IRREDUNDANCE IN THE QUEENS' GRAPH

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Abstract

The vertices of the queens' graph Q_n are the squares of an $n \times n$ chessboard and two squares are adjacent if a queen placed on one covers the other. Informally, a set I of queens on the board is irredundant if each queen in I covers a square (perhaps its own) which is not covered by any other queen in I. It is shown that the cardinality of any irredundant set of vertices of Q_n is at most $\lfloor 6n + 6 - 8\sqrt{n+3} \rfloor$ for $n \geq 6$. We also show that the bound is not exact since $IR(Q_8) \leq 23$.

1. Introduction

The lower (upper) domination numbers $\gamma(G), (\Gamma(G))$, independence numbers $i(G)$ $(\beta(G))$ and irredundance numbers $ir(G)$ $(IR(G))$ of a graph G are respectively the smallest (largest) cardinalities of minimal dominating, maximal independent and maximal irredundant vertex sets of G.

These six parameters are well-studied in the literature (see [3]) and satisfy the following chain of inequalities:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

In particular there has been considerable recent interest in the evaluation of these parameters for graphs defined from $n \times n$ chessboards ([2]). This is perhaps due to the fact that two of these problems, namely the determination of γ and i for the Queens' graph Q_n (defined in the following paragraph), have defied all efforts at solution for at least a hundred years (see [2]).

The Queens' graph Q_n has the n^2 squares of the chessboard as its vertex set and two vertices are adjacent if a queen placed on one covers the other, *i.e.* if the two squares are on the same *line* (row, column or diagonal) of the board.

The survey paper ([2]) gives an excellent account of recent results on the six parameters for Q_n. Since that paper was written, Weakley ([5])

and Burger, Cockayne and Mynhardt ([1]) have established new values of $\gamma(Q_n)$.

This paper is concerned with the upper irredundance number $IR(Q_n)$. Informally, a set I of queens on the board is irredundant if each queen in I covers a square (perhaps its own) which is not covered by any other queen in I. Weakley ([5]) has shown $\Gamma(Q_n)$ (and hence $IR(Q_n)$) $\geq 2n-5$ and McCrae ([4]) has used computer techniques to generate examples which show this lower bound is not exact. In the present work we show that $IR(Q_n) \leq \lfloor 6n+6-8\sqrt{n+3} \rfloor$ and show that our bound is also not exact since $IR(Q_8) \leq 23$.

2. The Upper Bound for $IR(Q_n)$

The following notation and terminology will be required. The rows and columns are numbered in obvious matrix fashion. The *sum-diagonal* numbered k contains the squares (i,j) such that $(i+j)-(n+1)=k$. The *difference-diagonal* numbered k contains the squares (i,j) satisfying $(i-j)=k$. There are $(2n-1)$ sum-diagonals (difference-diagonals) which are numbered $0, \pm 1, \pm 2, \ldots, \pm(n-1)$.

For a vertex v of Q_n, $r(v)$, $c(v)$, $d(v)$, $s(v)$ denote respectively the row, column, difference-diagonal and sum-diagonal which contain v. A set I of vertices of a graph G is *irredundant* if for each $v \in I$

$$N[v] - \bigcup_{u \in I-\{v\}} N[u] \neq \emptyset.$$

Let I be an irredundant vertex set of Q_n, A be the set of isolated vertices of $G[I]$ where $|A| = \alpha \leq n$ (since $\beta(Q_n)=n$) and $X = \{x_1, \ldots, x_t\} = I - A$. Since I is irredundant, for each $i = 1, \ldots, t$, x_i is adjacent to $y_i \in V - I$ (a *private neighbour of* x_i) which is not adjacent to any vertex of $I - \{x_i\}$. Vertices x_i and y_i are on a line ℓ_i. Let $\{y_1, \ldots, y_t\} = Y$, and $Z = V - (I \cup Y)$. The private neighbour property implies that $\ell_1, \ldots, \ell_t$ are distinct. Define $U = \{\ell_1, \ldots, \ell_t\}$.

We begin with a few simple propositions.

Proposition 1. *If x_i, x_j (or y_i, y_j) are adjacent on a line ℓ, then $\ell \notin U$.*

Proof. If $\{x_i, x_j\} \subseteq \ell$ and $\{x_k, y_k\} \subseteq \ell \in U$, then one of i, j (say i) is distinct from k. Then $\{x_i, x_k\} \subseteq \ell$ contradicting the private neighbour property of y_k. (Similar proof for $\{y_i, y_j\} \subseteq \ell$.) ∎

Proposition 2. *Let ℓ be any line. If $\{x_i, x_j\} \subseteq \ell$, then $\{y_k, y_\ell\} \nsubseteq \ell$ and conversely.*

Proof. Suppose $\{x_i, x_j, y_k, y_\ell\} \subseteq \ell$. Clearly the private neighbour property is contradicted. ▮

Proposition 3. *Let ℓ_i be the line defined by x_i, y_i. Then none of the other six lines which contain x_i, y_i are in U.*

Proof. Suppose $m \neq \ell_i$ contains x_i and $m \in U$. Then for some j, $\{x_j, y_j, x_i\} \subseteq m$, contrary to the private neighbour property of y_j. (Similar proof for $y_i \in m$.) ▮

Proposition 4. *If $v \in A$, then $\{r(v), c(v), d(v), s(v)\} \cap U = \emptyset$.*

Proof. Suppose $v \in A$ is on line $\ell \in U$. Then for some i, $\{x_i, y_i, v\} \subseteq \ell$, contrary to the private neighbour property. ▮

Proposition 5. *If $v \in A$, then $N(v) \subseteq Z$.*

Proof. Vertex v is isolated in $G[I]$ and is not adjacent to $y \in Y$ (for otherwise y is not a private neighbour). ▮

Proposition 6. *For each $i = 1\ldots, t$, $\ell_i - \{x_i, y_i\} \subseteq Z$.*

Proof. Let $w_i \in \ell_i - \{x_i, y_i\}$. If $w_i = x \in I$, then both x and x_i are adjacent to y_i. If $w_i = y_j \in Y$, then x_i is adjacent to both y_i and y_j. In each case the private neighbour property is contradicted. ▮

Now suppose that U contains r, c, s, d rows, columns, sum-diagonals and difference-diagonals respectively.

Proposition 7. *If $r + \alpha \geq n - 4$ (or $c + \alpha \geq n - 4$), then $|I| \leq 3n$.*

Proof. Since each line in U contains a private neighbor, the r rows of U are distinct. Since A is independent, the rows occupied by vertices of A are distinct, and these rows are distinct from the rows in U. By Propositions 5 and 6, Z contains $(n-2)$ elements of r rows of U and $(n-1)$ elements from α additional rows. Since $|Z| = n^2 - 2t - \alpha$,

$$r(n-2) + \alpha(n-1) \leq n^2 - 2t - \alpha.$$

Therefore

$$\begin{aligned} 2t &\leq n^2 - (r+\alpha)n + 2r \\ &\leq n^2 - (n-4)n + 2r \\ &= 4n + 2r. \end{aligned}$$

Hence $|I| = t + \alpha \leq 2n + (r + \alpha) \leq 3n$. ▮

We now establish the upper bound for $IR(Q_n)$.

Theorem 8. *For $n \geq 6$, $IR(Q_n) \leq \lfloor 6n + 6 - 8\sqrt{n+3} \rfloor$.*

Proof. If $r + \alpha$ (or $c + \alpha$) $\geq n - 4$, then by Proposition 7, $|I| \leq 3n \leq 6n + 6 - 8\sqrt{n+3}$ for $n \geq 6$. Hence assume $r + \alpha \leq n - 5$ and $c + \alpha \leq n - 5$. Assume, without loss of generality, that $d \leq s$ and re-label X, Y so that $\ell_1, \ldots, \ell_s$ are sum-diagonals. Let $r_1, \ldots, r_s$ (respectively $r'_1, \ldots, r'_s$) be the rows occupied by $x_1, \ldots, x_s$ ($y_1, \ldots, y_s$). Note that there may be repetitions among $r_1, \ldots, r_s$ and among $r'_1, \ldots, r'_s$, but no r_i is equal to an r'_j ($r_i \neq r'_i$ since x_i, y_i are on a sum-diagonal and $r_i = r'_j$ ($j \neq i$) contradicts the private neighbour property).

Suppose L is the set of lines which are neither in U nor pass through any vertex of A. Let λ be the largest multiplicity of a row in the sequence $r_1, \ldots, r_s$. Then there are at least $\lceil s/\lambda \rceil$ distinct rows in the sequence. These rows are in L (by Proposition 3 and the fact that no vertex of A is adjacent to vertices of $X \cup Y$). Further the λ vertices of X which are on the same row, occupy distinct columns and distinct difference-diagonals. These 2λ lines are also in L. Hence we have a set of lines $L_1 \subseteq L$ satisfying (using elementary calculus)

$$|L_1| = \left\lceil \frac{s}{\lambda} \right\rceil + 2\lambda \geq \frac{s}{\lambda} + 2\lambda \geq 2\sqrt{2s}.$$

Applying the same argument to the sequence $r'_1, \ldots, r'_s$, we obtain a set $L_2 \subseteq L$ with $|L_2| \geq 2\sqrt{2s}$ and $L_1 \cap L_2 = \emptyset$ (otherwise x_i, y_j are on same row, column or difference-diagonal which contradicts the private neighbour property). We conclude $|L| \geq 4\sqrt{2s}$.

The total number of lines is $6n - 2$. Hence

$$t + 4\alpha + 4\sqrt{2s} \leq 6n - 2$$

and

$$|I| = t + \alpha \leq 6n - 2 - 4\sqrt{2s} - 3\alpha.$$

Therefore

$$|I| \leq f_1(s) = 6n - 2 - 4\sqrt{2s}. \tag{1}$$

Moreover

$$\begin{aligned} |I| &= (r + \alpha) + c + s + d \\ &\leq (n - 5) + (n - 5) + 2s. \end{aligned}$$

Therefore

$$|I| \leq f_2(s) = (2n - 10) + 2s. \tag{2}$$

Hence

$$|I| \le \max_{1 \le s \le 2n-3} \Big(\min(f_1(s), f_2(s))\Big).$$

The maximum occurs where $f_1(s) = f_2(s)$.

Solving the quadratic for $\sqrt{2s}$, we find that the maximum occurs when $\sqrt{2s} = 2\sqrt{n+3} - 2$ and so from (1)

$$\begin{aligned} |I| &\le 6n - 2 - 4(2\sqrt{n+3} - 2) \\ &= 6n + 6 - 8\sqrt{n+3}. \quad \blacksquare \end{aligned}$$

3. An upper bound for $IR(Q_8)$

In this section we show that the bound of Section 2 is not exact. The bound for $IR(Q_8)$ is 27. However we prove

Theorem 9. $IR(Q_8) \le 23$.

Proof. Suppose I is an irredundant set of 24 vertices of Q_8. Then $t = 24 - \alpha$ and $|Z| = 64 - 2t - \alpha = 16 + \alpha$. Suppose $\alpha \ge 2$ and a_1, $a_2 \in A$. By Proposition 5, $N(a_1) \cup N(a_2) \subseteq Z$ and the minimum degree of Q_8 is 21. Hence

$$\begin{aligned} |Z| &\ge |N(a_1)| + |N(a_2)| - |N(a_1) \cap N(a_2)| \\ &= 42 - |N(a_1) \cap N(a_2)|. \end{aligned}$$

But for any n and non-adjacent v_1, $v_2 \in V(Q_n)$, $|N(v_1) \cap N(v_2)| \le 12$, hence $|Z| = 16 + \alpha \ge 30$, which is impossible since $\alpha \le \beta(Q_n) = 8$. We have shown that $\alpha = 0$ or 1.

Suppose U contains 4 or more lines which contain 8 squares (*i.e.* 4 or more rows, columns or major diagonals). Let four of these lines be $\ell_1, \dots, \ell_4$ and $Z_i = \ell_i - \{x_i, y_i\}$. By Proposition 6, $\bigcup_1^4 Z_i \subseteq Z$. Therefore

$$\begin{aligned} |Z| \ge \left|\bigcup_1^4 Z_i\right| &\ge \sum_1^4 |Z_i| - \sum_{1 \le i \le j \le 4} |Z_i \cap Z_j| \\ &\ge 24 - 6 = 18. \end{aligned}$$

Hence $16 + \alpha \ge 18$, a contradiction.

It follows (using $\alpha \in \{0, 1\}$) that U contains at least 20 lines from the set of sum-diagonals and difference-diagonals numbered $\pm 1, \dots, \pm 6$.

Without losing generality, U contains at least 10 sum-diagonals from this list say $s_1, \ldots s_{10}$ and these are disjoint. By Proposition 6

$$|Z| \geq \sum_{i=1}^{10} \left|s_i - \{x_i, y_i\}\right| \geq 2(0+1+2+3+4) = 20.$$

Therefore $16 + \alpha \geq 20$, a contradiction which shows that there is no 24-vertex irredundant set. ▮

Acknowledgements

The author gratefully acknowledges research support from the Canadian Natural Sciences and Engineering Research Council. He would also like to thank G. Grant, S.T. Hedetniemi, and A. McCrae for profitable and stimulating discussions concerning this work.

The work was completed while the author was enjoying the hospitality of the Department of Mathematics, Applied Mathematics and Astronomy of the University of South Africa during 1994.

References

1. A. Burger, E.J. Cockayne and C.M. Mynhardt. Dominating numbers for the Queens' graph, *Bull. Inst. Comb. and Applications* **10**(1994), 73–82.
2. G.H. Fricke, S.M. Hedetniemi, S.T. Hedetniemi, A.A. McCrae, C.K. Wallis, M.S. Jacobson, H.W. Martin, W.D. Weakley. Combinatorial problems on chessboards: A brief survey. In *Graph Theory, Combinatorics and Applications*, Vol. **1**. Proc. 7th Quadrennial Conference on the theory and applications of graphs (Alari and Schwenk, eds.), Western Michigan University (Wiley 1995), pp. 507–528.
3. S.T. Hedetniemi and R.C. Laskar. A bibliography on dominating sets in graphs and some basic definitions of dominating parameters, *Disc. Math.* **86**(1990), 257-277.
4. A. McCrae. (Private communication)
5. W.D. Weakley. Domination in the Queens' graph. In *Graph Theory, Combinatorics and Applications*, Vol. **2**. Proc. 7th Quadrennial Conference on the theory and applications of graphs (Alari and Schwenk, eds.), Western Michigan University (Wiley 1995), pp. 1223–1232.