Generator-Preserving contractions and a Min-Max result on the graphs of planar polyminoes

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ABSTRACT. In this paper, we deal with the *convex generators* of a graph $G = (V(G), E(G))$. A *convex generator* being a minimal set whose convex hull is $V(G)$, we show that it is included in the "boundary" of G. Then we show that the "boundary" of a polymino's graph, or more precisely the *seaweed's* "boundary", enjoys some nice properties which permit us to prove that for such a graph G, the minimal size of a convex generator is equal to the maximal number of hanging vertices of a tree T, obtained from G by a sequence of *generator-preserving* contractions.

1 Introduction

In order to make use of the geometric notion of convexity outside the context of euclidian spaces, one need to define an abstract *convexity .space.* A *convexity space* is a pair $(V, \mathcal{C})$, where V is a set and $\mathcal{C}$ a collection of V-subsets, called the *convex sets*, such that:

(c.1) $\emptyset$ and V are convex;

(c.2) the arbitrary inersection of convex sets is convex;

(c.3) the union of any family of convex sets totally ordered by inclusion is convex.

For any $A \subseteq V$ the *convex hull* of A is $\varphi(A) = \{C \mid A \subseteq C, C \in \mathcal{C}\}$. We will deal with convexity in graphs. For generalities about graphs we refer [4]. Given a connected graph $G = (V(G), E(G))$ we define the *interval-function* $I\colon V(G) \times V(G) \to 2^{V(G)}$ such that for any $x, y \in V(G), I_G(x, y)$ is the set of vertices of all shortest paths between x and y. We write $I(x,y)$ instead of $I_G(x,y)$ when there is no ambiguity. A subset $C \subseteq V(G)$ is convex if $I(x,y) \subseteq C$, for any $x,y \in C$. It is easy to see that the sets defined this way satisfy c.1, c.2 and c.3 thus, giving a convexity space, called *geodesic* convexity, widely used by many authors (see [11]). This is the convexity we use throughout this paper. There are several other ways to define intervals in a graph, but we will not treat this topic here. The interested reader can find good results and references in [6]. The interval notion is interesting because: firstly, it translates naturally the notion of the interval from the euclidian geometry and secondly, it gives an iterative way to calculate the convex hull. More precisely, for any subset A of $V(G)$ we have:

$$\varphi(A) = \bigcup_{k=0}^{\infty} I^k(A) \text{ where } I^0(A) = A \text{ and } I^{k+1}(A) = \bigcup_{x,y \in I^k(A)} I(x,y)$$

A *minimal* subset of vertices B of G such that $\varphi(B) = V(G)$ is called a *convex generator* of G.

Given a connected graph G, it is clear that we can always obtain the tree containing a single edge, by contracting a set of edges of G. The graph obtained from G by contracting a set $A \subseteq E(G)$ is denoted G/A. Contracting an edge xy of G means identifying x with y and omitting the loop created. If we allow only the *generator-preserving contractions*, then we may expect to find a tree T which reflects somehow the structure of G. This is indeed the case and we show that for any generator B of G and for any tree T obtained from G by a sequence of *gp*-contractions we have $|B| \geq |d_1(T)|$, where $d_1(T)$ is the set of hanging vertices of T. We can interpret this as a min-max inequality and the natural question is whether there are graphs for which the equality is attained. We show that by taking the generator-preserving contractions to be those obtained by contracting the edges of *Djokovic classes* the equality is attained for the *seaweeds* which are a subclass of *median* graphs. We have studied the seaweeds in [9] in the frame of image compression.

2 Generator-contractible graphs

All the graphs considered subsequently are connected. Given the graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, $f\colon V(G) \to V(H)$ is a *contracting* map if for any $x,y \in V(G)$ we have $d_G(x,y) \geq d_H(f(x), f(y))$. Here $d_G(x,y)$ denotes the *distance* between x and y that is, the number of edges of any shortest path between x and y. Using this notation we have $I_G(x,y) = \{z \in V(G) \mid d_G(x,y) = d_G(x,z) + d_G(z,y)\}$. We are interested only on those contracting maps obtained by the contraction of sets of edges.

It is clear that $f\colon \{a,b,c,d\} \to \{x,y,z\}$ such that $f(a) = x$, $f(c) = z$

and $f(b) = f(d) = y$ (see figure 1) is a contracting map but it can not be obtained by the contraction of any set of edges of C_4.

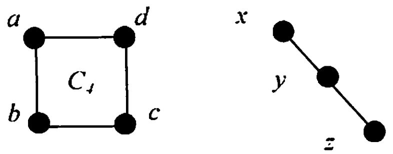


Figure 1

From now on we will write "generator" instead of "convex-generator". A graph G is *generator-contractible* onto H ($GgcH$) if there is a sequence of graphs $G = G_0, \ldots, G_p = H$ such that for any $i < p$ and for any generator B of G_i there is an onto contracting map $f\colon V(G_i) \to V(G_{i+1})$ such that $\varphi(f(B)) = V(G_{i+1})$. We denote by $\beta(G)$ the set of generators of G and the *g-rank* of G is $g(G) = \min\{|B| \mid B \in \beta(G)\}$. We will denote by $\theta(G)$ the set of trees T such that $GgcT$.

Lemma 1.2. *For any graph G we have $g(G) \geq \max\{|d_1(T)| \mid T \in \theta(G)\}$.*

Proof: Using the easy remark that any set A whose convex hull is $V(G)$ contains a convex generator, the result follows easily by induction on the length of the sequence of graphs $G = G_0, \ldots, G_p = T$ needed to contract G onto T. □

We show now that the generators of graphs and the transversals of hypergraphs are closely related. Given a graph $G = (V(G), E(G))$, we define the hypergraph $H_G = (V(H_G), E(H_G)$:

- $V(H_G) = V(G)$;
- $A \in E(H_G)$ if and only if $A \neq \emptyset$ and $V(G) - A$ is a maximal convex set in G.

With these notations we have the following:

Proposition 1.2. *B is a generator of G if and only if B is a transversal of H_G.*

Proof: Let B be a generator of G. Assume on the contrary that B is not a transversal of H_G. Then there is an edge A of H_G such that $A \cap B = \emptyset$. The fact that $V(G) - A$ is convex implies $B \subseteq \varphi(B) \subseteq V(G) - A \neq V(G)$, which is impossible.

Let now B be a transversal of H_G. Assume on the contrary that B is not a generator of G, that is $\varphi(B) \neq V(G)$. Clearly, $\varphi(B)$ is convex and $B \subseteq \varphi(B)$. This implies that $A = V(G) - \varphi(B)$ is an edge of H_G and $A \cap B = \emptyset$, which is impossible. □

Remark 2: Given the hypergraph $H = (V(H), E(H))$, with $V(H) = \{a, b, c, d\}$ and $E(H) = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{a, c\}\}$, one can easily verify that for any graph G on the vertices $\{a, b, c, d\}$, the hypergraph H_G defined above is different from H. This remark suggest the following:

Problem 1: Given a hypergraph H, under what conditions there is a graph G such that H_G is isomorphic to H?

Remark 3: The edges of H_G are what is left outside the maximal convex sets of G. Thus, it makes sense to consider these edges as parts of a somewhat "boundary" of G. So, the preceding proposition says that the generators of G are included in the "boundary" of G.

In the following section we will show that it is "easy", for a *median* graph G, to find generator preserving contractions and the "boundary" of G.

3 Median graphs

Introduced by Avann [1], these class of graphs have been intensively studied by many authors. The interested reader can find recent contributions in [2], [3], [9], [10], [11], [12], [13].

A graph G is *median* if for any $x, y, z \in V(G)$, $I(x, y, z) = I(x, y) \cap I(y, z) \cap I(z, x)$ is a singleton, or equivalently, there is a unique vertex $m = m_G(x, y, z)$, the *median* of x, y and z in G, such that: $d(x, y) = d(x, m) + d(m, y)$, $d(y, z) = d(y, m) + d(m, z)$ and $d(z, x) = d(z, m) + d(m, x)$.

All median graphs are bipartite. Trees are median graphs. Another important example of median graph is the *hypercube Q_n of dimension n*, having the $\{0, 1\}$-vectors of length n as vertices, two vertices being joined if they differ in exactly one coordinate. The 6-cycle and the graph $K_{2,3}$ are not median graphs.

Mulder [11] has shown that the median graphs are included in the class of graphs *isometrically-embeddable* in the hypercube. We will give the precise statement of this result later in this section after having introduced some terminology and the chracterisation of the graphs *isometrically-embeddable* in the hypercube. A graph G is *isometrically-embeddable* in a graph H if there is a map $f: V(G) \to V(H)$ such that $d_G(x, y) = d_H(f(x), f(y))$ for any pair of vertices $x, y \in V(G)$. If $V(G) \subseteq V(H)$ then G is an *isometrical subgraph* of H. The subgraph of G *induced* by the set $X \subseteq V(G)$, is the graph $G[X] = (X, E_1)$ such that $E_1 = \{uv \in E(G) \mid u, v \in X\}$. A *convex-subgraph* of G is the graph $G[C]$ induced by the convex set C. A convex-subgraph is always isometrical, the converse is not true in general.

For any edge ab of G we define the sets $Nab = \{x \in V(G) \mid d(x, a) < d(x, b)\}$, $Nba = \{x \in V(G) \mid d(x, b) < d(x, a)\}$ and $aNb = \{x \in V(G) \mid d(x, a) = d(x, b)\}$ that partition $V(G)$. It is clear that if G is *bipartite* then $aNb = \emptyset$. Unless explicitly stated, all the graphs encountered hereafter will be bipartite. Djokovic [5] has defined the relation δ on the edges of a bipartite graph as follows: if $e, f \in E(G)$, $e = ab$ and $f = xy$ then,

$$e\delta f \leftrightarrow \{x,y\} \cap Nab \neq \emptyset \text{ and } \{x,y\} \cap Nba \neq \emptyset.$$

The relation δ is always reflexive and symmetric but, in general, it is not transitive as one can easily verify for the graph $K_{2,3}$. We will use the terms *Djokovic classes* or δ-classes when δ is an equivalence relation. It is shown in [5] and [8] that δ is *transitive if and only if Nab and Nba are convex for any edge ab of G*. Djokovic has shown:

Theorem 1.3. [5]: *G is isometrically-embeddable in a hypercube Q if and only if G is bipartite and Nab and Nba are convex for any $ab \in E(G)$.*

Now we can state the following result of Mulder:

Theorem 2.3. [11]: *G is median if and only if it is an isometric subgraph of a hypercube Q such that the median in Q of any three vertices of G is also a vertex of G.*

Figure 2 illustrates the fact that the 6-cycle C_6 (the bold edges in the figure) is an isometric subgraph of Q_3 but, the median m in Q_3 of x, y and z is not in C_6 thus, showing once more that C_6 is not a median graph.

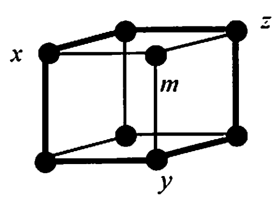


Figure 2. C_6 is an isometric subgraph of Q_3

For a graph isometrically embeddable in a hypercube, we will note by *Dab* the δ-class containing the edge *ab* and *Bab* the set of vertices of *Nab* having a neighbour in *Nba*. Here we list some properties of median graphs, shown in [11], that we will need subsequently.

Theorem 3.3. *If G is a median graph then:*

(m.1) *For any edge ab of G the sets Nab, Nba, Bab and Bba are convex;*

(m.2) *The equivalence class Dab is a cut and a matching between Bab and Bba and the mapping $f\colon Bab \to Bba$, defined by $f(x) = x'$ whenever $xx' \in Dab$ is an isomorphism between $G[Bab]$ and $G[Bba]$.*

(m.3) *The graph $G' = G/Dab$ is median for any edge ab of G.*

One can find the proof of the following results in [9] and [12]:

Proposition 1.3. *If G is a median graph then:*

(i) *for any edge $ab \in E(G)$, G is generator-contractible onto G/Dab;*

(ii) *$C \neq V(G)$ is a maximal convex set if and only if there is an edge $uv \in E(G)$ such that $C = Nuv$ and $Nvu = Bvu$.*

4 Polyminoes' graphs and seaweeds

Golomb [7] defined the polyminoes as *shapes made by connecting certain number of equalsized squares, each joined together with at least one other square along an edge.* Without loss of generality, the squares of this definition can be taken to be the *unit squares* of the plane, that are the sets $K = [a, a'] \times [b, b']$ with $a' = a \pm 1$, $b' = b \pm 1$ and a, b are integers. Of course, all the pixels images are polyminoes. Given a set A of unit squares we construct the graph $G^*(A)$ by taking the centers of the squares of A as its vertices, two vertices being joined by an edge if and only if the corresponding squares have eactly one edge in common. Thus we can define a *plane polymino* as a set π of unit squares such that $G^*(\pi)$ is connected.

Let now define the *gluing operation* on the graphs. Let G_1 and G_2 be any two graphs. Let $f\colon V(H_1) \to V(H_2)$ be an isomorphism between an induced subgraph H_1 of G_1 and an induced subgraph H_2 of G_2. We can obtain a new graph G by *gluing G_1 with G_2 along H_1* and H_2 as follows: 1. Delete all the edges of H_1 from G_1; 2. Identify all the vertices x of H_1 with their images $f(x)$ of H_2.

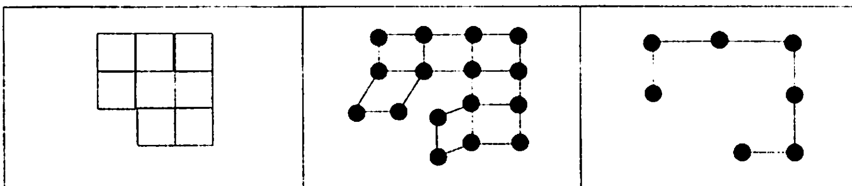


Fig. 3.a A polymino π. | Fig. 3.b The graph $G(\pi)$. | Fig. 3.c The graph $G^*(\pi)$.

Given a polymino π we construct the *polyminos' graph* $G(\pi)$ by associating a 4-cycle $C(K)$ to any square K of π and we glue two such cycles $C(K_1)$ and $C(K_2)$ along an edge if and only if the corresponding squares K_1 and K_2 have that edge in common (see figure 3).

We deal with the polyminoes without holes that are those polyminoes π for which $G^*(\pi)$ has no isometric cycles of length > 4 (see [9]). As a natural generalisation of the polyminoes without holes, we define the class $\mathbf{W}$ of the *seaweeds* by the following inductive procedure:

Step 0: $\mathbf{W} = \{C_4\}$[1]

Step 1: For any $G_1, G_2 \in \mathbf{W}$, let G be the graph obtained by gluing G_1 with G_2 along P_1 and P_2 verifying:

[1] C_4 is the cycle of length 4

(a) P_1 and P_2 are convex paths of the same length ≥ 1 of G_1 and G_2;
(b) Any edge of P_1 (P_2) is contained in exactly one 4-cycle of G_1 (G_2);
(c) Every interior vertex of P_1 (P_2) is of valency 3 in G_1 (G_2).

Step 2: Add G to **W**. Goto **Step 1.**

We have shown [9] that if π is a polymino without holes, then $G(\pi)$ is a seaweed. We have chracterized the seaweeds and we have shown that we can recognize them and the graphs of plane polyominoes in polynomial time $O(mn)$, where m is the number of edges and n is the number of vertices. Here we list some properties of the seaweeds needed for the next section. The detailed proofs of these properties can be found in [9].

Theorem 1.4. *If G is a seaweed then:*

(1) *G is a median graph, for any edge uv of G the subgraph $G[Buv]$ is a convex path and the number of 4-cycles containing uv is 1 or 2.*

(2) *for any 4-cycle C_0 of G and for any unit square K_0 in the plane, there is an unique embedding $\mu\colon V(G) \to Z^3$ (Z is the set of integers) verifying:*

  (i) *the projection of $\mu(C_0)$ on Z^2 is the boundary of K_0;*
  (ii) *for any 4-cycle C of G, the projection of $\mu(C)$ on Z^2 is the boundary of a unit square K;*
  (iii) *if the 4-cycles C_1 and C_2 of G have one edge in common then the projections K_1 and K_2 of $\mu(C_1)$ and $\mu(C_2)$ on Z^2 have exactly one edge in common.*

For a fixed embedding μ of G, an edge xy is *horizontal (vertical)* if the projection of $\mu(xy)$ on Z^2 is horizontal (vertical). As a seaweed G is a median graph, all the edges of the hypergraph H_G will be of the form Buv (proposition 1.3), where uv is some edge of G. By the preceding theorem, the subgraph $P = G[Buv]$ is a convex path that we call an *extremal segment* of G. Let ES be the set of all extremal segments of G, that is the set of all G's subgraphs, induced by the vertices of the edges of H_G. In [9] we have shown:

Lemma 1.4. *Let $P \in ES$ and let e be an edge of P. If e is horizontal (vertical), then all the edges of P and Duv are horizontal (vertical).*

This lemma allows us to classify the segments of ES in vertical and horizontal ones.

Lemma 2.4. *For any $P, P' \in ES$ we have $|V(P) \cap V(P')| \leq 1$ and when $V(P) \cap V(P') = \{x\}$, then x is an extremity of P and P' and either P is horizontal and P' vertical, or P is vertical and P' horizontal.*

5 A min-max result on seaweeds

Given a seaweed G, we fix one of its embeddings given by theorem 1.4. Let R be a maximal *chain* of extremal segments, that is a maximal sequence $P_1, \ldots, P_k$ of segments of ES such that $|V(P_i) \cap V(P_{i+1})| = 1$ for $i = 1, \ldots, k-1$. We will write "chain" for "maximal chain". It is clear from the lemma 2.4 that in any chain the horizontal segments alternate with the vertical ones. Let HH (VV) be the set of all the chains whose first and last segments are horizontal (vertical). We note by HV the set of all the other chains. It is clear that any segment is included in exactly one chain. For any chain R we denote with $h(R)$ ($v(R)$) the number of its horizontal (vertical) segments. With these notations we have:

Lemma 1.5. *If G is a seaweed, then $g(G) = \sum_{R \in HH} h(R) + \sum_{R \in VV} v(R) + \sum_{R \in HV} h(R)$.*

Proof: By proposition 1.2, we need to find a minimum size transversal of H_G. It is clear that for any hypergraph a minimum size transversal is the union of minimum size transversals of its connected components. The connected components of H_G are the chains, defined above, contained in the sets HH, HV and VV, which form a partition of ES. For any chain R we denote by 1 and 2 the extremities of its first segment, 2 and 3 the extremities of its second segment and so on. Let now $\{1, 2, \ldots, k\}$ be the set of these extremities for a chain R. It is easy to see that if R is in VV (in HH) then k is even and the set $\{1, 3, \ldots, k-1\}$ is a minimum size transversal of R and $|\{1, 3, \ldots, k-1\}| = v(R)$ ($|\{1, 3, \ldots, k-1\}| = h(R)$). If R is a chain of HV then k is odd and the set $\{2, 4, \ldots, k-1\}$ is a minimum size transversal of R and the equalities $|\{2, 4, \ldots, k-1\}| = h(R) = v(R)$ are verified. This shows the lemma. □

Theorem 1.5. *If G is a seaweed then $g(G) = \max\{|d_1(T)| \mid T \in \theta(G)\}$.*

Proof: Let B be the minimum generator of G found in the preceding lemma. By the lemma 1.2 we have $|B| \geq |d_1(T)|$ for any tree T obtained from G by a sequence of generator-preserving contractions. We will find a sequence of δ-classes $D_1, \ldots, D_q$, each containing at least two edges, and a sequence of graphs $G = G_0$, $G_i = G_{i-1}/D_i$, $i = 1, \ldots, q$, such that G_q is a tree T and $|B| = |d_1(T)|$.

Let R be a chain of HH and $\{1, 3, \ldots, k-1\}$ a minimum size transversal of R (see the proof of the preceding lemma.) We note by $P_1, \ldots, P_k$, the horizontal segments of this chain. Let $D_1, \ldots, D_r$ be the δ-classes defined by the edges of $E(P_1) \cup \cdots \cup E(P_{k-1})$. We have $|D_i| \geq 2$, because any edge of G is contained in at least one 4-cycle (theorem 1.4.). Let G_r be the graph obtained from G by contracting $D_1, \ldots, D_r$. The order of contractions is insignificant, because the contraction is a commutative operation. It is clear that after these contractions, every horizontal segment of R is reduced into a vertex of valency one in G_r. Thus, $|d_1(G_r)| \geq h(R)$. We write "$\geq$" because some other horizontal segments of G could have been reduced into vertices of valency one in G_r.

Let now, $D_1, \ldots, D_h$ be the δ-classes defined by the edges of *all* the horizontal segments of *all* the chains in $HH \cup HV$. Let G_h be the graph obtained from G by contracting these classes. By repeating the preceding argument for all the chains in $HH \cup HV$, we have $|d_1(G_h)| \geq \sum_{R \in HH \cup HV} h(R)$.

Remember that G_h is median ((*m*.3) th. 3.3) and all the contracted edges up to now were horizontal (lemma 1.4.)

Let now P be any vertical segment of a chain $R \in VV$. Assume that P is defined by the edge uv of G, that is $P = G[Buv]$, $Buv = Nuv$ (see figure 4.) Let a and a' be the extremities of P. The class Duv is not in $\{D_1, \ldots, D_h\}$ otherwise, at least one of the edges ab or $a'b'$ is contained in a horizontal segment P' of a chain $R' \in HH \cup HV$. But then, the segment P is contained in R and in R', which is impossible by the maximality of chains. Thus, the vertical segments of the chains VV of G are not altered by the contraction of $D_1, \ldots, D_h$. Let $D_{h+1}, \ldots, D_v$ be the δ-classes defined by *all* the edges of *all* the vertical segments of the chains in VV.

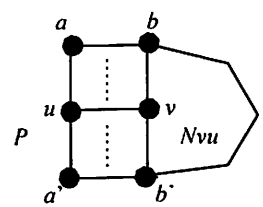


Figure 4

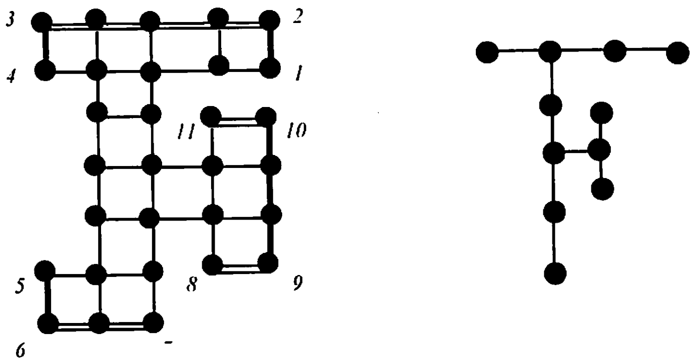
It is easy to deduce by what precedes, that all these classes have at least 2 edges in G_h. Thus, if x is a vertex of valency 1 in G_h and xy is the edge of G_h incident to x, then xy is not included in any of D_i, $i = h+1, \ldots, v$. This implies that the vertices of $d_1(G_h)$ remain of valency 1 in the graph G_v obtained from G_h by the contraction of $D_{h+1}, \ldots, D_v$.

All the vertical segments of the chains VV being reduced in vertices of valency 1 in G_v, we have $|d_1(G_v)| \geq |d_1(G_h)| + \sum_{R \in VV} v(R) \geq |B|$.

If G_v contains any cycle, this will be of even length because G_v is median and so, bipartite. From this follows that anyone of the δ-classes $D_{v+1}, \ldots, D_q$ defined by the edges of all the cycles of G_v contains at least two edges. Thus, the vertices of $d_1(G_v)$ remain of valency 1 in the tree G_q

obtained from G_v by the contraction of $D_{v+1}, \dots, D_q$. This implies that $|d_1(G_q)| \geq |d_1(G_v)| \geq |B|$, and the proof is complete. □

The following example illustrates the constructive proof of the preceding theorem.



(a) G is a seaweed (b) A tree obtained from G by contracting δ-classes

Figure 5

In this figure, the vertical (horizontal) segments are presented in bold (double line).There are three chains: one chain $R \in VV$ containing the segments P_1 with extremities 1 and 2 and we write $P_1(1,2)$ and similarly, $P_2(2,3)$ and $P_3(3,4)$; one chain $R' \in VH$ containing the segments $P_4(5,6)$ and $P_5(6,7)$; one chain $R'' \in HH$ containing the segments $P_6(8,9)$, $P_7(9,10)$ and $P_8(10,11)$. It is easy to find the transversals $\{1,3\}$ for R, $\{6\}$ for R' and $\{8,10\}$ for R''. A minimum generator of G is then $B = \{1,3,6,8,10\}$ thus, $g(G) = 5$. The contraction of three classes defined by the edges of the horizontal segments of R' and R'' and the contraction of the classes defined by the edges of the vertical segments of R gives a graph which has 5 vertices of valency 1, but is not yet a tree. It suffices then to contract one δ-class defined by the edges of the cycle of this graph to obtain the tree in 5.b.

Remark 4: In the proof of the preceding theorem we found some very special generator-preserving contractions, obtained by the contraction of δ-classes. This is due to the structure of the seaweeds which are a special subclass of median graphs. One may wonder whether these contractions work for the graphs of 3-dimensional polyminoes or the median graphs in general. The answer is negative for both the cases, as one can verify for the graph of a 3-dimensional polyminoe π, such that $G^*(\pi)$ is drawn in the figure 6. A minimum generator of this median graph is $\{a,b,c\}$.

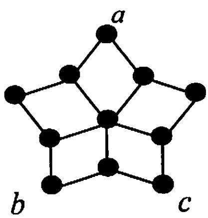


Figure 6. The graph G^* of a 3-dimensional polyminoe

It is not hard to verify that for all the trees T obtained by contracting δ-classes on this graph, we have $|d_1(T)| \leq 2$ thus showing the falsity of the following conjecture of Mulder [12] saying:

Mulder's Conjecture: *If G is a median graph then* $g(G) = \max\{|d_1(T)| \mid T \in \theta(G)\}$.

This remark suggests the following:

Problem 1: Chracterize the median graphs for which the min-max equality can be obtained by contracting only δ-classes. We conjecture that these graphs are the class of seaweeds obtained by the procedure in 4, with the step 1 (a) relaxed as follows:

Step 1: (a') P_1 and P_2 are convex paths of the same length ≥ 0 of G_1 and G_2;

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