Double Domination in Graphs

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Abstract. Each vertex of a graph $G = (V, E)$ is said to dominate every vertex in its closed neighborhood. A set $S \subseteq V$ is a double dominating set for G if each vertex in V is dominated by at least two vertices in S. The smallest cardinality of a double dominating set is called the double domination number $dd(G)$. We initiate the study of double domination in graphs and present bounds and some exact values for $dd(G)$. Also, relationships between $dd(G)$ and other domination parameters are explored. Then we extend many results of double dominati. a to multiple domination.

1 Introduction

Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. Each vertex of a graph is said to *dominate* every vertex in its closed neighborhood. A set $S \subseteq V$ is a *dominating set* if each vertex in V is dominated by some vertex of S. The *domination number* $\gamma(G)$ is the minimum cardinality of dominating set. Many domination related parameters have been defined. For a comprehensive work on the subject see [7, 8]. Here we introduce another variant of the domination concept. Set S is a *double dominating set* for G if every vertex in V is dominated by at least two vertices in S. The minimum cardinality of a double dominating set is the *double domination number*, denoted $dd(G)$. We refer to a minimum dominating set as a γ-set and a minimum double dominating set as a dd-set. Obviously, every dd-set is also a dominating set. Note that the concept of double domination can be extended to *multiple domination (h-tuple domination)* by requiring that each vertex in V be dominated at least h times.

Cockayne, Dawes and Hedetniemi [3] defined a similar concept, which they called "total domination", described here as "open domination". A set S is an *open dominating set* if each vertex in V has at least one neighbor

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in S. *A fortiori* every dd-set is an open dominating set. The open (total) domination number $\gamma_t(G)$ is the minimum cardinality of a open dominating set.

For examples, consider the graphs in Figure 1. Set $\{v_1, v_3\}$ is a γ-set and set $\{v_1, v_2, v_3, v_4\}$ is a dd-set and a γ_t-set for the cube Q_3. So the cube has $\gamma(Q_3) = 2$, and $dd(Q_3) = \gamma_t(Q_3) = 4$. For the subdivided star $K^*_{1,k}$, set $S = \{u_1, u_2, ..., u_k\}$ is a γ-set, and $S \cup \{w\}$ is a γ_t-set. However, any dd-set must include each endvertex v_i and its neighbor u_i. Thus $dd(G) = 2k$.

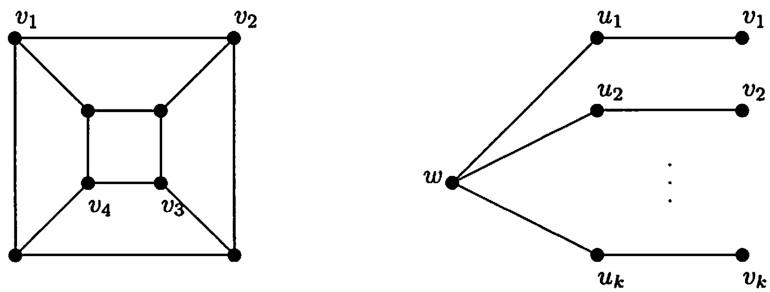

Figure 1: Graphs Q_3 and $K^*_{1,k}$.

Many applications of domination in graphs can be extended to double domination. For example, if we think of each vertex in a dominating set as a fileserver for a computer network, then each computer in the network has direct access to a fileserver. It is sometimes reasonable to assume that this access be available even when one of the fileservers goes down. A double dominating set provides the desired fault tolerance for such cases because each computer has access to at least two fileservers and each of the fileservers has direct access to at least one backup fileserver.

In general, we follow the notation and terminology of [6]. We use $\langle S \rangle$ to denote the subgraph induced by the set of vertices S and $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v, respectively. Let $deg(v)$ be the degree of vertex v and as usual $\delta(G)$ the minimum degree and $\Delta(G)$ the maximum degree.

Results involving $dd(G)$ are presented in Section 2. Then relationships between $dd(G)$ and other domination parameters are explored in Section 3. Finally, the concept of double domination and many of the results are extended to multiple domination in Section 4.

2 Double Domination Number

Obviously, a graph with an isolated vertex cannot have a double dominating set. We ask the natural question regarding the existence of double dominating sets.

Theorem 1 *Every graph with no isolated vertices has a double dominating set and hence a double domination number.*

Proof. Without loss of generality, let $G = (V, E)$ be connected. Then V itself is a double dominating set as each vertex is considered to dominate itself, and since G has no isolates, G is nontrivial so each vertex v is adjacent to some other vertex u. Thus both u and v dominate v. Now that we know G has a double dominating set, we may remove one vertex at a time from V if and only if the remaining subset of V is still a double dominating set. This will give a minimal double dominating set. Among all the minimal double dominating sets, each of the smallest sets has cardinality $dd(G)$. □

Henceforth we consider only graphs with no isolated vertices. Before presenting bounds on $dd(G)$, we give examples of double domination in specific families of graphs. These easily computed values of $dd(G)$ are stated without proof. Here K_n is the complete graph, C_n is the cycle, W_n is the wheel with $n-1$ spokes, P is the Petersen graph, $K_{1,m}$ is the star, and $K_{r,s}$ is the complete bipartite graph.

Examples:

(1) $dd(K_n) = 2$.

(2) $dd(C_n) = \lceil \frac{2n}{3} \rceil$.

(3) $dd(W_n) = 1 + \left\lceil \frac{(n-1)}{3} \right\rceil$.

(4) $dd(P) = 6$.

(5) $dd(K_{1,m}) = m + 1 = n$.

(6) $dd(K_{r,s}) = 4$ for $r \geq 3, s \geq 3$.

Next we state straightforward upper and lower bounds for $dd(G)$.

Theorem 2 *Let G be a graph with no isolated vertices. Then $2 \leq dd(G) \leq n$ and these bounds are sharp.*

Proof. Let G be a graph with no isolated vertices. For a vertex to be dominated twice, there must be at least two vertices in any dd-set. We have already seen that V forms a double dominating set, so any dd-set contains at most n vertices.

These bounds are sharp as can be seen with nontrivial complete graphs and stars achieving the lower and upper bounds, respectively. □

We now specify all graphs G having V as a unique dd-set.

Lemma 3.1 *If vertex v has degree one, then both v and its neighbor must be in every double dominating set of G.* □

Theorem 3 *A graph G has V as its unique dd-set if and only if for each $v \in V$, there is a vertex with degree one in $N[v]$.*

Proof. If there is a vertex of degree one in $N[v]$ for every $v \in V$, then Lemma 3.1 implies that V is the unique dd-set of G. Suppose G has V as its unique dd-set, that there exists $v \in V$ with $deg(v) \geq 2$ and for all $x \in N(v), deg(x) \geq 2$. Then $V - \{v\}$ is a double dominating set for G with order less than $dd(G)$, a contradiction. □

Corollary 3.1 *If there exists $v \in V$ such that for all $x \in N[v]$ $deg(x) \geq 2$, then $dd(G) \leq n - 1$.* □

Considering the lower bound of Theorem 2, we make an observation.

Observation 1 *A graph G has $dd(G) = 2$ if and only if there exist vertices $u, v \in V$ such that $deg(u) = deg(v) = n - 1$.* □

Obviously, $\gamma(G) \leq dd(G)$ for all G without isolated vertices. This lower bound can be improved slightly.

Theorem 4 *For any graph G with no isolated vertices, $\gamma(G) + 1 \leq dd(G)$ and this bound is sharp.*

Proof. Let S be a dd-set of G. Then $N[x] \subseteq N[S - \{x\}]$ for all $x \in S$. Therefore, $S - \{x\}$ dominates G and $\gamma(G) \leq dd(G) - 1$. The complete bipartite graph $K_{2,t}$, $t \geq 2$, achieves this lower bound with $\gamma(K_{2,t}) = 2$ and $dd(K_{2,t}) = 3$. □

Observation 2 *Let G be a graph with no isolated vertices and two vertex disjoint γ-sets. Then*

$$\gamma(G) + 1 \leq dd(G) \leq 2\gamma(G)$$

and these bounds are sharp.

Sharpness for both the lower and upper bounds is realized by nontrivial K_n. In fact, $dd(G) = 2$ implies that $dd(G) = 2\gamma(G)$. To see that we need the condition of vertex disjoint γ-sets in our observation, let G be the star $K_{1,m}$ for $m \geq 2$. Then G has a unique γ-set, $\gamma(G) = 1$ and $dd(G) = n$. Hence the difference in $\gamma(G)$ and $dd(G)$ can be made arbitrarily large. The next result is another lower bound on $dd(G)$.

Theorem 5 *Let G be a graph with no isolated vertices. Then*

$$dd(G) \geq \frac{4n - 2m}{3}$$

and this bound is sharp.

Proof. Let S be a dd-set of G. Then each vertex of $V - S$ is adjacent to at least two vertices in S. Further, each vertex of S must have at least one neighbor in S. Thus the number of edges

$$m \geq 2|V - S| + dd(G)/2 = 2n - 2dd(G) + dd(G)/2.$$

Cycles on $3k$ vertices realize the sharp lower bound. □

The well known result [9],

$$\gamma(G) \geq \left\lceil \frac{n}{\Delta(G) + 1} \right\rceil$$

relates the domination number to the maximum degree. We establish a similar result for $dd(G)$.

Theorem 6 *If G has no isolated vertices, then*

$$dd(G) \geq \frac{2n}{\Delta(G) + 1}$$

and this bound is sharp.

Proof. Let G have no isolated vertices and let S be a dd-set for G. Further, let t denote the number of edges in G having one vertex in S and the other in $V - S$. Since $\Delta(G) \geq deg(v)$ for all $v \in S$ and each vertex in S is adjacent to at least one member of S, we have

$$t \leq (\Delta(G) - 1)|S| = (\Delta(G) - 1)(dd(G)).$$

Also, since each vertex in $V - S$ is adjacent to at least two members of S, we have

$$t \geq 2|V - S| = 2[n - dd(G)].$$

Hence

$$2(n - dd(G)) \leq \Delta(G)dd(G) - dd(G),$$

which reduces to the bound of the theorem. Both the 1-regular graphs mK_2 and the nontrivial K_n achieve the lower bound. □

We now turn our attention to upper bounds on $dd(G)$. The next result relates $dd(G)$ with the maximum number of independent edges $\beta_1(G)$.

Theorem 7 *If G has $\delta(G) \geq 2$, then $dd(G) \leq 2\beta_1(G)$.*

Proof. Let G be a graph with $\delta(G) \geq 2$ and M be a maximum independent set of edges in G. Let S be the vertices in the set of edges of M. Since $V - S$ is an independent set, each $v \in V - S$ must have at least two neighbors in S. Further the vertices of S double dominate themselves, so S is a double dominating set for G. □

Requiring that G has no endvertices allows us to improve the upper bound of Corollary 3.1. Our main and final result for this section gives a best possible upper bound for $dd(G)$ when $\delta(G) \geq 2$. The proof relies on the result from Ore [11] that a graph G with no isolated vertices has $\gamma(G) \leq n/2$.

Theorem 8 *Let G be a graph with $\delta(G) \geq 2$. Then*

$$dd(G) \leq \begin{cases} \lfloor n/2 \rfloor + \gamma(G) & \text{for } n = 3 \text{ and } n = 5 \\ \lfloor n/2 \rfloor + \gamma(G) - 1 & \text{otherwise.} \end{cases}$$

Furthermore, these bounds are sharp.

Proof. Let G be a graph with $\delta(G) \geq 2$ and S be a minimum dominating set having the smallest number of isolates in $\langle S \rangle$ among all γ-sets of G. It is easy to verify the bound of the theorem for $n \leq 6$, so assume $n \geq 7$.

Construct a double dominating set for G in the following way. For each isolate in $\langle S \rangle$, select a neighbor in $V - S$. Let X be a smallest set of these neighbors. Note that $|X| \leq \gamma(G)$ and that $S \cup X$ double dominates itself. Let W be the set of vertices in $V - (S \cup X)$ that are not double dominated by $S \cup X$. Since $\delta(G) \geq 2$, $v \in W$ implies that $|N(v) \cap (V - (S \cup X))| \geq 1$. Let $S' \subseteq V - (S \cup X)$ be a minimum set which dominates the vertices of W. Since $|W| \leq |V - (S \cup X)|$ and no vertex of W is isolated in the induced subgraph $\langle V - (S \cup X) \rangle$, from Ore's theorem we have $|S'| \leq \frac{|V-(S \cup X)|}{2}$. Furthermore,

$S \cup X \cup S'$ is a double dominating set for G. Hence $dd(G) \leq |S| + |X| + |S'|$. We consider the cardinality of X. If $|X| \leq \gamma(G) - 2$, then

$$dd(G) \leq \gamma(G) + \gamma(G) - 2 + \left\lfloor \frac{n - 2\gamma(G) + 2}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + \gamma(G) - 1.$$

Further, if two or more vertices in S are adjacent, then $|X| \leq \gamma(G) - 2$. Thus let S be an independent set and consider the cases $|X| = \gamma(G)$ and $|X| = \gamma(G) - 1$. Let t be the number of vertices in $V - (S \cup X)$ which are double dominated by $S \cup X$. In either case if $t \geq 2$, then

$$dd(G) \leq \gamma(G) + |X| + \left\lfloor \frac{n - \gamma(G) - |X| - t}{2} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor + \gamma(G) - 1.$$

Moreover, if $|X| = \gamma(G) - 1$ and $t = 1$, we again have the bound of the theorem.

Therefore assume that $t \leq 1$ and label the vertices of S as $y_1, y_2, \ldots, y_{\gamma(G)}$, the vertices of X as $x_1, x_2, \ldots, x_{|X|}$, and the vertices of $V - (S \cup X)$ as $u_i, 1 \leq i \leq |V - (S \cup X)|$.

Case 1. Let $|X| = \gamma(G)$. Then $t = 0$ or $t = 1$ and each vertex $y_i \in S$ has exactly one neighbor $x_i \in X$ and at least one neighbor, say $u_i \in V - (S \cup X)$. Then minimality of X implies that $N(u_i) \cap S = 1$. Without loss of generality, if $t = 0$, then x_1 is adjacent to x_2 and u_1 is adjacent to u_2. Then $(S \cup X) - \{x_1\} \cup \{u_1\}$ is a set with the same cardinality as $S \cup X$ that double dominates at least the two vertices x_1 and u_2 in $S \cup X - \{x_1\} \cup \{u_1\}$. Hence the upper bound of the theorem holds in this case.

If $t = 1$ and $n \geq 7$, then it can be shown that there exists at least one pair of adjacent vertices in $\langle V - (S \cup X) \rangle$ and we can repeat the above argument.

Case 2. Let $|X| = \gamma(G) - 1$. Then $t = 0$ and exactly one vertex of X, say x_1 is adjacent to exactly two vertices of S, say y_1 and y_2. If $\gamma(G) \geq 3$, we can use an argument similar to the one above. Hence assume $\gamma(G) = 2$. Then both y_1 and y_2 are adjacent to x_1. If S' can be chosen so that $S \cup S'$ is a double dominating set for G, then

$$dd(G) \leq \gamma(G) + \left\lfloor \frac{n - \gamma(G) - 1}{2} \right\rfloor \leq 2 + \left\lfloor \frac{n - 3}{2} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor + \gamma(G) - 1.$$

If not, then S' cannot dominate both y_1 and y_2. That is, S' can only be chosen from neighbors of y_1, say. We may assume that $|S'| \geq 2$ since the theorem holds if $|S'| = 1$. Observe that any neigbhor (other than x_1) of y_2, say z, is dominated by some neighbor of y_1, say $d \in S'$. Furthermore, d is necessary in S' to dominate at least one other vertex in $V - (S \cup X)$, say z, else d could be replaced by z in S', contradicting that S' does not dominate y_2. Hence

$$|S'| \leq \left\lfloor \frac{n - \gamma(G) - 1 - 1}{2} \right\rfloor.$$

Then $S \cup S' \cup \{x_1\}$ is a double dominating set of G and

$$dd(G) \leq |S \cup S' \cup \{x_1\}|$$

$$= \gamma(G) + \left\lfloor \frac{n - \gamma(G) - 2}{2} \right\rfloor + 1 = \left\lfloor \frac{n-4}{2} \right\rfloor + 3 \leq \left\lfloor \frac{n}{2} \right\rfloor + \gamma(G) - 1.$$

To see that this bound is sharp, consider a graph G on $3+2r$ vertices for $r \geq 2$, say $\{y_1, y_2, x, u_1, u_2, ...u_r, v_1, v_2, ...v_r\}$, with $E(G) = \{y_1x, y_2x, y_1u_i, y_2v_i, u_iv_i \mid 1 \leq i \leq r\}$. The graph G has $\gamma(G) = 2$ and $dd(G) = r + 2 = \lfloor n/2 \rfloor + \gamma(G) - 1$. □

3 Relationships Between Double Domination Number and Other Domination Parameters

As mentioned in the introduction, there are many different types of domination. We define a few selected domination invariants and consider relationships between each of them and $dd(G)$.

A set $E' \subseteq E$ is an *edge-edge dominating set* if each edge in $E - E'$ is adjacent to an edge in E'. Let $\gamma'(G)$ denote the size of a minimum edge dominating set.

Theorem 9 *If $\delta(G) \geq 2$, then $dd(G) \leq 2\gamma'(G)$.*

Proof. Let E' be a minimum edge dominating set for G and let S denote the set of vertices incident with an edge in E'. Obviously, each vertex in S is doubly dominated by S. Since $\delta(G) \geq 2$, any vertex $v \in V - S$ must be incident to at least two edges and these edges must be adjacent to edges in E'. Hence S is a dd-set for G. □

Since any dd-set for G is also an open dominating set for G, we have the next statement.

Observation 3 *In a graph G with no isolated vertices, $\gamma(G) \leq \gamma_t(G) \leq dd(G)$.*

Allan, Laskar and Hedetniemi [1] established the following inequalities for a graph G with no isolates:

$$\gamma(G) \le \gamma_t(G) \le 2\gamma(G).$$

The upper bound is not always good for $dd(G)$ as was seen with the star $K_{1,m}$ for $m \ge 2$. However, we have found the place for $dd(G)$ in this sequence of inequalities for the case when G has two vertex disjoint minimum dominating sets.

Observation 4 *If G has no isolates and two vertex disjoint minimum dominating sets, then*

$$\gamma(G) \le \gamma_t(G) \le dd(G) \le 2\gamma(G).$$

Sampathkumar and Walikar [12] define a *connected dominating set S* to mean $\langle S \rangle$ is connected. The minimum cardinality taken over all connected dominating sets is called the *connected domination number of G* $\gamma_c(G)$. Obviously,

$$\gamma(G) \le \gamma_t(G) \le \gamma_c(G).$$

However, the following examples show that no particular inequality holds between $dd(G)$ and $\gamma_c(G)$.

Examples:

(1) $dd(C_n) = \lceil 2n/3 \rceil < n - 2 = \gamma_c(C_n)$ for $n \ge 9$.

(2) $dd(K_{1,m}) = m + 1 = n > 1 = \gamma_c(K_{1,m})$.

(3) $dd(C_6) = 4 = \gamma_c(C_6)$.

4 Multiple Domination

We now generalize from domination and double domination to multiple domination. First we consider two equivalent definitions of a dominating set. A set $S \subseteq V$ is a *dominating set* if each vertex in $V - S$ is adjacent to a vertex in S. Hence a vertex can be said to dominate itself and all vertices adjacent to it. The definition of dominating set may also be written as follows. A set $S \subseteq V$ is a *dominating set* if the union of the closed neighborhoods of the vertices in S is V. These obviously equivalent definitions lend themselves to different generalizations.

In [4] Fink and Jacobson introduced *multiple domination* as a generalization of the first definition. A set $S \subseteq V$ is an *h-dominating set* if each vertex in $V - S$ is adjacent to at least h vertices in S. The size of a minimum h-dominating set is the *h-domination number*, denoted by $\gamma_h(G)$. In their definition the vertices of S do not need to be multiply dominated.

We generalize the second definition of a dominating set as follows. A set $S \subseteq V$ is an *h-tuple dominating set* for G if each vertex in V is dominated by at least h vertices in S. The *h-tuple domination number*, denoted $\gamma_{\times h}(G)$, is the smallest number of vertices in an h-tuple dominating set. Here each vertex in S does need to be multiply dominated. Trivially, $\gamma(G) = \gamma_1(G) = \gamma_{\times 1}(G)$, $dd(G) = \gamma_{\times 2}(G)$, and $\gamma(G) \leq \gamma_h(G) \leq \gamma_{\times h}(G)$. Furthermore, $\gamma(G) < dd(G) \leq \gamma_{\times h}(G)$ for $h \geq 2$.

Many of the results obtained in Section 2 for double domination are easily extended to multiple domination. Since the proofs are simple generalizations of the corresponding proofs for double domination, they are omitted.

Theorem 10 *Every graph with $\delta(G) \geq h - 1$ has an h-tuple dominating set and hence an h-tuple domination number.* □

Now that we know that graphs with $\delta(G) \geq h - 1$ have multiple dominating sets, we consider the difficulty of finding the h-tuple domination number of G.

Observation 5 *The decision problem of determining whether a graph has an h-tuple dominating set of size t or less is NP-complete.*

Proof. Restrict $h = 1$. The problem is DOMINATING SET [5]. □

Next we give bounds for $\gamma_{\times h}(G)$ which are direct generalizations of the double domination results.

Theorem 11 *Let G be a graph with $\delta(G) \geq h - 1$. Then*

$$h \leq \gamma_{\times h}(G) \leq n$$

and these bounds are sharp. □

Both the upper and lower bounds in the above theorem are realized by the complete graph K_h.

Lemma 12.1 *If vertex v has degree $h - 1$, then $N[v]$ must be in every h-tuple dominating set of G.* □

Theorem 12 *A graph G has V as its unique h-tuple dominating set if and only if for each $v \in V$, there is a vertex of degree $h - 1$ in $N[v]$.* □

Any $(h-1)$–regular graph will have V as its unique h–tuple dominating set.

Corollary 12.1 *If there exists $v \in V$ such that for all $x \in N[v]$, $deg(x) \geq h$, then $\gamma_{\times h}(G) < n$.* □

Observation 6 *A graph G has $\gamma_{\times h}(G) = h$ if and only if there exists at least h vertices in V having degree $n-1$.*

We have shown that $\gamma_{\times h}(G) > \gamma(G)$ for $h \geq 2$. Using a theorem from Fink and Jacobson [4], we determine a better bound for $\gamma_{\times h}(G)$ when $h \geq 3$.

Theorem A [4] *If G is a graph with $\Delta(G) \geq h \geq 2$, then $\gamma_h(G) \geq \gamma(G) + h - 2$.*

Since $\gamma_{\times h} \geq \gamma_h(G)$, we have the following.

Theorem 13 *If $\Delta(G) \geq h \geq 2$, then $\gamma_{\times h}(G) \geq \gamma(G) + h - 2$.* □

The final two theorems extend the best possible lower bounds for $dd(G)$ to best possible lower bounds for $\gamma_{\times h}(G)$. We note that K_h achieves both lower bounds.

Theorem 14 *Let G have $\delta(G) \geq h-1$. Then*

$$\gamma_{\times h}(G) \geq \frac{hn}{\Delta(G)+1}$$

and this bound is sharp. □

Theorem 15 *Let G have $\delta(G) \geq h-1$. Then*

$$\gamma_{\times h}(G) \geq \frac{2hn - 2m}{h+1}$$

and this bound is sharp. □

5 Concluding Remarks

We introduced the concept of double domination in graphs and derived sharp bounds on $dd(G)$. We also explored relationships between $dd(G)$ and other domination parameters. Finally, we extended the concept of double domination to multiple domination (h–tuple domination) in graphs. We noted that finding the h–tuple domination number of a graph is a difficult problem and presented sharp bounds on $\gamma_{\times h}(G)$.

In general, all the questions asked about domination can be asked about multiple domination and, in particular, double domination. We are investigating several of these questions as well as new related types of domination. We close with two of the many open problems concerning double domination.

(1) Characterize the graphs for which $dd(G) = 2\gamma(G)$.

(2) Bange, Barkauskas and Slater [2] called a dominating set S of a graph G *efficient* (or *exact* or *perfect*) if each vertex in V is dominated by exactly one vertex in S. Livingston and Stout [10] studied the existence and construction of perfect dominating sets in several families of graphs. Analogously, we say a double dominating set S is *efficient* if each vertex in V is dominated by exactly two vertices in S. Characterize the graphs which have efficient dd-sets.

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