Stratified Graphs and Distance Graphs

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ABSTRACT. We address questions of Chartrand et al. about k-stratified graphs and distance graphs. A *k-stratified graph G* is a graph whose vertices have been partitioned into k distinct color classes, or strata. An *underlying graph G'* is obtained by ignoring the colors of G. We prove that for every pair of positive integers k and l, there exists a pair of 2-stratified graphs with exactly k greatest common stratified subgraphs such that their underlying graphs have exactly l greatest common subgraphs.

A *distance graph* $D(\mathcal{A})$ has vertices from some set $\mathcal{A}$ of $0-1$ sequences of a fixed length and fixed weight. Two vertices are adjacent if one of the corresponding sequences can be obtained from the other by the interchange of a 0 and 1. If G is a graph of order m that can be realized as the distance graph of $0-1$ sequences, then we prove that the $0-1$ sequences require length at most $2m-2$. We present a list of minimal forbidden induced subgraphs of distance graphs of $0-1$ sequences. A *distance graph* $D(\mathcal{G})$ has vertices from some set $\mathcal{G}$ of graphs or k-stratified graphs. Two vertices are adjacent if one of the corresponding graphs can be obtained from the other by a single edge rotation. We prove that K_n minus an edge is a distance graph of a set of graphs. We fully characterize which radius one graphs are distance graphs of $0-1$ sequences and which are distance graphs of graphs with distinctly labelled vertices.

1 Introduction

In [1], Chartrand et al. raised several questions about k-stratified graphs and distance graphs. We answer some of these open questions and present results that contribute to the solutions of most of the remaining questions.

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A *k-stratified graph* G is a graph whose vertex set has been partitioned into k distinct color classes, or strata. Stratified graphs were introduced in response to problems in VLSI design [5], but have become the focus of theoretical works. We denote the k colors of G by $1, 2, \ldots, k$, and if there are n_i vertices of color i for $1 \leq i \leq k$, then $(n_1, n_2, \ldots, n_k)$ is the *color vector* of G. The *underlying graph* of G, denoted G', is obtained by ignoring the colors assigned to the vertices of G. In Section 2, we answer one open question from [1] by proving that for every pair of positive integers k and l, there exists a pair of 2-stratified graphs with exactly k greatest common stratified subgraphs such that their underlying graphs have exactly l greatest common subgraphs.

A graph (respectively k-stratified graph) G_a can be *rotated* into a graph (respectively k-stratified graph) G_b if G_a contains vertices u, v, and w such that $uv \in E(G_a)$, $uw \notin E(G_a)$ and $G_b = G_a - uv + uw$ [1]. Chartrand et al. define a *distance graph* $D(\mathcal{G})$ of a set $\mathcal{G}$ of stratified graphs with fixed order and fixed color vector as a graph whose vertices are the elements in $\mathcal{G}$, where two vertices are adjacent if the corresponding graphs can be obtained from each other by a single edge rotation [1]. An analogous distance graph for unstratified graphs has also been defined, and determining which graphs are distance graphs of some set of graphs has been studied in detail by several authors [1] [2] [4]. Chartrand et al. conjecture that all graphs are distance graphs of some set of graphs [2]. In Section 3, we prove that K_n minus an edge is such a distance graph. Section 3 is also concerned with determining which graphs are distance graphs $D(\mathcal{A})$ of some set $\mathcal{A}$ of $0-1$ sequences with fixed length and fixed weight. These distance graphs have vertices from $\mathcal{A}$, and two vertices are adjacent if the corresponding sequences can be obtained from each other by the interchange of a 0 and 1. If G is a graph of order m that can be realized as the distance graph of $0-1$ sequences, then we prove that the sequences require length at most $2m-2$, thereby answering another open question from [1]. We also present a list of minimal forbidden induced subgraphs of distance graphs of $0-1$ sequences.

In Section 4, we study radius one graphs. We fully classify which radius one graphs are distance graphs of $0-1$ sequences and which are distance graphs of graphs with distinctly labelled vertices. We present an example of a graph that can not be realized as the distance graph of a set of graphs with distinctly labelled vertices.

2 Stratified Graphs

When two graphs are not isomorphic, it is often useful to have some way of measuring how different they are. In [1], Chartrand et al. define a distance between two k-stratified graphs G_a and G_b in terms of a *greatest common stratified subgraph*, or more simply a *greatest stratified subgraph*,

H of G_a and G_b. A greatest stratified subgraph H is an l-stratified graph, $1 \le l \le k$, which has maximum size, no isolated vertices, and is isomorphic to stratified subgraphs of both G_a and G_b. Its l colors are any subset of the k colors assigned to G_a and G_b. Chartrand et al. analogously define the *greatest common subgraph* H_0 of the underlying graphs G'_a and G'_b. They present several results concerning greatest stratified subgraphs and greatest common subgraphs. One example that will be useful for our purposes is that the size of the greatest stratified subgraph is less than or equal to the size of the greatest common subgraph. Our first theorem answers one of the open questions in [1].

Theorem 1. *For every pair of positive integers k and l, there exists a pair of 2-stratified graphs G_a and G_b containing exactly k distinct greatest stratified subgraphs, such that the underlying graphs G'_a and G'_b contain exactly l distinct greatest common subgraphs.*

Proof: Given a pair of positive integers k and l, choose an integer x so that $k \le (x+1)l$. Let G'_1 be the one-point union of x copies of C_{2l+2} obtained by identifying exactly one vertex from each cycle.

We will construct the disjoint graph G'_2 such that it has l distinct isomorphism types of components, $H_1, H_2, \ldots, H_l$. Let H_i be composed of x paths P_{i+1} and x paths P_{2l+2-i}, all originating at a common center vertex. Let G'_2 be a graph with $\max\{k, l\}$ components where each component is isomorphic to some H_i in such a way that all l of the H_i's occur at least one time and no more than $x+1$ times. Figure 1 illustrates G'_1 and the two isomorphism types of G'_2 for $k = 5$ and $l = 2$.

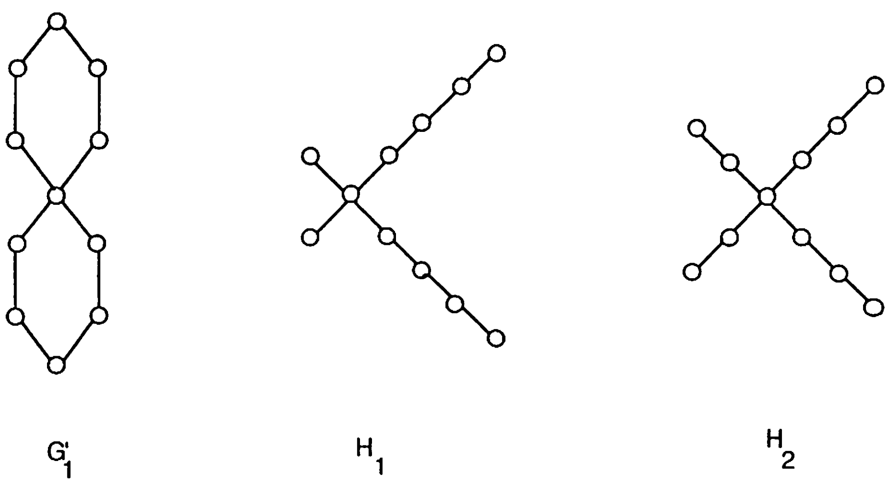


Figure 1.

We claim that representatives from these l isomorphism types $H_1, H_2, \ldots,$ H_l, are the l unique distinct greatest common subgraphs of G'_1 and G'_2.

Clearly, the representatives of the l different isomorphism types are common subgraphs of G'_1 and G'_2. First, we show that the size q of the greatest common subgraphs of G'_1 and G'_2 is $2lx$. Notice that each component in G'_2 has size $2lx$ by virtue of its construction. Thus, $q \geq 2lx$. Now, note that G'_1 has x cycles, and G'_2 has no cycles. Thus, any common subgraphs can have no cycles. In order to break the cycles in G'_1, we must remove at least x edges since no cycles share an edge. The graph G'_1 consists of $x(2l+1)$ edges, and so any common subgraph has size $q \leq x(2l+1) - x = 2lx$. Therefore, the size of any greatest common subgraph is $q = 2lx$, and so $H_1, H_2, \ldots, H_l$ are greatest common subgraphs of G'_1 and G'_2.

We now show that representatives from the l isomorphism types are the unique distinct greatest common subgraphs of G'_1 and G'_2. Suppose that a graph B is a greatest common subgraph of G'_1 and G'_2. Since B is a common subgraph, it must have no cycles. We again remove at least x edges of G'_1, one from each cycle, leaving a graph with size $q \leq 2lx$. Since B must have size $q = 2lx$, we remove exactly one edge from each cycle, and so B is connected. Hence, B is necessarily isomorphic to a component of G'_2. Thus, the l unique distinct greatest common subgraphs of G'_1 and G'_2 are components of G'_2; Specifically, they are the l distinct isomorphism types $H_1, H_2, \ldots, H_l$.

We now color the underlying graphs G'_1 and G'_2 to obtain G_1 and G_2. Color the center vertex of G'_1 with color 1. In each cycle, color exactly one of the vertices that is adjacent to the center vertex with a 1. Color all remaining vertices with 2's.

We color G'_2 by considering the following two cases.

Case 1. If $k \leq l$, then G'_2 has l components. Choose any k of these. Color the k center vertices with 1's. For each of these k components, choose x of the paths and color the vertices adjacent to the center vertex with 1's. Color all remaining vertices with 2's. For the remaining $l - k$ components, color the center vertices with 2's. Color the paths arbitrarily; for example, use all 2's.

So, exactly k of these components are stratified subgraphs of G_1 and G_2. They are greatest stratified subgraphs since the underlying subgraphs are greatest common subgraphs, and the size of a greatest stratified subgraph is less than or equal to the size of a greatest common subgraph of a pair of graphs.

Case 2. If $k > l$, we have k components in G'_2. Then, some of the l isomorphism types H_i for $1 \leq i \leq l$ will have more than one representative. For each isomorphism type H_i, we have at most $x + 1$ representatives. Denote representative n_i for $0 \leq n_i \leq x$ of isomorphism type H_i as H_{i,n_i}. For each representative H_{i,n_i}, color the center vertex with a 1. Notice that each representative H_{i,n_i} has x shorter paths and x longer paths. Color n_i

of the shorter paths so that the vertex adjacent to the center vertex has color 1. Color $x - n_i$ of the longer paths so that the vertex adjacent to the center vertex has color 1. Color all remaining vertices with 2's. Notice that for every isomorphism type H_i, there are $x+1$ distinct ways to do this coloring scheme. Clearly, we have l distinct greatest common subgraphs and k distinct greatest stratified subgraphs. Figure 2 shows G_1 and G_2 for $k = 5$ and $l = 2$. □

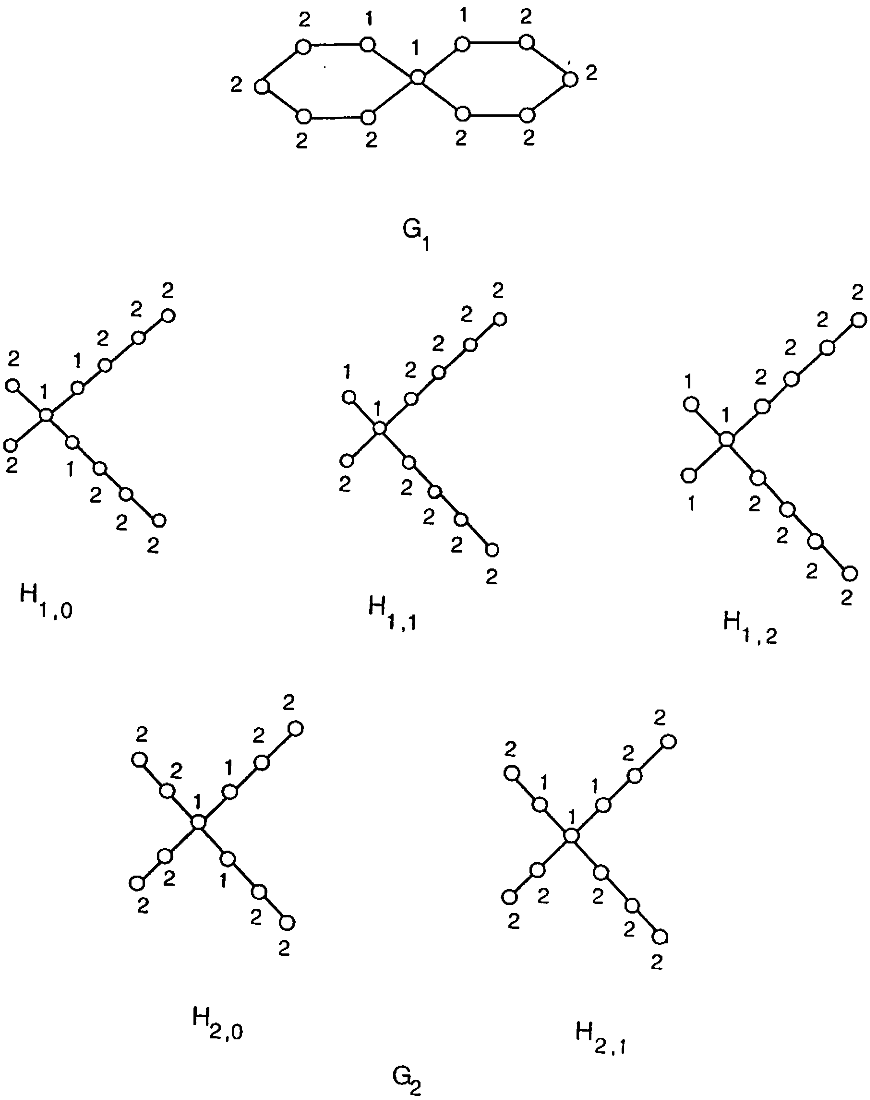


Figure 2.

This new relationship between greatest stratified subgraphs and greatest common subgraphs aids in the understanding of the connections between stratified graphs and their underlying graphs.

3 Distance Graphs

In [1], Chartrand et al. introduced two types of distance graphs. The first type relates to $0-1$ sequences. A $0-1$ sequence is a binary string of a prescribed length and weight. The *distance graph* $D(\mathcal{A})$ of a set of $0-1$ sequences of a fixed length and fixed weight is the graph whose vertices are the sequences in $\mathcal{A}$; two vertices s_1 and s_2 are adjacent if one sequence can be obtained from the other by the interchange of a 0 and 1. The *distance graph* $D(\mathcal{G})$ of a set of stratified graphs $\mathcal{G}$ of a fixed size and fixed color vector is the graph whose vertices are the graphs in $\mathcal{G}$; two vertices G_1 and G_2 are adjacent if the corresponding graphs can be obtained from each other by a single edge rotation. Distance graphs of unstratified graphs have also been studied [2]. We only consider stratification as a significant construct when some of the k color classes have more than one vertex. If a graph of order k is k-stratified, then, in keeping with convention, we call this a graph with distinctly labelled vertices. We are interested in knowing which graphs are distance graphs of $0-1$ sequences and which graphs are distance graphs of graphs.

Lemma 1 formalizes a statement made by Chartrand et al. concerning the relationship between distance graphs of $0-1$ sequences and distance graphs of graphs [1].

Lemma 1. *If $G = D(\mathcal{A})$ for some set $\mathcal{A}$ of $0-1$ sequences, then $G = D(\mathcal{G})$ for some set $\mathcal{G}$ of graphs with distinctly labelled vertices.*

Proof: Let G be a graph of order m. Suppose that $G = D(\mathcal{A})$ for some set $\mathcal{A}$ of $0-1$ sequences. For each $0-1$ sequence s_i for $1 \leq i \leq m$, we must construct a distinctly labelled graph G_i that preserves the adjacencies of the $0-1$ sequences. Suppose that each $0-1$ sequence has length k. Construct a series of bipartite graphs $G_1, G_2, \ldots, G_m$ where one partite set has a single vertex labelled v_0, and the other partite set has k vertices labelled $v_1, v_2, \ldots, v_k$. To construct the graph G_i, draw an edge from the vertex labelled v_0 to the vertex labelled v_j $(1 \leq j \leq k)$ if bit j is equal to 1 in s_i. Thus, if each of the $0-1$ sequences have weight l (for some $l < k$), then the corresponding bipartite graphs will each have size l. It is clear that this construction preserves all adjacencies. □

The converse of Lemma 1 is not true. Chartrand et al. state that the graph H shown in Figure 3 is not a distance graph of $0-1$ sequences but assert that it may still be the distance graph of a set of stratified graphs [1]. In Figure 3, we present a set of distinctly labelled graphs of which it is the distance graph.

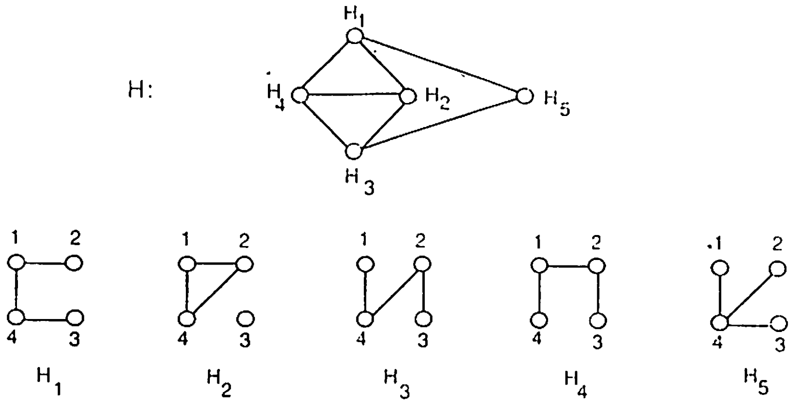


Figure 3

If a graph G can be realized as the distance graph of some set of $0-1$ sequences, it would be useful to have an upper bound for the minimum required length of the sequences. Given a distance graph G of $0-1$ sequences, we can think of the sequences as the rows in a matrix. To determine an upper bound for the minimum length of the sequences, it suffices to determine the number of non-constant columns in this matrix. We can ignore any constant columns while preserving all adjacencies among the sequences.

The following theorem and corollary use this idea to answer open questions posed by Chartrand et al. [1].

Theorem 2. *Let G be a connected graph of order $m \geq 2$. If G can be realized as the distance graph of some set of $0-1$ sequences, then the matrix of the sequences has at most $2m-2$ non-constant columns.*

Proof: Let G be a connected distance graph of some set of $0-1$ sequences. We will prove that the matrix of the sequences has at most $2m-2$ nonconstant columns by induction on m.

Since G is connected, we can label the vertices as $v_1, v_2, \ldots, v_m$ such that each v_i $(1 < i \leq m)$ is adjacent to some v_j $(j < i)$. Let s_i be the $0-1$ sequence that corresponds to vertex v_i. Suppose that $m = 2$. Then, v_1 is adjacent to v_2, and therefore s_1 and s_2 must differ in exactly two bits, and thus their matrix has $2 = 2m-2$ non-constant columns.

Suppose that the statement holds for k vertices. Then the matrix of sequences $s_1, s_2, \ldots, s_k$ has at most $2k-2$ non-constant columns.

Consider the vertex v_{k+1}. Since v_{k+1} is adjacent to at least one previous vertex v_j $(j < k+1)$ the sequence s_{k+1} differs from s_j in exactly two bits. So, s_{k+1} agrees with s_j in all but at most two of the constant columns. Therefore, at most two of the constant columns are no longer constant after adding the row s_{k+1}. Thus, by our inductive hypothesis, the matrix of sequences $s_1, s_2, \ldots, s_{k+1}$ has at most $(2k-2)+2 = 2(k+1)-2$ non-constant columns. □

Corollary 1. *Let G be a connected graph of order $m \geq 2$. If G can be realized as the distance graph of some set of $0-1$ sequences, then an upper bound for the minimum required weight of the sequences is $m-1$.*

Proof: Theorem 2 shows that the sequences require length at most $2m-2$, and what is accomplished by assigning 1's to more than half of these bits can be accomplished by simply switching the roles of the 0's and 1's. □

Theorem 2 gives a finite algorithm for determining whether a given graph can be realized as the distance graph of $0-1$ sequences. For example, given a graph of order m, we can check all possible $0-1$ sequences of length less than or equal to $2m-2$ with weight less than or equal to $m-1$. Using this idea, we looked for minimal forbidden induced subgraphs of distance graphs of $0-1$ sequences. There is a natural connection between distance graphs of $0-1$ sequences and line graphs; all distance graphs of $0-1$ sequences of weight two are isomorphic to line graphs. Thus, we looked at the nine minimal forbidden induced subgraphs of line graphs [3, p. 75]. We proved that exactly four of these are minimal forbidden induced subgraphs of distance graphs of $0-1$ sequences. See Figure 4.

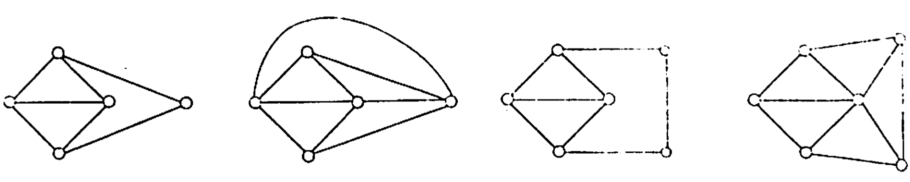


Figure 4

In addition, we proved that the complete bipartite graph $K_{2,3}$ is a minimal forbidden induced subgraph of distance graphs of $0-1$ sequences. Thus, except for $K_{1,n}$ and $K_{2,2}$, which can be realized as distance graphs of $0-1$ sequences through simple constructions, all complete bipartite graphs are unattainable as distance graphs of $0-1$ sequences. In contrast, Jarrett showed that all complete bipartite graphs are distance graphs of some set of graphs [4].

Several families of graphs have been shown to be the distance graphs of sets of $0-1$ sequences and therefore of sets of graphs. It is known that complete graphs, cycles, trees, unicyclic graphs, and line graphs are distance graphs of $0-1$ sequences [1]. It is conjectured by Chartrand et al. that all graphs are distance graphs of some set of graphs [2].

Next, we provide an additional family of graphs that are distance graphs of graphs.

Theorem 3. *For all n, the graph K_n minus an edge is the distance graph of a set of graphs.*

Proof: We may assume that $n \geq 5$. Construct $n-2$ disjoint copies of the cycle C_{n-2}. For convenience, we label the $n-2$ consecutive vertices of each cycle as $v_1, v_2, \ldots, v_{n-2}$. The labels are only considered for the purposes of attaching pendant edges and are then ignored. For each copy, attach two pendant edges to vertex v_1, no pendant edges to vertex v_2, and single pendant edges on the remaining $n-4$ vertices. Make these $n-2$ copies into distinct graphs $H_1, H_2, \ldots, H_{n-2}$ by adding a vertex v such that graph H_i has an edge that connects the vertex v_i to the vertex v. We distinguish v by attaching two pendant edges to it. Clearly, these $n-2$ graphs are mutually adjacent. Now let H_{n-1} be the graph obtained from the cycle C_{n-2} by attaching single pendant edges at all vertices and an edge connecting a vertex v (distinguished with two pendant edges) to one of the vertices of the cycle. Clearly H_{n-1} is adjacent to all previous graphs. Now construct the final graph H_n by attaching pendant edges to C_{n-2} exactly as those on graphs H_1 through H_{n-2}, but H_n has an edge connecting a vertex v (distinguished with two pendant edges) to an end vertex of one of the two pendant edges attached to vertex v_1. This graph H_n is adjacent to the first $n-2$ graphs, but it is not adjacent to graph H_{n-1}. Thus, we have constructed a set of graphs for which K_n minus an edge is the distance graph. □

Notice that we distinguished the vertex v in the above construction by attaching two pendant edges to it, but if we were considering K_n minus as edge as the distance graph of stratified graphs, we could distinguish v by coloring it a different color from the other vertices.

Generalizations of the construction employed in the proof of Theorem 3 could lead to the classification of more graphs as distance graphs of graphs or of stratified graphs.

4 Radius One Graphs

A *radius one graph* G is a graph that has at least one vertex v_c, which we call a *center*, such that every other vertex in G is adjacent to v_c. In this section, we determine which radius one graphs are distance graphs of $0-1$ sequences and which radius one graphs are distance graphs of graphs with distinctly labelled vertices. Our next theorem provides the motivation for our study of radius one graphs.

Theorem 4. *All graphs are distance graphs of some set of graphs (respectively, stratified graphs) if and only if all radius one graphs are distance graphs of some set of graphs (respectively, stratified graphs).*

Proof: The forward direction is obvious, so we consider the reverse direction. Suppose that every radius one graph is a distance graph of some set of graphs. Then, for any graph H, $H \vee K_1$ (where $\vee$ denotes the join of two graphs) is a distance graph of some set of graphs. Since every induced subgraph of $H \vee K_1$ is also a distance graph, H is a distance graph of some set of graphs. □

We now introduce some notation. Choose positive integers m and n. Let s be the $0-1$ sequence of length $m+n$ with the first m bits equal to 1 and the next n bits equal to 0. All sequences that are adjacent to s are obtained by the interchange of a 1 and a 0, and each can be denoted as $s_{i,j}$ to represent the sequence obtained by interchanging a 1 from the ith bit position of s (for $i \in \{1, 2, \ldots, m\}$) with a 0 from the jth bit position of s (for $j \in \{m+1, m+2, \ldots, m+n\}$). We define the graph S to have vertex set $\{s_{i,j} \mid i \in \{1, 2, \ldots, m\}, j \in \{m+1, m+2, \ldots, m+n\}\}$. Clearly, all vertices in S are $0-1$ sequences that are adjacent to s. Since all vertices in S are sequences of the same weight, two vertices are adjacent if they differ in exactly two bits.

Notice that for a fixed $i_0 \in \{1, 2, \ldots, m\}$, we can define an induced subgraph of S as $S_{i_0} = \{s_{i_0,j} \mid j \in \{m+1, m+2, \ldots, m+n\}\}$. All n vertices in S_{i_0} are mutually adjacent, and thus S_{i_0} is isomorphic to K_n. Similarly, we can define an induced subgraph S_{j_0} that is isomorphic to K_m.

Lemma 2. *The graph S is isomorphic to $K_m \times K_n$.*

Proof: For simplicity, we denote $V(K_m) = \{1, 2, \ldots, m\}$ and denote $V(K_n) = \{m+1, m+2, \ldots, m+n\}$. Then, $V(K_m \times K_n) = \{(i, j) \mid i \in V(K_m), j \in V(K_n)\}$. Now, let $\Phi\colon S \to K_m \times K_n$ be defined by $\Phi(s_{i,j}) = (i, j)$ where $i \in \{1, 2, \ldots, m\}$ and $j \in \{m+1, m+2, \ldots, m+n\}$. The mapping Φ is clearly a bijection. Two elements $(i, j), (i', j') \in K_m \times K_n$ are adjacent if $i = i'$ and $j \neq j'$ or $j = j'$ and $i \neq i'$. Two elements in S are adjacent if the sequences differ in exactly 2 bits. Two sequences $s_{i,j}$ and $s_{i',j'}$ differ in 0 bits among the first m bits if $i = i'$, and they differ in exactly 2 bits among the first m bits if $i \neq i'$. Similarly for the last n bits. Thus, $s_{i,j}$ and $s_{i',j'}$ differ in a total of 2 bits if either $i = i'$ and $j \neq j'$ or $i \neq i'$ and $j = j'$. Thus, Φ preserves adjacencies, and the proof is complete. □

Theorem 5. *Given a radius one graph G with a center vertex v_c, let $G_c = G - \{v_c\}$. Then G is a distance graph of a set of $0-1$ sequences of length $m+n$ and weight m if and only if G_c is an induced subgraph of $K_m \times K_n$.*

Proof: First, suppose that G_c is an induced subgraph of $K_m \times K_n$. Construct a sequence s with bits 1 through m equal to 1, and bits $m+1$ through $m+n$ equal to 0. This sequence s will correspond to v_c.

By Lemma 2, all elements in G_c can be represented by a vertex of S (as defined above) such that all adjacencies are preserved. Furthermore, all sequences corresponding to elements in G_c are adjacent to s. Thus, G is the distance graph of some set of $0-1$ sequences of length $m+n$ and weight m.

Next, suppose that G is a distance graph of $0-1$ sequences of length $m+n$ and weight m. Reorder all columns of all sequences such that the sequence s which corresponds to v_c has bits 1 through m equal to 1, and bits $m+1$ through $m+n$ equal to 0. Since all other sequences are adjacent to s, they must be vertices from the graph S. Thus, G_c is an induced subgraph of S, and it now follows from Lemma 2 that G_c is an induced subgraph of $K_m \times K_n$. □

Corollary 2. *Given a radius one graph G with a center vertex v_c, let $G_c = G - \{v_c\}$. Then G is a distance graph of a set of $0-1$ sequences of length $m+n$ and weight m if and only if G_c is a line graph of a bipartite graph.*

Proof: We will need the following facts. First, the line graph of the complete bipartite graph $K_{m,n}$ is isomorphic to $K_m \times K_n$. Second, induced subgraphs of the line graph $L(H)$ of some graph H are isomorphic to the line graphs of subgraphs of H. Since subgraphs of $K_{m,n}$ are bipartite graphs, it follows that line graphs of bipartite graphs are isomorphic to induced subgraphs of $K_m \times K_n$ With these results, the corollary follows directly from Theorem 5. □

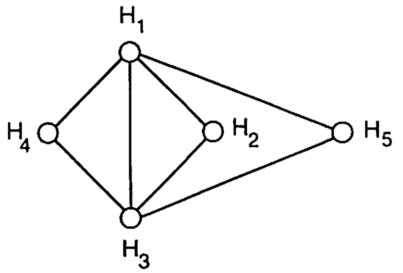
As a result of Theorem 5, the following graphs are unattainable as distance graphs of $0-1$ sequences: $K_1 \vee C_{2l+1}$ for $l \geq 2$, $K_1 \vee K_{1,1,2}$, and $K_1 \vee K_{1,3}$.

In fact, C_{2l+1} for $l \geq 2$, $K_{1,1,2}$, and $K_{1,3}$ are exactly the minimal forbidden induced subgraphs of line graphs of bipartite graphs [6]. Notice that all nine of the minimal forbidden induced subgraphs of line graphs contain some of the above as induced subgraphs. For instance, $K_1 \vee K_{1,1,2}$ and $K_1 \vee C_{2l+1}$ for $l = 2$ are in the list of minimal forbidden induced subgraphs of line graphs that are also minimal forbidden induced subgraph of $0-1$ sequences. See Figure 4.

Corollary 3. *Given a radius one graph G with a center vertex v_c, let $G_c = G - \{v_c\}$. If G_c is an induced subgraph of $K_m \times K_n$, then G is a distance graph of a set of distinctly labelled graphs.*

Proof: Suppose that G_c is an induced subgraph of $K_m \times K_n$. Then G is a distance graph of a set of $0-1$ sequences by Theorem 5. It follows that G is a distance graph of a set of distinctly labelled graphs by Lemma 1. □

The converse of Corollary 3 is false. As a counterexample, we proved that $K_{1,1,3}$ is the distance graph of a set of graphs with distinctly labelled vertices, but, as stated above, $K_{1,3}$ is not an induced subgraph of $K_m \times K_n$. See Figure 5.



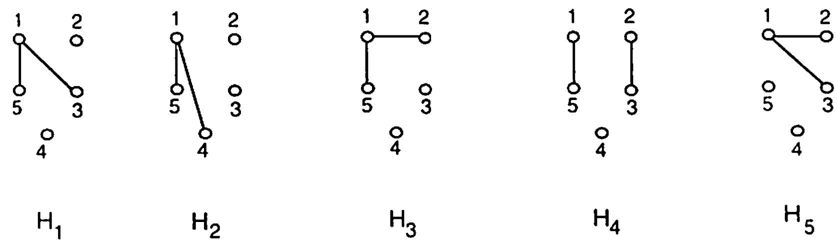


Figure 5

We now introduce some additional notation. Let E be a subset of the vertex set of the line graph of a complete graph, $V(L(K_n))$, and let $G(E)$ be the subgraph of $L(K_n)$ induced by the vertices in E. Let E^C be the complement of E, so $E^C = V(L(K_n)) \setminus E$. Then, $G(E^C)$ is the subgraph of $L(K_n)$ induced by the vertices in E^C.

Define $R(E)$ as the graph with vertices (v, w) such that $v \in E$ and $w \in E^C$, where v is adjacent to w in $L(K_n)$. Define the adjacencies in $R(E)$ as follows: (v_i, w_j) is adjacent to $(v_{i'}, w_{j'})$ if $v_i = v_{i'}$ and w_j is adjacent to $w_{j'}$ or $w_j = w_{j'}$ and v_i is adjacent to $v_{i'}$. Clearly by these adjacencies, $R(E)$ is an induced subgraph of $L(K_n) \times L(K_n)$. In particular, $R(E)$ is an induced subgraph of $G(E) \times G(E^C)$.

Now, let H be any graph with n distinctly labelled vertices. Then the line graph $L(H)$ is an induced subgraph of $L(K_n)$. Let $E = E(H)$, where $E(H)$ denotes the edge set of H. All graphs that are adjacent to H can be obtained by a single edge rotation. We denote any graph that is adjacent to H by $H(v_i, w_j)$ where the edge $v_i \in E$ is rotated to the position $w_j \in E^C$ (an edge that is not in H). Clearly, v_i must be adjacent to w_j in $L(K_n)$. We define the graph A_H as the graph with vertex set $\{H(v_i, w_j) \mid v_i \in E, w_j \in E^C$, and v_i is adjacent to w_j in $L(K_n)\}$. So, the vertices in A_H are distinctly labelled graphs that are adjacent to H. Two vertices in A_H are adjacent if the corresponding graphs are adjacent.

Lemma 3. *Let H be a graph with distinctly labelled vertices. Then, for $E = E(H)$, the graph $R(E)$ is isomorphic to the graph A_H.*

Proof: Let Φ: $R(E) \to A_H$ be given by $\Phi(v_i, w_j) = H(v_i, w_j)$ where $v_i \in E$, $w_j \in E^C$, and v_i is adjacent to w_j in $L(K_n)$. Clearly, Φ is a bijection. By the adjacencies defined for $R(E)$, two elements (v_i, w_j) and $(v_{i'}, w_{j'})$ are adjacent if $v_i = v_{i'}$ and w_j is adjacent to $w_{j'}$ or $w_j = w_{j'}$ and v_i is adjacent to $v_{i'}$. Two vertices $H(v_i, w_j)$ and $H(v_{i'}, w_{j'})$ of A_H are adjacent if one of the corresponding graphs can be obtained from the other by a single edge rotation. This clearly implies that they are adjacent if $v_i = v_{i'}$ and w_j is adjacent to $w_{j'}$ or $w_j = w_{j'}$ and v_i is adjacent to $v_{i'}$. Thus Φ preserves adjacencies, completing the proof. □

Theorem 6. *Given a radius one graph G with a center vertex v_c, let $G_c = G - \{v_c\}$. Then, G is a distance graph of a set of distinctly labelled graphs if and only if G_c is isomorphic to an induced subgraph of $R(E)$ for some set $E \subset V(L(K_n))$.*

Proof: First suppose that G_c is isomorphic to an induced subgraph of $R(E)$ for some set $E \subset V(L(K_n))$. Construct a graph H such that $E(H) = E$. Then H has size at least 1, and has no isolated vertices. The graph H will correspond to v_c.

By Lemma 3, G_c is isomorphic to an induced subgraph of A_H. So, all vertices of G_c can be represented by a distinctly labelled graph such that all adjacencies are preserved, and all graphs corresponding to vertices in G_c are adjacent to H. Thus, G is the distance graph of some set of distinctly labelled graphs.

Next, suppose that G is the distance graph of a set $\mathcal{G}$ of distinctly labelled graphs. Then, there exists a graph $H \in \mathcal{G}$ that corresponds to v_c. The line graph $L(H)$ is an induced subgraph of $L(K_n)$, where n is equal to the order of H. Choose the set $E \subset L(K_n)$ such that $E = E(H)$. Then, all graphs adjacent to H can be represented by vertices of the graph A_H. So, G_c is an induced subgraph of A_H, and by Lemma 2, G_c is an induced subgraph of $R(E)$. □

Corollary 4. *Let G be a graph of radius one with a center vertex v_c, and let $G_c = G - \{v_c\}$. If G_c is not an induced subgraph of $L(K_n) \times L(K_n)$, then G is not a distance graph of a set of graphs with distinctly labelled vertices.*

Proof: This follows directly from Theorem 6 using the fact that $R(E)$ is an induced subgraph of $L(K_n) \times L(K_n)$. □

We now have a way to determine which graphs can not be obtained as the distance graph of a set of graphs with distinctly labelled vertices. For example, the graph $K_{1,1,5}$ is not attainable as such a distance graph, since $K_{1,5}$ is not an induced subgraph of $L(K_n) \times L(K_n)$. Of course, it may still be attainable as the distance graph of some set of stratified graphs where some vertices are assigned the same color.

The complete classification of radius one graphs which are distance graphs of $0-1$ sequences and which are distance graphs of graphs with distinctly labelled vertices provides insight for the question of which graphs can be attained as distance graphs.

5 Further Research

In order to obtain a classification for those radius one graphs that are distance graphs of graphs with distinctly labelled vertices, we introduced a graph called $R(E)$. Two open problems are to determine exactly which graphs are isomorphic to an $R(E)$ for some set E and which graphs are minimal forbidden induced subgraphs of $R(E)$ for all E.

It remains an open question to see which additional graphs can be realized as distance graphs when stratification plays a significant role. I am interested to know which graphs are distance graphs when exactly one of the $k > 1$ color classes has more than one vertex. Would this stratification scheme be sufficient to obtain all graphs as distance graphs? Must we consider unlabelled graphs (that is, graphs with all vertices in the same color class) in order to realize all graphs as distance graphs?

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