } w = ST.$$

Thus, the edge set of the cross graph could be viewed as consisting of all those edges in G with which no cross vertices are associated, and every other edge uv of G replaced by a "path", $P(u, v)$, whose interior vertices are cross vertices associated with uv. Note that there are many cross graphs for a given graph, depending on the choice of Y and on the ordinals associated with each edge in the above definition. It is convenient to change notation henceforth in order to describe the edges of these paths, which we call "segments". For an edge (u, v) of G, the "ith segment" encountered when traveling along $P(u, v)$ from u to v in G^* is labeled $(X^i_{u,v}, X^{i+1}_{u,v})$. It is considered to be the union of the directed segments $[X^i_{u,v}, X^{i+1}_{u,v}]$ and $[X^{i+1}_{u,v}, X^i_{u,v}]$, while these directed segments are referred to as *mirrors* of each other.

An *alternating neighborhood, $x - nbhd$*, of a cross vertex y in a cross graph G_x is a cyclic permutation $\pi_x(y)$ of the neighbors of y such that if y is associated with the edge pair (i, j) and (u, v), with (a, y) and (y, b) in $P(i, j)$, and (c, y) and (y, d) in $P(u, v)$, then $\pi_x(y) = (a, c, b, d)$ or $\pi_x(y) = (a, d, b, c)$. An *alternating neighborhood set, $x - N_\pi(G_x)$*, of a cross graph G_x is a neighborhood set $N_\pi(G_x)$ such that all cross vertices in $V(G_x)$ have $x - nbhds$. The resultant embedding of G_x is called an *alternating embedding.* In this embedding two directed segments are said to be *consecutive* on a region if neither segment's mirror occurs between the two segments when using a rotational scheme for "tracing" the boundary of the region. For notational brevity, edges or parts of edges on the boundary of a region will be called "in the region".

Let T be an embedding of a graph H on some surface S with a vertex v of degree 4 whose cyclic permutation $\pi(v) = (v_1, v_2, v_3, v_4)$ describes the neighbors of v in counterclockwise order about v in T. Assume that neither v_1v_3 nor v_2v_4 are in the edge set of H. To *cross lift* v, x-lift, from T is to mark a point v_s on S at the location of v in T and then create $H^* = (V^*, E^*)$, with drawing D^*, by $V^* = V(H) \setminus \{v\}$ and $E^* = E(H) \cup Q - P$, where $Q = \{(v_1, v_3), (v_2, v_4)\}$, $P = \{(v, v_1), (v, v_2), (v, v_3)\}$, and the edges of Q are drawn with a crossing at v_s. (One could "*tangent lift*" by including the set $\{(v_1, v_2), (v_3, v_4)\}$ in E^* making a tangent point in D^* at vs, but we have no use for that here, since it does not create a good drawing.) We will use the term "lifting" of vertices in this paper to indicate x-lifting. Notice that the condition on non adjacencies is always satisfied if the vertex of degree four is a cross vertex in an alternating embedding of a cross graph.

Now, given a drawing of a graph G which has crossings, if we use a reversal of this lifting process, which we could call "placing", we have put a new vertex at each crossing in a drawing and have obtained a new embedded graph, G^*. If one considers each of the new vertices to be cross vertices in the above definition, and uses as the ordinal correspondence an ordering of the cross vertices on edge ij, from i to j, $i < j$, then G^* is a cross graph of G. By taking a counterclockwise cyclic rotation at each vertex of G^*, an alternating neighborhood set corresponding to the embedding of G^* results. Likewise, when we imagine x-lifting all cross vertices of an embedded cross graph simultaneously, we can see that there is an easy correspondence between alternating neighborhood sets of cross graphs and drawings of the graph.

For a given cross graph derived from a drawing of a graph G, the embedding corresponding to this neighborhood set is called "the embedding corresponding to the drawing of G". Thus, an alternating neighborhood set of a cross graph of G is a combinatorial representation for a good drawing of G. The alternating neighborhood set structure has been used to show the following result [5].

Theorem 2.1. *Given a connected graph G and an integer k, $0 \leq k \leq \Theta(G)$, there exists a good drawing of G with k crossings on some surface, where $\Theta(G)$ is the order of $X(G)$.*

The bound $\Theta(G)$ in Theorem 2.1 is easily derived [6] and is calculated by the following sum.

$$\Theta(G) = \sum_{uv \in E} (|E| - \deg u - \deg v + 1)/2.$$

3 The Cross Addition Theorem

We are now ready for the main theorem. It is important to note that this theorem and Theorem 4.1 are "prescriptive". That is, although the cases given are rather technical, when one has an explicit drawing, it is possible to determine which case is applicable and actually form a new drawing. Accordingly, all cases are described in detail. In what follows, the compact orientable surface of genus n is denoted S_n.

Theorem 3.1. *Given a connected graph $G = (V, E)$ and a good drawing D of G on S_h with k crossings, $k < \Theta(G)$, there exists a good drawing of*

G with $k+1$ crossings on S_n where $n = h$, $h+1$, or $h+2$. Furthermore, the value of n can be explicitly determined according to cases as given in the proof.

Proof: Since $k < \Theta(G)$, there exists a pair of parallel edges in D. For all cases, we let G_x be the cross graph associated with the drawing D. Let (a,b) and (c,d) be those parallel edges with arbitrarily chosen segments $[X^i_{a,b}, X^{i'}_{a,b}]$ and $[X^j_{c,d}, X^{j'}_{c,d}]$, respectively. We construct a new cross graph G^*_x of G_x by letting $Y = \{X^*\}$, where X^* is the cross vertex associated with the edge pair $([X^i_{a,b}, X^{i'}_{a,b}], [X^j_{c,d}, X^{j'}_{c,d}])$. Let $x - N_\pi(G_x)$ be the alternating neighborhood set of G_x corresponding to D, giving an embedding $\mathcal{E}$ on a surface of genus h. Construct an alternating neighborhood set of G^*_x, $x - N_\pi(G^*_x)$, with embedding $\mathcal{E}^*$ of genus n, by leaving all cyclic permutations of the vertices in $x - N_\pi(G_x)$ the same except for $\pi_x(X^i_{a,b})$, $\pi_x(X^{i'}_{a,b})$, $\pi_x(X^j_{c,d})$, and $\pi_x(X^{j'}_{c,d})$, which are changed by,

$$\text{in } \pi_x(X^i_{a,b}), X^{i'}_{a,b} \text{ is replaced by } X^*,$$
$$\text{in } \pi_x(X^{i'}_{a,b}), X^{i}_{a,b} \text{ is replaced by } X^*,$$
$$\text{in } \pi_x(X^j_{c,d}), X^{j'}_{c,d} \text{ is replaced by } X^*,$$
$$\text{in } \pi_x(X^{j'}_{c,d}), X^{j}_{c,d} \text{ is replaced by } X^*,$$

and then let $\pi_x(X^*) = (X^i_{a,b}, X^{j'}_{c,d}, X^{i'}_{a,b}, X^j_{c,d})$.

The regions of $\mathcal{E}^*$ will be the regions of $\mathcal{E}$ except that those containing $[X^i_{a,b}, X^{i'}_{a,b}]$, $[X^{i'}_{a,b}, X^i_{a,b}]$, $[X^j_{c,d}, X^{j'}_{c,d}]$, and $[X^{j'}_{c,d}, X^j_{c,d}]$ on their boundaries will be replaced by new region(s) according to several cases. Since $\mathcal{E}^*$ is the embedding corresponding to an alternating neighborhood set of a cross graph of G, we can lift the cross vertices giving a good drawing of G.

There are twenty-four ways in which these two segments and their mirrors can occur on one, two, three, or four regions in $\mathcal{E}$. These twenty-four cases can be reduced to the following eight. Each of these cases in turn is proven in a manner similar to Case 1; thus, the proofs are omitted. (All details are available on request.)

Case 1. If the segments and their mirrors occur on the same region, the segments are consecutive on the boundary of that region, and no segment and its mirror occur consecutively on the region, then $n = h$.

Let R be the region containing the segments and their mirrors in $\mathcal{E}$. A schematic of the region R is given in Figure 1. Schematics are used to represent the cyclic permutations of the neighbors of a vertex; therefore, geometric regions in a schematic are meaningless. A region in a schematic and the boundary of that region which is defined by the embedding are considered to be the same. When using a schematic, we always proceed counterclockwise from an edge entering the vertex to the next edge to exit it.

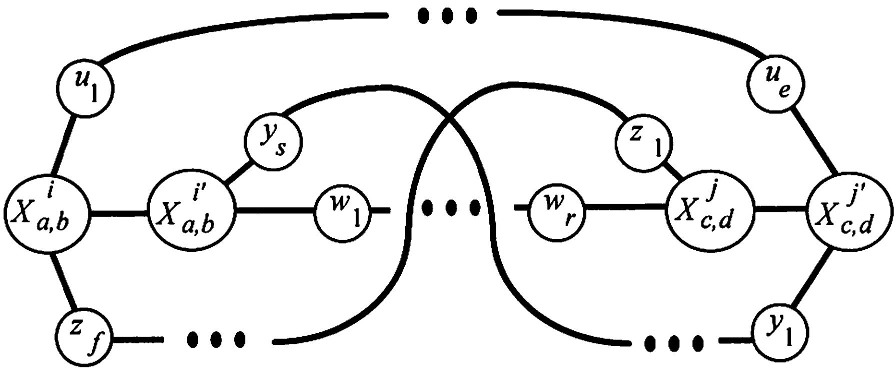

Figure 1.
A Schematic of the Region R Containing the Segments and their Mirrors from Case 1

Without loss of generality, we can represent R as:

$$R\colon X^{i}_{a,b} - X^{i'}_{a,b} - w_1 - \cdots - w_r - X^{j}_{c,d} - X^{j'}_{c,d} - y_1 - \cdots - y_s - X^{i'}_{a,b} - X^{i}_{a,b} - u_1 - \cdots - u_e - X^{j'}_{c,d} - X^{j}_{c,d} - z_1 - \cdots - z_f - X^{i}_{a,b}.$$

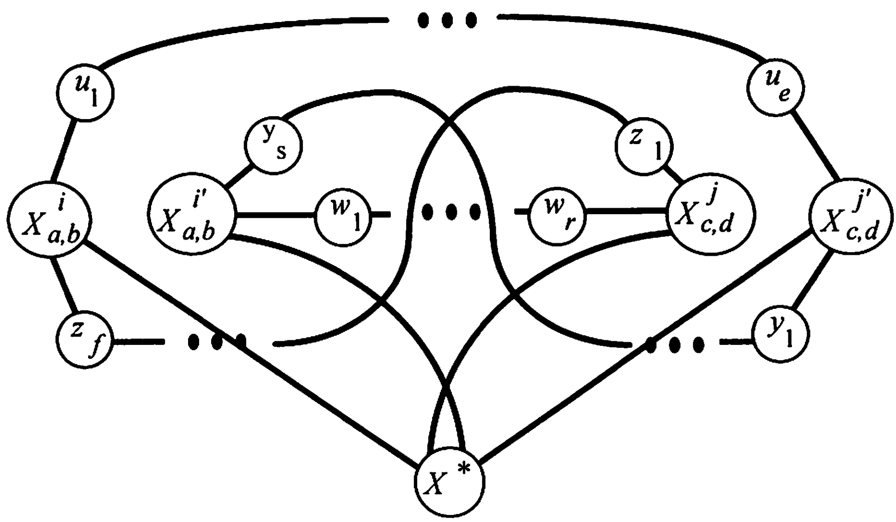

Figure 2.
A Schematic of the Replacement of the Region R in Case 1

As illustrated in Figure 2, the regions of $\mathcal{E}^*$ will be the regions of $\mathcal{E}$ except that R will be replaced by the two regions, R_1 and R_2, represented as:

$$R_1\colon X^i_{a,b} - X^* - X^{j'}_{c,d} - y_1 - \cdots - y_s - X^{i'}_{a,b} - X^* - X^j_{c,d} - z_1 - \ldots$$
$$- z_f - X^i_{a,b}$$

and

$$R_2\colon X^j_{c,d} - X^* - X^i_{a,b} - u_1 - \cdots - u_e - X^{j'}_{c,d} - X^* - X^{i'}_{a,b} - w_1 - \ldots$$
$$- w_r - X^j_{c,d}.$$

By Euler's formula, we see that $x - N_\pi(G^*_x)$ corresponds to a 2-cell embedding of G^*_x on S_h.

The remaining seven cases are given below and describe exactly the surface upon which the drawing is created.

Case 2. If the segments and their mirrors occur on the same region and each segment and its mirror are consecutive on the boundary of that region, then $n = h$.

Case 3. If the segments and one of their mirrors occur on the same region, then $n = h$.

Case 4. If the segments occur on one region and the mirrors occur on another (same) region, then $n = h$.

Case 5. If the segments occur on one region and the mirrors occur on two different regions, then $n = h + 1$.

Case 6. If each segment and its mirror occur on the same region and these two regions are different, then $n = h + 1$.

Case 7. If one segment and its mirror occur on the same region and the other segment and its mirror occur on two different regions, then $n = h + 1$.

Case 8. If the segments and their mirrors occur on four distinct regions, then $n = h + 2$. □

4 The Cross Deletion Theorem

The method used to add a crossing to an existing drawing can be reversed to delete a crossing from a drawing. However, since the crossing is added using exactly one of the two possible x-nbhds for its associated cross vertex, not all cases in the Cross Deletion Theorem are reversals of cases in the Cross Addition Theorem. All case reversals will be given along with one non-reversal case.

Theorem 4.1. *Given a connected graph $G = (V, E)$ and a good drawing D of G on S_h with k crossings, $k > 0$, there exists a good drawing of G with $k - 1$ crossings on S_n where $n = h - 2$, $h - 1$, h, or $h + 1$. (It follows from the theory of rotation schemes that the first two cases do not occur* *unless $h \geq 2$, or $h \geq 1$, respectively.) Furthermore, the value of n can be explicitly determined according to cases as given in the proof.*

Proof: Let G_x be the cross graph associated with the drawing D and $x - N_\pi(G_x)$ be the alternating neighborhood set of G_x corresponding to D, giving an embedding $\mathcal{E}$. Since there is a crossing in D, then there is a cross vertex, say X^*, in $V(G_x)$. Using our earlier notation for segments, suppose X^* is the cross vertex associated with the edge pair (a, b) and (c, d) and the cyclic permutation of the neighbors of X^* in $x - N_\pi(G_x)$ is given by $\pi_x(X^*) = (X^i_{a,b}, X^{j'}_{c,d}, X^{i'}_{a,b}, X^j_{c,d})$. Then, locally, X^* can be thought of as the cross vertex associated with the crossing of the segments $(X^i_{a,b}, X^{i'}_{a,b})$ and $(X^j_{c,d}, X^{j'}_{c,d})$. Construct a new graph G^*_x by $V(G^*_x) = V(G_x) \setminus \{X^*\}$ and $E(G^*_x) = E(G_x) \setminus \{(X^i_{a,b}, X^*), (X^{i'}_{a,b}, X^*), (X^j_{c,d}, X^*), (X^{j'}_{c,d}, X^*)\} \cup \{(X^i_{a,b}, X^{i'}_{a,b}), (X^j_{c,d}, X^{j'}_{c,d})\}$. Construct an alternating neighborhood set of G^*_x, $x - N_\pi(G^*_x)$, with embedding $\mathcal{E}^*$, by leaving all cyclic permutations of the vertices in $x - N_\pi(G_x)$ the same except $\pi_x(X^*)$, $\pi_x(X^i_{a,b})$, $\pi_x(X^{i'}_{a,b})$, $\pi_x(X^j_{c,d})$, and $\pi_x(X^{j'}_{c,d})$ in $\mathcal{E}$ are changed by the following:

in $\pi_x(X^i_{a,b}), X^*$ is replaced by $X^{i'}_{a,b}$,
in $\pi_x(X^{i'}_{a,b}), X^*$ is replaced by $X^i_{a,b}$,
in $\pi_x(X^j_{c,d}), X^*$ is replaced by $X^{j'}_{c,d}$,
in $\pi_x(X^{j'}_{c,d}), X^*$ is replaced by $X^j_{c,d}$,
and delete $\pi_x(X^*)$.

The regions of $\mathcal{E}^*$ will be the regions of $\mathcal{E}$ except that those regions which contain the four segments and their four mirrors with X^* as one endpoint will be replaced by new region(s) according to the cases which follow in the proof.

For brevity of notation, let the path $X^j_{c,d}$, X^*, $X^i_{a,b}$ be called the *segment path*, path $X^{j'}_{c,d}$, X^*, $X^{i'}_{a,b}$ be called the *mirror path*, path $X^i_{a,b}$, X^*, $X^{j'}_{c,d}$ be called the *segment-mirror path* and, lastly, the path $X^{i'}_{a,b}$, X^*, $X^j_{c,d}$ be called the *mirror-segment path*. The paths are illustrated in Figure 3. Two of these four paths are said to be *consecutive* on a region if neither of the two remaining paths occurs between them on the boundary of that region, when following the rotation scheme to trace the boundary.

There are twenty-four possibilities for which these four paths can occur on one, two, three, or four regions in $\mathcal{E}$ which reduce to nine cases. In the first we'll illustrate a case which reduces to one from the Cross Addition Theorem.

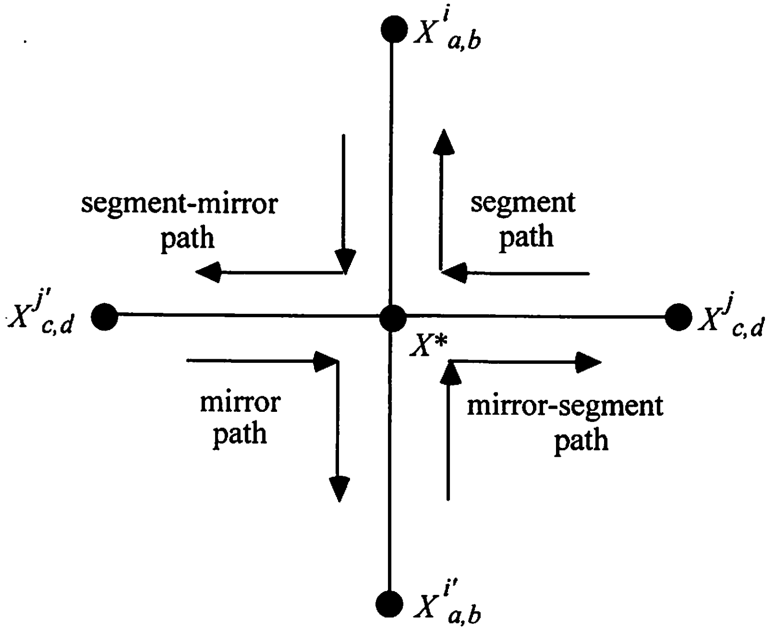

Figure 3.
The Paths Surrounding the Cross Vertex X^*

Case 1. If X^* is on the boundary of one region, the segment path and the mirror path are not consecutive on the region and the segment-mirror path and the mirror path are consecutive on the boundary of that region, then $n = h - 1$.

Let R be the region containing X^* in E. Without loss of generality, we can represent R as:

$$R\colon X^{i}_{a,b} - X^* - X^{j'}_{c,d} - y_1 - \cdots - y_s - X^{j'}_{c,d} - X^* - X^{i'}_{a,b} - w_1 - \cdots - w_r - X^{i'}_{a,b} - X^* - X^{j}_{c,d} - z_1 - \cdots - z_f - X^{j}_{c,d} - X^* - X^{i}_{a,b} - u_1 - \cdots - u_e - X^{i}_{a,b}.$$

In this case, the alternating neighborhood set which we have created is the resulting alternating neighborhood set in Case 6 of the Cross Addition Theorem (CAT), Theorem 3.1. Thus, we have that the regions of $\mathcal{E}^*$ will be the regions of $\mathcal{E}$ except that R will be replaced by the two regions R^i and R^{ii} represented as:

$$R^i\colon X^{i}_{a,b} - X^{i'}_{a,b} - w_1 - \cdots - w_r - X^{i'}_{a,b} - X^{i}_{a,b} - u_1 - \cdots - u_e - X^{i}_{a,b}$$

and

$$R^{ii}\colon X^{j}_{c,d} - X^{j'}_{c,d} - y_1 - \cdots - y_s - X^{j'}_{c,d} - X^{j}_{c,d} - z_1 - \cdots - z_f - X^{j}_{c,d}.$$

By Euler's formula, we see that $\mathcal{E}^*$ is an embedding on S_{h-1}.

Case 2. If X^* is on the boundary of one region and the segment path and the mirror path are consecutive on that region, then $n = h - 1$.

(This case is not a "reversal" of a cross addition case because the region described is not obtained as a result of any case there.) Let R be the region containing X^* in E. Without loss of generality, we can represent R as:

$$\begin{aligned} R\colon & X^{i}_{a,b} - X^* - X^{j'}_{c,d} - y_1 - \cdots - y_s - X^{i'}_{a,b} - X^* - X^{j}_{c,d} - z_1 - \ldots \\ & - z_f - X^{j'}_{c,d} - X^* - X^{i'}_{a,b} - w_1 - \cdots - w_r - X^{j}_{c,d} - X^* \\ & - X^{i}_{a,b} - u_1 - \cdots - u_e - X^{i}_{a,b}. \end{aligned}$$

The regions of $\mathcal{E}^*$ will be the regions of $\mathcal{E}$ except that R will be replaced by the two regions R^i and R^{ii} represented as:

$$\begin{aligned} R^{i}\colon & X^{i}_{a,b} - X^{i'}_{a,b} - w_1 - \cdots - w_r - X^{j}_{c,d} - X^{j'}_{c,d} - y_1 - \ldots \\ & - y_s - X^{i'}_{a,b} - X^{i}_{a,b} - u_1 - \cdots - u_e - X^{i}_{a,b} \end{aligned}$$

and

$$R^{ii}\colon X^{j'}_{c,d} - X^{j}_{c,d} - z_1 - \cdots - z_f - X^{j}_{c,d}.$$

By Euler's formula, we see that $\mathcal{E}^*$ is an embedding on S_{h-1}.

The proofs to Cases 3-8 are all similar to Case 1.

Case 3. If X^* is on the boundary of one region, the segment path and the mirror path are not consecutive on the region and the segment-mirror path and the segment path are consecutive on the boundary of that region, then $n = h - 2$.

Case 4. Suppose X^* is on the boundary of two regions and one region contains three of the four paths. If on that region the paths which occur are in the order: the segment path, the segment-mirror path, the mirror path, the mirror-segment path, then $n = h$.

Case 5. Suppose X^* is on the boundary of two regions. If one region contains three of the four paths and on that region the paths which occur are not in the order: the segment path, the segment-mirror path, the mirror path, the mirror-segment path; or, if each of the two regions contains two paths and the segment path and the mirror path do not occur on the same region, then $n = h - 1$.

Case 6. If X^* is on the boundary of two regions, each region contains two paths, and the segment path and the mirror path occur on the same region, then $n = h$.

Case 7. If X^* is on the boundary of three regions and either the segment path or the mirror path (but not both) occur on the region containing two paths, then $n = h$.

Case 8. If X^* is on the boundary of three regions and either the segment path and the mirror path occur on the same region or the segment-mirror path and the mirror-segment path occur on the same region, then $n = h$.

Case 9. If X^* is on the boundary of four regions, then $n = h + 1$.

This case is not a reversal of a case of the cross addition theorem and proceeds like Case 2. □

5 Upper Bounds on the Thrackle Genus

The theoretical use of these results can be illustrated by applying them to the well studied "thrackle" conjecture. Conway, quoted by Woodall, defined a *thrackle* of a graph to be a good drawing D on the plane in which all pairs of nonadjacent edges cross, i.e. D contains no parallel edges [7]. If a thrackle of G exists, then G is said to be *thrackleable*. But not all graphs are thrackleable on the plane. (The "thrackle conjecture" asserts that only a graph whose number of edges does not exceed its number of vertices can be so drawn [7].) However, in [5] we show that every graph can be thrackled on some orientable surface. Hence, one could define the *thrackle genus* of G, $\gamma_T(G)$, to be the minimum among all genera of surfaces on which G is thrackleable.

The *crossing number* of a graph G, $\nu(G)$, and the *maximum crossing number*, $\nu_M(G)$, are the minimum and maximum number of crossings among all good drawings of G on the plane, respectively. The upper bound $\Theta(G)$ on the maximum crossing number, introduced in Section 2, is called the *thrackle bound* [6].

The Cross Addition Theorem can be applied to build a thrackle drawing of a graph on some orientable surface. This leads to upper bounds on the thrackle genus.

Theorem 5.1. *Given a good drawing D of a graph G with c crossings on S_h, $0 \le c \le \Theta(G)$, $\gamma_T(G) \le h + 2[\Theta(G) - c]$.*

Proof: If there are no parallel edges in D, then $c = \Theta(G)$ and $\gamma_T(G) \le h$; otherwise, $c < \Theta(G)$ and, as long as there are parallel edges in a given drawing of G, the Cross Addition Theorem [CAT] can be invoked to create a new drawing of G with one more crossing. We can continue using the CAT until we have created a thrackle of G on a surface with at most $h + 2[\Theta(G) - c]$ handles, since we add at most two handles to our surface with each use. □

Since we know that all cycles, except the cycle on four vertices, are thrackleable on the plane [8], we know that this bound is tight given any thrackle drawing of a cycle on the plane. The same is true for any plane thrackle drawing of a graph; however, determining the plane thrackleability of a graph is an open problem. Since the thrackle genus is known for so few graphs, no sophisticated examples can be given of the tightness of the bound in the theorem. However, the theorem does lead us to more bounds on the thrackle genus.

Corollary 5.2. *Given a graph G, $\gamma_T(G) \leq \gamma(G) + 2[\Theta(G)]$ where $\gamma(G)$ is the genus of G.*

Corollary 5.3. *Given a planar graph G, $\gamma_T(G) \leq 2[\Theta(G)]$.*

Corollary 5.4. *If there exists a good drawing of graph G on a surface S_h, $\gamma_T(G) \leq h+2[\Theta(G)-v_h(G)]$ where $v_h(G)$ is the minimum crossing number of G on S_h. (This minimum means the minimum number of crossings among all good drawings on the surface.)*

Proof: Since there exists a good drawing of G on S_h, there exists a minimum drawing of G on S_h, i.e. a drawing of G on S_h with $v_h(G)$ crossings. Thus, the result follows directly from Theorem 5.1. □

Corollary 5.5. *Given a graph G, $\gamma_T(G) \leq 2[\Theta(G) - \nu_M(G)]$.*

Another upper bound for the maximum crossing number is given in [6] called the *subthrackle bound*, $\Theta'(G)$, where $\Theta'(G) = \Theta(G) - N + M$ and N and M are the number of nonidentical C_4's and K_4's in G, respectively. A graph is called *subthrackleable* if it has a good drawing on the plane with $\Theta'(G)$ crossings.

Corollary 5.6. *Given a subthrackleable graph G, $\gamma_T(G) \leq 2(N - M)$ where N and M are the number of nonidentical C_4's and K_4's in G, respectively.*

Proof: By Corollary 5.5, $\gamma_T(G) \leq 2[\Theta(G) - \Theta'(G)] = 2\{\Theta(G) - [\Theta(G) - N + M]\} = 2(N - M)$. □

There is so little known about thrackles that the bounds given in Corollaries 5.2 and 5.3 are not known to be tight. However, it is known that the bound given in Corollary 5.4 is tight for the toroidal thrackle of C_4, as it would be for any thrackle drawing of a graph G on a surface of genus $\gamma_T(G)$. The bound is tight in Corollary 5.5 given a plane thrackleable graph. Once again, it is unknown as to whether the bound in Corollary 5.6 is tight.

Final Remarks. The Cross Addition Theorem and the Cross Deletion Theorem provide methods for moving from one good drawing of a graph to another. We call this "moving" between drawings *derivation*. Work on these derivations is underway and could have implications for finding both minimum and maximum drawings.

Software has been developed which builds a thrackle of a graph from a good drawing and which derives one good drawing from another, using the methods of this paper [9]. Since they depend on combinatorial representations, the algorithms are not polynomial.

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