On the genus of the star graph

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ABSTRACT. The star graph S_n is a graph with S_n the set of all permutations over $\{1, \ldots, n\}$ as its vertex set; two vertices π_1 and π_2 are connected if π_1 call be obtained form π_2 by swapping the first element of π_1 with one of the other $n-1$ elements. In this paper we establish the genus of the star graph. We show that the genus, g_n of S_n, is exactly equal to $n!(n-4)/6+1$ by establishing a lower bound and inductively giving a drawing on a surface of appropriate genus.

1 Introduction

In [1] Akers, Harel and Krishnamurthy introduced the star graph as a computational network for parallel computing. It was also shown that the star graph has many properties which are desirable for practical networks; such as low diameter, low degree, symmetry and low fault diameter. They also designed efficient routing algorithms on the star graph. Since then some work has been done to design efficient parallel algorithms for the star graph [6]. In this paper we are concerned with the genus of this graph. Akers, Harel and Krishnamurthy [1, 2] conjectured that the genus of the star graph is $n-3$. We show that this conjecture is far from the truth: in fact the genus, g_n of S_n, is given by

$$g_n = \frac{n!(n-4)}{6} + 1.$$

2 Cayley graphs

Given Γ a group and $\rho_1, \rho_2, \ldots, \rho_k$ a set of generators for Γ, the *Cayley graph* $G(\Gamma; \rho_1, \ldots, \rho_k)$ is obtained by taking Γ as the vertex set; two vertices π_1,

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π_2 are connected if and only if $\pi_1 = \rho_j \pi_2$ for some j. Many well known interconnection networks can be expressed as Cayley graphs. For example, we can realize the *n-cube* as a Cayley graph by taking Γ as the product of n copies of Z_2, that is,

$$\Gamma = \oplus_{i=1}^{n} Z_2.$$

The generators of Γ are given by $(1,0,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,0,\ldots,1)$. In [2] it has been suggested that in general Cayley graphs make a good choice for interconnection networks.

We want to study Cayley graphs of familiar groups. Therefore, $\mathcal{S}_n$ the group of permutations of $\{1,\ldots,n\}$ is a natural choice. We shall restrict our attention to Cayley graphs obtained from a set of transpositions as generators of $\mathcal{S}_n$. Given a set of transpositions T, we can construct a *transposition graph* G_T on n vertices $\{1,\ldots,n\}$, such that $\{i,j\}$ is an edge in G_T if and only if $(i,j) \in T$. The following theorem, first given by Cayley himself, characterizes when T is a minimal set of generators for $\mathcal{S}_n$.

Theorem 2.1. *—em (Cayley) A set of transpositions T is a minimal set of generators for $\mathcal{S}_n$ if and only if the transposition graph G_T is a tree spanning $\{1,\ldots,n\}$.*

The *star graph* $\mathcal{S}_n$ is a Cayley graph obtained by taking $\Gamma = \mathcal{S}_n$, the group of all permutations of $\{1,2,\ldots,n\}$ with set of generators $T_n = \{(1,2),(1,3),\ldots,(1,n)\}$. Thus there is an edge between two permutations π_1 and π_2, if π_2 can be obtained from π_1 by swapping the first element of π_1 with one of the other $n-1$ elements. The transposition graph G_{T_n} can be drawn to look like a star (This explains the reason for the name "star graph"). The star graph $\mathcal{S}_n$ is $n-1$ regular with $n!$ vertices and $n!(n-1)/2$ edges.

Figure 1 shows the star graphs $\mathcal{S}_3$ and $\mathcal{S}_4$. We have labeled the vertices by appropriate permutations written in straight line notation (In straight line notation a permutation π is written as $(\pi(1),\pi(2),\ldots,\pi(n))$.) which we will use henceforth. Note that $\mathcal{S}_4$ consists of four copies of $\mathcal{S}_3$, where the ith copy of $\mathcal{S}_3$ is obtained by considering vertices with $\pi(4) = i$. We can generalize this decomposition as follows. Let $\mathcal{S}_n^i$ be the subgraph of $\mathcal{S}_n$ induced by all the permutations π with $\pi(n) = i$. Then it is easy to see that $\mathcal{S}_n^i$ is isomorphic to $\mathcal{S}_{n-1}$. Hence $\mathcal{S}_n$ consists of n copies of $\mathcal{S}_{n-1}$. In order to understand the genus of the star graph we will have to further decompose $\mathcal{S}_n$ into smaller star graphs. For every $\pi = (\pi(1),\ldots,\pi(n))$, we write $\pi = (\alpha_\pi, \beta_\pi, \gamma_\pi, \delta_\pi)$, where $\alpha_\pi = (\pi(1),\pi(2))$, $\beta_\pi = \pi(3)$, $\gamma_\pi = (\pi(4),\ldots,\pi(n-1))$, and $\delta_\pi = \pi(n)$. Hence we view every permutation π as illustrated below:

$$\pi = (\overbrace{\pi(1),\pi(2)}^{\alpha_\pi},\overbrace{\pi(3)}^{\beta_\pi},\overbrace{\pi(4),\ldots,\pi(n-1)}^{\gamma_\pi},\overbrace{\pi(n)}^{\delta_\pi})$$

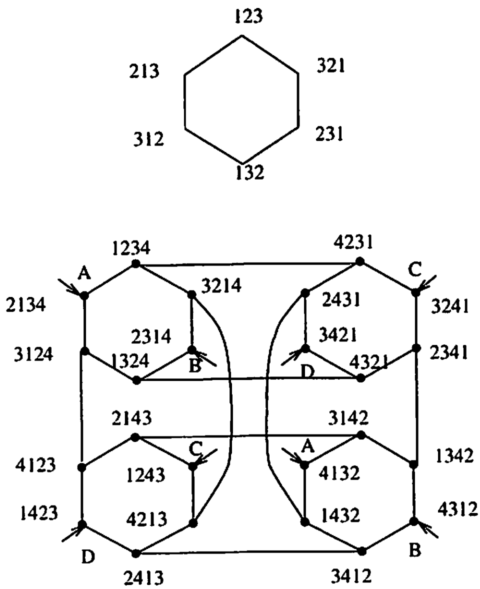

Figure 1. Star Graphs $\mathcal{S}_3$ and $\mathcal{S}_4$

Let $\mathcal{S}_n[\gamma = \gamma_0; \delta = i]$ denote the graph induced by all the permutations π such that $\gamma_\pi = \gamma_0$ and $\delta_\pi = i$; that is, $\mathcal{S}_n[\gamma = \gamma_0; \delta = i]$ consists of all the permutations whose last $n-3$ entries are fixed as (γ_0, i). Similarly, we can define $\mathcal{S}_n[\gamma = \gamma_0]$ etc. For example, $\mathcal{S}_6[\gamma = (2, 6)]$ is the graph induced by all permutations of the form $(*, *, *, 2, 6, *)$. It can easily be seen that for any γ_0 and i, $\mathcal{S}_n[\gamma = \gamma_0; \delta = i]$ is isomorphic to $\mathcal{S}_3$ and $\mathcal{S}_n[\gamma = \gamma_0]$ is isomorphic to $\mathcal{S}_4$. Lastly, we also note that $\mathcal{S}_n^i = \mathcal{S}_n[\delta = i]$.

3 The genus of the star graph

The genus of a graph is the minimal genus of an orientable surface on which the graph can be drawn without any edge crossings. The genus of a graph may be considered as a measure of non-planarity. Hence if we can draw a graph on a plane without any edge crossings then its genus is 0. Graphs with high genus are "extremely non-planar." For more formal treatment of this subject see [3, 4]. In this section our goal is to prove the following theorem:

Theorem 3.1. *For $n \geq 3$, the genus of the star graph $\mathcal{S}_n$ is given by,*

$$g_n = \frac{n!(n-4)}{6} + 1.$$

Our method is based on that of Beineke and Harary [5], who computed the genus of the n-cube. To show the above result we first establish a lower bound for g_n which follows from the Euler characteristic formula.

Theorem 3.2. *Let g_n denote the genus of $\mathcal{S}_n$, then for $n \geq 3$,*

$$g_n \geq \frac{n!(n-4)}{6} + 1.$$

Proof: Consider any optimal embedding of $\mathcal{S}_n$ in a orientable surface of genus g_n. Let F_n be the number of faces of $\mathcal{S}_n$ on the surface. Further, let V_n and E_n denote the number of vertices and edges in $\mathcal{S}_n$. Notice, that for $n \geq 3$, the *girth*[1] of $\mathcal{S}_n$ is 6. Therefore, it follows that each face in the embedding has at least 6 edges. Thus,

$$6F_n \leq 2E_n. \tag{1}$$

Now, according to Euler characteristic formula we have,

$$2g_n = E_n - F_n - V_n + 2.$$

Using inequality (1) we get,

$$2g_n \geq 2/3E_n - V_n + 2.$$

Since $\mathcal{S}_n$ has $n!$ vertices and $n!(n-1)/2$ edges, a simple calculation gives us the desired result. □

To obtain the upper bound we have to do a little more work. We start by observing that $n!(n-4)/6+1$ is a unique solution to the following recursion:

$$f_n = \begin{cases} 0 & \text{if } n = 3 \\ nf_{n-1} + \frac{n!}{6} - n + 1 & \text{otherwise} \end{cases} \tag{2}$$

To prove Theorem 3.1 it suffices to show that $g_3 = 0$ and $g_n \leq ng_{n-1} + n!/6 - n + 1$. $\mathcal{S}_3$ is a hexagon; hence $g_3 = 0$. However, $\mathcal{S}_4$ is more interesting. We want to embed $\mathcal{S}_4$ in a surface of genus 1. We think of a torus (a surface of genus 1) as a square whose opposite sides are identified appropriately. Therefore Figure 2 shows that $g_4 = 1$ by drawing it on a square with opposite sides identified. This figure will play an important role in the proof for the upper bound. We observe that in Figure 3(a) every $\mathcal{S}_4[\delta = i]$ is a face. Also, every $\mathcal{S}_4[\beta = i]$ is a face (see Figure 3(b)).

[1]The girth of a graph G is the length the smallest cycle in G.

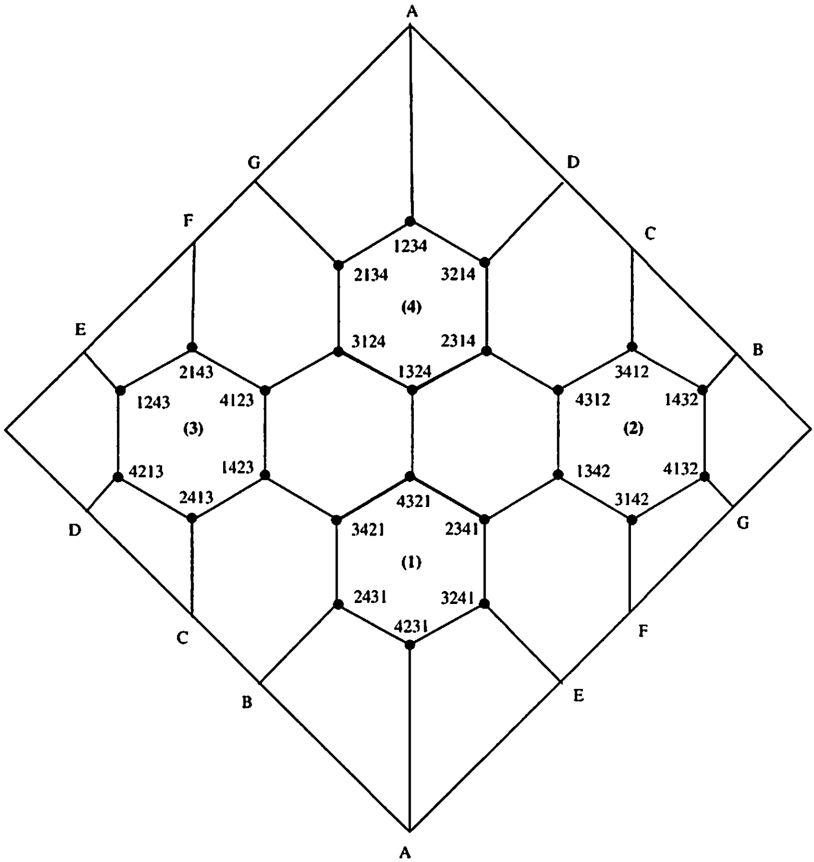

Figure 2. The Genus of $\mathcal{S}_4$ is 1

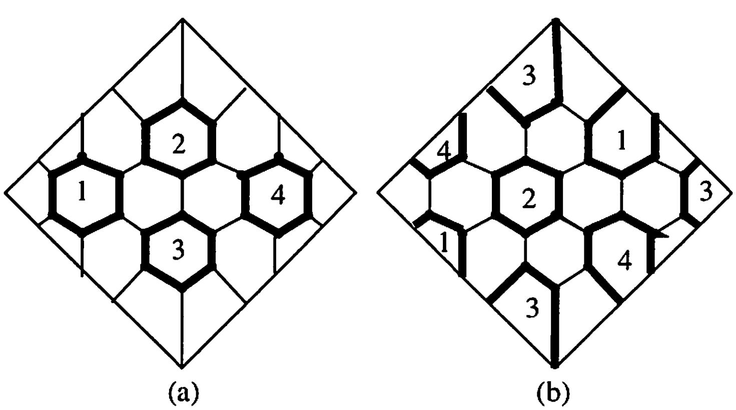

Figure 3. Each $\mathcal{S}_4[\delta = i]$ and $\mathcal{S}_4[\beta = i]$ is a face

Figure 4 is obtained by simply sliding each $\mathcal{S}_4$ to align them in a line. Figure 5 shows that we can think of a torus as four spheres with four handles attached between them. Hence our original drawing of Figure 2, can be used to obtain a drawing of $\mathcal{S}_4$ such that each $\mathcal{S}_4$ is drawn on a sphere (Figure 6). Four handles have been attached between the spheres to complete the drawing. Further, every face in Figure 2 is still a face in Figure 6. For simplicity, the edges between the different $\mathcal{S}_4^i$ are not drawn.

To embed $\mathcal{S}_n$ in an orientable surface we will take n embedded copies of $\mathcal{S}_{n-1}$ in a surface of genus g_{n-1} such that $\mathcal{S}_n[\gamma = \gamma_0; \delta = i]$ is a face for every γ_0 and i. For every γ_0, we will then use four handles as in Figure 6 to connect the rest of the edges in $\mathcal{S}_n[\gamma = \gamma_0]$. Therefore by attaching a total of $n!/6$ handles we will obtain a drawing of $\mathcal{S}_n$. The next lemma and recursion (2), tell us that $n!/6$ handles are sufficient for our purpose.

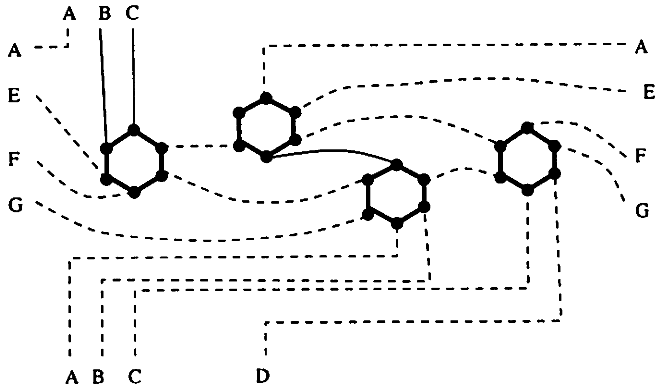

Figure 4. Another drawing of $\mathcal{S}_4$

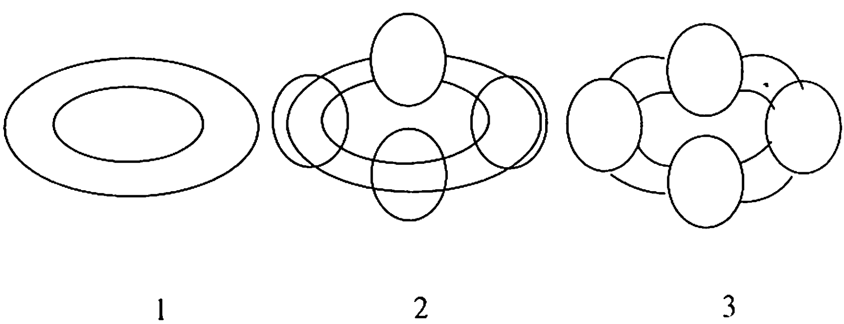

Figure 5. The torus shown as four spheres with four handles attached

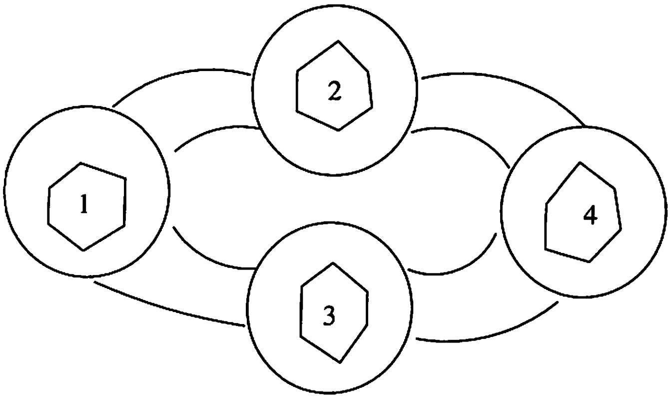

Figure 6. Drawing a $\mathcal{S}_4$ on a torus

Lemma 3.1. *Let $M_1, \ldots, M_n$ be orientable surfaces, all of genus g. If we add $k \geq n-1$ handles between them to make a connected orientable surface M, then the genus of M is $ng + k - n + 1$.*

Proof: Consider two disconnected surfaces of genus g_1 and g_2. If the two surfaces are connected by a handle then the resulting surface has genus $g_1 + g_2$. Further, if we add a handle to a connected surface we increase its genus by one. Now the result follows from noting that if we attach k handles between n surfaces one by one to obtain M, then exactly $n-1$ handles will be used to connect disconnected surfaces. □

Comparing the above lemma and recursion for $n!(n-4)/6+1$, we notice that if we start with $\mathcal{S}_n$ and embed every $\mathcal{S}_{n-1}$ inductively in n surfaces of genus g_{n-1} then we are allowed to attach $n!/6$ handles between the n surfaces. Call an edge in $\mathcal{S}_n$ *old* if it connects two vertices ln $\mathcal{S}_n^i$ for some i. Further, an edge is *new* if it is not old. Hence all the new edges of $\mathcal{S}_n$ connect some vertex in $\mathcal{S}_n^i$ to some vertex in $\mathcal{S}_n^j$ for $i \neq j$. For example, the thin edges in Figure 2(a) are the new edges of $\mathcal{S}_4$. The following lemma gives us a nice drawing of $\mathcal{S}_4$ provided we have a nice drawing of its old edges.

Lemma 3.2. *Assume we have a drawing of $\mathcal{S}_4$ which contains the old edges of $\mathcal{S}_4$ on some (not necessarily connected) surface M. Further, in this drawing $\mathcal{S}_4^i$ is a face for every i. Then we can obtain a drawing of $\mathcal{S}_4$ by attaching four handles to this surface such that every $\mathcal{S}_4[\beta = i]$ is a face for every i.*

Proof: Given a drawing in which every $\mathcal{S}_4^i$ is a face we can "inflate" the surfaces under each $\mathcal{S}_4^i$ to form spheres over $\mathcal{S}_4^i$. Now four handles can be attached to these spheres as shown in Figure 6 to route all the new edges. Since this new Figure is similar to Figure 6, it can be readily seen that every $\mathcal{S}_4[\beta = i]$ is a face. □

Now we are ready to prove our upper bound. The proof is by induction on a slightly stronger statement.

Theorem 3.3. *For $n > 3$ there exists an embedding of $\mathcal{S}_n$ on a surface of genus $n!(n-4)/6+1$ such that every $\mathcal{S}_n[\gamma = \gamma_0; \delta = i]$ is a face.*

Proof: For $n = 4$ the result follows from Figure 2. Assume that $n \geq 5$; then by induction we can embed every $\mathcal{S}_n^i$ in a surface of genus g_{n-1} such that every $\mathcal{S}_n[\gamma = \gamma_0; \delta = i]$ is a face. Hence we have a drawing of all the old edges of $\mathcal{S}_n$. Fix a γ_0 and consider $\mathcal{S}_n[\gamma = \gamma_0]$ which is isomorphic to $\mathcal{S}_4$. The new edges of $\mathcal{S}_n[\gamma = \gamma_0]$ correspond to the thin edges of $\mathcal{S}_4$ in Figure 3(a). Further, for every i, $\mathcal{S}_n[\gamma = \gamma_0; \delta = i]$ corresponds to $\mathcal{S}_4^i$. Since each $\mathcal{S}_n[\gamma = \gamma_0; \delta = i]$ is a face by Lemma 3.2 we can add all the new edges in $\mathcal{S}_n[\gamma = \gamma_0]$ by attaching only four handles. Notice that every new edge of $\mathcal{S}_n$ is a new edge in $\mathcal{S}_n[\gamma = \gamma_0]$ for some unique γ_0. Hence, if we repeat this process for all choices of γ_0 we get a embedding of $\mathcal{S}_n$. Furthermore, in this drawing every $\mathcal{S}_4[\beta = i; \gamma = \gamma_0]$ is a face. There are $\binom{n}{4}(n-4)!$ choices for γ_0, and each requires 4 handles. We have added $4\binom{n}{4}(n-4)! = \frac{n!}{6}$ handles. Hence by Lemma 3.1 the genus of the new surface is $ng_{n-1} + n!/6 - n + 1$. Further, from our construction this drawing of $\mathcal{S}_n$ has every $\mathcal{S}_n[\beta = i; \gamma = \gamma_0]$ as a face. To continue the induction on this stronger statement we must obtain a drawing of $\mathcal{S}_n$ such that every $\mathcal{S}_n[\gamma = \gamma_0; \delta = i]$ is a face. Hence by renaming every vertex $\pi = (\alpha_\pi, \beta_\pi, \gamma_\pi, \delta_\pi)$ to $(\alpha_\pi, \delta_\pi, \gamma_\pi, \beta_\pi)$ we get the desired drawing. □

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