A Note on the Hall-Condition Number of a Graph

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ABSTRACT. In this note a conjecture of P. Johnson Jr., on the Hall condition number is disproved.

We follow [1] for terminologies and notations and we consider finite undirected simple graphs. We say that (G, C), where C is a list assignment of G, satisfies Hall's condition if the following holds:

$$\sum_i t(H, C, i) \geq |E(H)|,$$

here H is an arbitrary subgraph of G and $t(H, C, i)$ denotes the maximum number of independent vertices of H having the color i in their lists, and i ranges over $\cup_{e \in E(H)} C(e)$.

For a graph G, let $s(G)$ denote the smallest integer m with the property that (G, C) satisfies Hall's condition for every list assignment C of G with $|C(v)| \geq m$ for all $v \in V(G)$.

The following theorem has appeared in [2] and [3]:

Theorem A. *For every graph G we have:*

$$s(G) = \max\{\lceil \frac{|V(H)|}{\alpha(H)} \rceil : H \text{ is a subgragh of } G\}$$

Hilton and Johnson in [2] posed a problem stating that: Does there exist a graph G with $s(G) \leq \chi(G) - 2$? And how much less than $\chi(G)$ can $s(G)$ be? In [3] Johnson solved this problem by constructing a family of graphs which showed that the set $\{\chi(G) - s(G)\}$ is unbounded. In the same paper he posed the following conjecture:

Conjecture. *For every graph G, $\chi(G) < \frac{3}{2}s(G)$.*

In this note we disprove the above conjecture. We construct a graph G with $\chi(G) = 5$ and $s(G) = 3$.

Let M be the graph shown in Figure 1. Note that $\{b_1, \ldots, b_5\}$ is an independent set of size 5 in M and $|V(M)| = 11$. We have $\chi(M) = 4$ and M is triangle-free. Actually, M is one of the graphs constructed by Mycielski (see [1]).

Proposition. *Let G be a graph obtained by joining a new vertex* o *to the vertices of M. Then $\chi(G) = 5$ and $s(G) = 3$.*

Proof: It is clear that $\chi(G) = 5$. Also, since G has an odd cycle as a subgraph, we have $s(G) \geq 3$. By Theorem A, it suffices to show that for every subgraph H of G with $|V(H)| = 3k + 1$, where $1 \leq k \leq 3$, we have $\alpha(H) \geq k+1$. But, since for any such H we have $|V(H) \cap V(M)| \geq 3k$, thus it is sufficient to show that for every subgraph L of M with $|V(L)| = 3k$ $(1 \leq k \leq 3)$, we have $\alpha(L) \geq k + 1$. We know that M is triangle-free and from Ramsey theory we have $r(3,2) = 3$, $r(3,3) = 6$ and $r(3,4) = 9$, as desired. □

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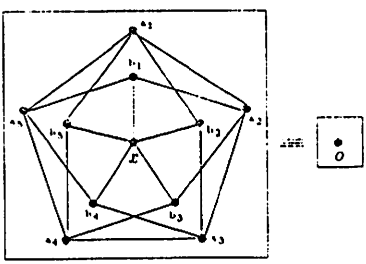

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Figure 1. A graph G with $\chi(G) = 5$ and $s(G) = 3$.

References

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[3] P.D. Johnson Jr., The Hall Condition Number of a Graph, *Ars Combinatoria* **37** (1994), 183–190.