On Determining the 2-Packing and Domination Numbers of the Cartesian Product of Certain Graphs

B.L. Hartnell
Saint Mary's University
Halifax, Nova Scotia
Canada B3H 3C3

ABSTRACT. In this paper we investigate the nature of both the 2-packing number and the minimum domination number of the cartesian product of graphs where at least one of them has the property that every vertex is either a leaf or has at least one leaf as a neighbour.

Introduction

Let $G = (V, E)$ be a finite, undirected, simple graph. If A and B are subsets of V we say A *dominates* B if each vertex $x \in B - A$ is adjacent to at least one vertex $y \in A$. In particular, if $A \subseteq V$ dominates V we call A a *dominating set* for the graph G. The minimum cardinality of any such dominating set for G is the *domination number*, $\gamma(G)$, of G, and such a set will be called a minimum dominating set of G.

If $A \subseteq V$ has the property that whenever x and y are distinct vertices in A they have no neighbours in common, then A is called a *2-packing* in G. $P_2(G)$, the 2-packing number of G, is the maximum cardinality of a 2-packing in G. Since any dominating set of G must have a nonempty intersection with every closed neighbourhood in G, it follows directly that $\gamma(G) \geq P_2(G)$.

If G and H are graphs, the *Cartesian product* of G and H, denoted $G \times H$, is the graph with vertex set $V(G \times H) = \{(x, y) \mid x \in V(G) \text{ and } y \in V(H)\}$ and edge set $E(G \times H) = \{(x_1, y_1)(x_2, y_2) \mid (x_1 = x_2 \text{ and } y_1y_2 \in E(H)) \text{ or } (x_1x_2 \in E(G) \text{ and } y_1 = y_2)\}$. For ease of reference we let H_u denote the subgraph $\{u\} \times H$ of $G \times H$ and similarly G_v will denote $G \times \{v\}$.

This investigation was prompted by attempts to settle a conjecture due to Vizing ([5]) in 1963 that $\gamma(G \times H)$ is always at least as large as $\gamma(G)\cdot$ $\gamma(H)$. One of the earliest observations was that if one of the graphs, say

G, was such that it admitted a 2-packing of size $\gamma(G)$, then $\gamma(G \times H)$, for any H, must be at least as large as $\gamma(G)$. $\gamma(H)$. For instance, if G were a graph in which every vertex was either a leaf or had a single leaf attached, then $P_2(G) = \gamma(G) = |V(G)|/2$ and $\gamma(G \times H) \geq \gamma(G)\gamma(H)$ for any H. One of the recurring (and annoying!) observations has been that very often $\gamma(G \times H)$ will, in fact, be considerably larger than the conjectured bound. In this paper we investigate the nature of both the 2-packing number and the domination number of the product of graphs where at least one of them has the property that every vertex is either a leaf or has at least one leaf attached. In particular, one of the results obtained shows that if both G and H are such that every vertex is either a leaf or has exactly k leaves attached, then $P_2(G \times H) = \gamma(G \times H) = k\gamma(G)\gamma(H)$.

An immediate corollary, first obtained by Fink et al ([1]), is that when $k = 1$, equality is achieved for Vizing's conjectured bound. So, in some sense, this result is a generalization of one of their results (Theorem 4) in [1]. This paper may also be considered as a partial response to the problem (raised by Jacobson and Kinch in [3]) of determining the domination number of $G \times H$ for other classes of graphs as well as the question recently posed by Fisher ([2]) of determining the 2-packing number of $G \times H$ for various families of graphs.

Main Results

To simplify the statement of the first theorem we define a $[k, n]$-packable graph H, with $\gamma(H) = n \geq 2$ as follows. H has $kn + n$ vertices and a minimum dominating set, say $S^* = \{s_1^*, s_2^*, \ldots, s_n^*\}$, where the graph induced by S^* has no isolated vertices. For each $i \in \{1, 2, \ldots, n\}$, s_i^* has exactly k neighbours not belonging to S^* where these neighbours are $\{h[i,1], h[i,2], \ldots, h[i,k]\}$. Furthermore, $\{h[1,j], h[2,j], \ldots, h[n,j]\}$ yields a 2-packing of size n for each choice of j from $\{1, 2, \ldots, k\}$. That is, we have k disjoint sets of size n, each of which is a 2-packing. For example, the graph H illustrated in Figure 1 is a $[2, 4]$-packable graph. Note that the vertices labelled "a" yield a 2-packing of size 4 as do the vertices labelled "b".

Theorem 1. *Let G and H, where $\gamma(G) = m \geq 2$ and $\gamma(H) = n \geq 2$, each be connected graphs such that in G every vertex is either a leaf or has precisely k_1 leaves attached and H is a $[k_2, n]$-packable graph where $k_1 \leq k_2$.*

Then $k_1\gamma(G)\gamma(H) \leq P_2(G \times H)$ and $\gamma(G \times H) \leq k_2\gamma(G)\gamma(H)$.

Proof: Let G and H be connected graphs as described in the statement of the theorem. Let G have $m \geq 2$ vertices, say $s_1, s_2, \ldots, s_m$, each with exactly $k_1 \geq 1$ leaves attached, where the leaves of s_i, say, are labelled $g[i,1], g[i,2], \ldots, g[i,k_1]$.

Let H have $n \geq 2$ vertices, say $s_1^*, s_2^*, \ldots, s_n^*$, forming the set S^*, each with exactly k_2 neighbours outside S^*, where these neighbours of s_i^*, say, are labelled $h[i,1], h[i,2], \ldots, h[i,k_2]$.

First we observe that the 2-packing number of $G \times H$ is at least k_1mn. This can be achieved as follows. In $H_{g[i,1]}$ (the copy of H in the first leaf of stem s_i of G), we pack at the first neighbour of each stem of H (this yields n vertices). In $H_{g[i,2]}$ (the copy of H in the second leaf of stem s_i of G), we pack at the second neighbour of each stem of H. In general, in $H_{g[i,j]}$ (the copy of H in the jth leaf of stem s_i of G), we pack at the jth neighbour of each stem of H. Thus the leaves of s_i yield a total of k_1n vertices at which we have packed. Repeating over all m stems of G, we obtain k_1mn. More specifically, for any vertex in G in the set $\{s_1, s_2, \ldots, s_m\}$, say s_i, consider the k_1n vertices of $G \times H$ in $\{(g[i,r], h[t,r]) \mid r$ takes on each value 1 to k_1 and, for each value of r, t ranges from 1 to $n\}$. These k_1n vertices admit a 2-packing and, furthermore, for each i ranging from 1 to m, all such k_1mn vertices are neighbourhood disjoint.

Next we show that $\gamma(G \times H)$ is at most k_2mn.

Partition $S^* = \{s_1^*, s_2^*, \ldots, s_n^*\}$ into 2 subsets R and B, say, such that R dominates B and B dominates R. That such a partition is always possible is easily seen. For instance, one could choose R as a minimal dominating set and B as $S^* - R$. Since R is minimal, each member is adjacent to a member of B so B is also a dominating set.

Similarly, partition $S = \{s_1, s_2, \ldots, s_m\}$ into 2 subsets, say S_1 and S_2, such that each of S_1 and S_2 dominate S. Now we form a dominating set, say D, of $G \times H$.

For each $s_i \in S_1$, place into D the k_2 $|R|$ vertices of the set $\{(s_i, h[j,r]) \mid$ where j takes on each of the values that $s_j^* \in R$ and, for each such value j, r ranges from 1 to $k_2\}$.

For each such i, for each value of r, as r runs from 1 to k_1, in the corresponding $H_{g[i,r]}$,we place the $|B|$ vertices of the set $\{(g[i,r], s_j^*) \mid s_j^* \in B\}$ into D. Observe that, for any r, $H_{g[i,r]}$ is dominated since the vertices of $R \cup B = S^*$ are dominated by B, the vertices not belonging to S^* but adjacent to vertices in B are dominated by B and the vertices not in S^* but adjacent to vertices of R are dominated by members of D in H_{s_i}. This yields $|S_1|K_2|R| + |S_1|K_1|B|$ vertices in D so far.

For each $s_i \in S_2$, place the k_2 $|B|$ vertices of the set $\{(s_i, h[j,r]) \mid$ where j takes on each of the values that $s_j^* \in B$ and, for each such value j, r ranges from 1 to $k_2\}$ into D.

For each such i, for each value of r, as r runs from 1 to k_1, in the corresponding $H_{g[i,r]}$, we place the $|R|$ vertices of the set $\{(g[i,r], s_j^*) \mid s_j^* \in R\}$ into D. Note that, for any r, $H_{g[i,r]}$ is dominated since the vertices of $R \cup B = S^*$ are dominated by R, the vertices not in S^* but adjacent to vertices in R are dominated by R and the vertices not belonging to S^* but adjacent to vertices of B are dominated by members of D in H_{s_i}. This gives another $|S_2|K_2|B| + |S_2|K_1|R|$ vertices in D.

Finally, since each element of S_1 is adjacent to an element of S_2 and each member of S_2 is adjacent to an element of S_1, we note that H_{s_i} is also dominated.

Hence $|D| = |S_1|K_2|R| + |S_1|K_1|B| + |S_2|K_2|B| + |S_2|K_1|R|$.

But $K_1 \leq K_2$ so $|D| \leq |S_1|K_2|R| + |S_1|K_2|B| + |S_2|K_2|B| + |S_2|K_2|R| = |S_1|K_2 n + |S_2|K_2 n = K_2 mn$.

Therefore $\gamma(G \times H) \leq k_2 mn$.

Thus $k_1 mn \leq P_2(G \times H) \leq \gamma(G \times H) \leq k_2 mn$, and the theorem is established. □

As an immediate consequence of Theorem 1, setting $k_1 = k_2$, we obtain the following corollary.

Corollary 1. *Let G and H, where $\gamma(G) = m \geq 2$ and $\gamma(H) = n \geq 2$, each be connected graphs such that in G every vertex is either a leaf or has precisely k leaves attached and H is a $[k, n]$-packable graph.*

Then $P_2(G \times H) = \gamma(G \times H) = k\gamma(G)\gamma(H)$.

In the case the k neighbours of each of the n vertices of a dominating set of H are leaves, then we obtain the next result.

Corollary 2. *Let G and H, where $\gamma(G) \geq 2$ and $\gamma(H) \geq 2$, each be connected graphs such that every vertex is either a leaf or has exactly k leaves attached. Then $P_2(G \times H) = \gamma(G \times H) = k\gamma(G)\gamma(H)$.*

As an illustration of Corollary 1, let G and H be as in Figure 1. Note that $\gamma(G) = m = 3$, $\gamma(H) = n = 4$ and $k = 2$.

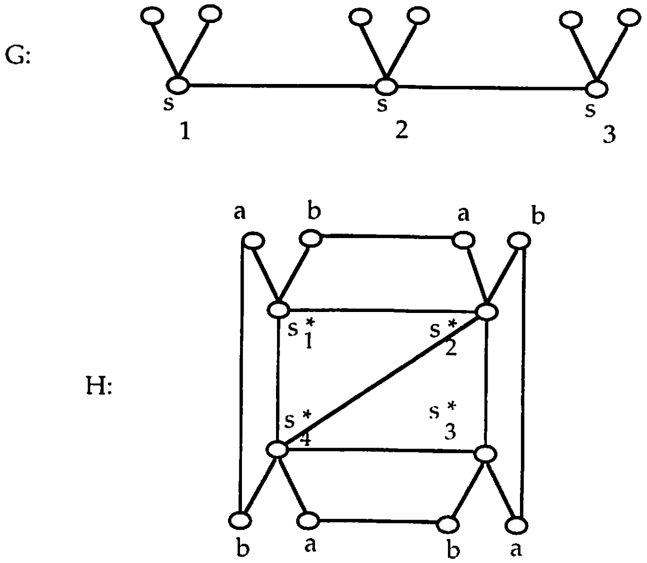

Figure 1

The 2-packing of $G \times H$ would be the set $\{(g[1,1],h[1,1]),(g[1,1],h[2,1]),$ $(g[1,1],h[3,1]),(g[1,1],h[4,1]),(g[1,2],h[1,2]),(g[1,2],h[2,2]),(g[1,2],h[3,2]),$ $(g[1,2],h[4,2]),(g[2,1],h[1,1]),(g[2,1],h[2,1]),(g[2,1],h[3,1]),(g[2,1],h[4,1]),$ $(g[2,2],h[1,2]),(g[2,2],h[2,2]),(g[2,2],h[3,2]),(g[2,2],h[4,2]),(g[3,1],h[1,1]),$ $(g[3,1],h[2,1]),(g[3,1],h[3,1]),(g[3,1],h[4,1]),(g[3,2],h[1,2]),(g[3,2],h[2,2]),$ $(g[3,2],h[3,2]),(g[3,2],h[4,2])\}$.

To dominate $G \times H$, first let $R = \{s_2^*\}$ and $B = \{s_1^*, s_3^*, s_4^*\}$. Also let $S_1 = \{s_1, s_3\}$ & $S_2 = \{s_2\}$.

Now form the dominating set given by $\{(s_1,h[2,1]),(s_1,h[2,2]),(s_3,h[2,1]),$ $(s_3,h[2,2,]),(g[1,1],s_1^*),(g[1,1],s_3^*),(g[1,1],s_4^*),(g[1,2],s_1^*),(g[1,2],s_3^*),$ $(g[1,2],s_4^*),(g[3,1],s_1^*),(g[3,1],s_3^*),(g[3,1],s_4^*),(g[3,2],s_1^*),(g[3,2],s_3^*),$ $(g[3,2],s_4^*),(s_2,h[1,1]),(s_2,h[1,2]),(s_2,h[3,1]),(s_2,h[3,2]),(s_2,h[4,1]),$ $(s_2,h[4,2]),(g[2,1],s_2^*),(g[2,2],s_2^*)\}$.

One possible direction in which we can attempt to generalize Theorem 1 and its corollaries is to allow a different number of leaves at the various stems in G as well as in H. In the special case that every stem of H has substantially more leaves than any stem of G we can obtain the following.

Theorem 2. *Let G and H be connected graphs with $\gamma(G) = m \geq 2$ and $\gamma(H) = n \geq 2$ where every vertex is either a leaf or has at least one leaf attached. Let k_{MAX} be the largest number of leaves attached to any vertex in G and let k^*_{MIN} be the smallest number of leaves attached to any vertex* *in H. Furthermore say $m + k_{MAX} \leq k^*_{MIN}$. Then $P_2(G \times H) = \gamma(G \times H) = \gamma(H)$. $|V(G)| = n(m + |L(G)|)$ where $|L(G)|$ represents the number of leaves of G.*

Proof: Let G and H be connected graphs as described in the hypothesis of the theorem. For each stem, say s_i, of G, in the copy of H in the first leaf at that stem, we pack at the first leaf of each stem of H (this yields n vertices). In the copy of H in the second leaf at that stem s_i, we pack at the second leaf of each of the stems of H (this gives another n vertices). Repeating this at each leaf of the stem s_i of G, we obtain a total of n times the number of leaves at that stem. Then repeating this process at all stems of G, we obtain $n|L(G)|$ vertices, where $|L(G)|$ represents the total number of leaves in G.

Observe that in $G \times H$, the copies of H at the m stems of G, have at most the first k_{MAX} leaves of each stem of H with neighbours in the 2-packing set so far. Hence, since $K^*_{\text{MIN}} - K_{\text{MAX}} \geq m$, we can enlarge our 2-packing by packing at the $(K_{\text{MAX}} + 1)$st leaf of each stem of H in stem s_1 of G (giving another n vertices). In the copy of H at stem s_2 of G, we pack at the $(K_{\text{MAX}} + 2)$nd leaf of each stem of H.

Continuing this process, we eventually pack at the $(K_{\text{MAX}} + m)$th leaf of each stem of H in the copy of H in stem s_m of G. Thus we have a total of $n|L(G)| + mn$ vertices in our 2-packing and $P_2(G \times H) \geq n|V(G)|$.

To conclude, we note that we can certainly dominate $G \times H$ by dominating each copy of H separately. Thus $\gamma(G \times H) \leq \gamma(H)$. $|V(G)| = n|V(G)|$.

Hence, the theorem is established. □

We illustrate Theorem 2 with the graphs in Figure 2. Here $\gamma(G) = m = 4$, $\gamma(H) = n = 5$, $K_{\text{MAX}} = 2$, and $K^*_{\text{MIN}} = 6$. It follows that $\gamma(G \times H) = P_2(G \times H) = 5(10) = 50$.

We also observe that Theorem 2 gives collections of graphs which actually achieve the maximum possible value for the 2-packing number and the domination number since

$$P_2(G \times H) \leq \gamma(G \times H) \leq \min\{\gamma(G)|V(H)|, \gamma(H)|V(G)|\}.$$

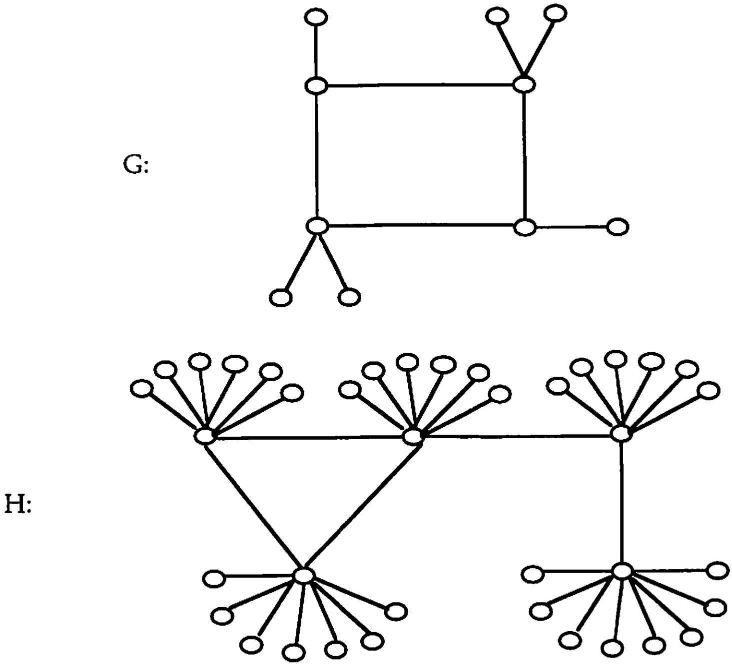

Figure 2

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