Maximum packings of $K_v - K_u$ with triples

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ABSTRACT. We construct, for all positive integers u and v with $u \leq v$, a decomposition of $K_v - K_u$ (the complete graph on v vertices with a hole of size u) into the maximum possible number of edge disjoint triangles.

1 Introduction

A *partial triangle decomposition* with *leave* L of a graph G, $PT(G)$, is a triple (V, B, L) where V is the vertex set $V(G)$ of G, B is a collection of edge-disjoint triangles in G and L is the subgraph (of G) containing those edges of G not covered by the triangles of B. When we do not wish to specify the leave of a $PT(G)$ we may just write the pair (V, B) instead of the triple (V, B, L). A *maximum partial triangle decomposition* of a graph G, $MPT(G)$, is a $PT(G)$ with the property that no other $PT(G)$ contains more triangles.

Let K_v denote the complete graph on v vertices. When $v \equiv 1$ or 3 (mod 6), an $MPT(K_v)$ is equivalent to a Steiner triple system of order v. In [9] and [10] an $MPT(K_v)$ is constructed for all positive integers v. Let $K_v - K_u$ be the complete graph on v vertices with a hole of size u; that is the graph formed from K_v by removing all edges which have both their vertices in a subset of size u of the vertex set of K_v. In this paper we construct an $MPT(K_v - K_u)$ for all positive integers u and v ($u \leq v$). This problem closely resembles the problem of embedding $MPT(K_v)$'s. An $MPT(K_u)$, (U, A) say, is said to be embedded in an $MPT(K_v)$, (V, B) say, if $U \subset V$ and $A \subset B$.

In 1983, Mendelsohn and Rosa [8] considered the following problem. For which values of u and v can any $MPT(K_u)$ be embedded in an $MPT(K_v)$?

This problem is a generalisation of the 1973 result of Doyen and Wilson on embeddings of Steiner triple systems [2]. Mendelsohn and Rosa proved necessary conditions for all u and v and sufficient conditions except in the cases where precisely one of u and v is 4 or 5 (mod 6). Following further progress on the problem by Hartman, Mendelsohn and Rosa [6] and Hartman [5], the problem was recently settled by Fu, Lindner and Rodger [4]. Their paper proves the following two theorems.

Theorem 1.1. [4] *Let $v > u$. Any $MPT(K_u)$ can be embedded in an $MPT(K_v)$ if and only if*

- *if $u = 6$ then $v = 7$ or $v \geq 10$,*
- *if $u > 6$ and u is even then $v = u + 1$ or $v \geq 2u$, and*
- *if $u > 6$ and u is odd then $v \geq 2u$.*

Theorem 1.2. [4] *Let $v > u$. Any $MPT(K_u)$ (U, A, L_1) can be embedded in an $MPT(K_v)$ (V, B, L_2) such that $L_1 \subseteq L_2$ if and only if*

- *if $u \equiv 0$ or $2 \pmod 6$ then v is even,*
- *if $u \equiv 4 \pmod 6$ then $v \equiv 4 \pmod 6$,*
- *if $u \equiv 5 \pmod 6$ then $v \equiv 5 \pmod 6$, and*
- *$v \geq 2u$, with strict inequality if $u \equiv 0, 2, 4$ or $5 \pmod 6$.*

Embeddings of the type described in the Theorem 1.2 are equivalent to $MPT(K_v - K_u)$'s and so we have the following corollary.

Corollary 1.1. *For $v > 2u$ and $(u, v) \equiv (0.0), (0,2), (0,4), (2,0), (2,2), (2,4), (4,4)$ or $(5,5) \pmod 6$, and for $v \geq 2u$ and $u \equiv 1$ or $3 \pmod 6$, there is an $MPT(K_v - K_u)$ with leave L as shown in Figure 1.*

Note that the leaves shown for the cases $(u, v) \equiv (1,4), (1,5), (3,4), (3,5) \pmod 6$ result from the constructions used in [4] and are not the only possible leaves. It is worth remarking that for some congruence classes of u and v (for example $(u, v) \equiv (0,1) \pmod 6$) the number of possible (as permitted by the degrees of the vertices and the number of edges in $K_v - K_u$) nonisomorphic leaves in an $MPT(K_v - K_u)$ approaches infinity as u and v become arbitrarily large.

We need the following notation. Let $\langle \mathbb{Z}_w, \{d_1, d_2, \ldots, d_t\} \rangle$ denote the graph G with vertex set $V(G) = \mathbb{Z}_w$ and edge set $E(G) = \{\{x, y\} : d_i \equiv x - y$ or $y - x \pmod w$ and $i \in \{1, 2, \ldots, t\}\}$. If $G_1, G_2, \ldots, G_t$ are edge disjoint subgraphs of a graph G such that $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_t)$ then we write $G = G_1 + G_2 + \cdots + G_t$. Let G^c denote the complement of the graph G and for vertex-disjoint graphs G_1 and G_2, denote by $G_1 \vee G_2$ the graph with vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{\{x,y\}: x \in V(G_1), y \in V(G_2)\}$.

2 Preliminary results

We begin this section with the following crucial lemma. Parts (1) and (3) of this lemma can also be found in [3], Lemma 6.3 and 6.4.

Lemma 2.1.

(1) *Let $w = 6m+4$, $m \geq 1$, and let $G = \langle \mathbb{Z}_w, \{1,2,3\} \rangle$. Then $G = F_1 + F_2 + F_3 + K_4 + 2mK_3 + 3mK_2$ where F_1, F_2 and F_3 are 1-factors of G and where the K_4 and the $3m$ copies of K_2 are pairwise vertex disjoint (and hence cover all the vertices of G).*

(2) *Let $w = 6m+4$, $m \geq 1$, and let $G = \langle \mathbb{Z}_w, \{1,2,3\} \rangle$. Then $G = F + P_3 + (4m+2)K_3 + (3m+1)K_2$ where F is a 1-factor of G, P_3 is a path with three edges, and the endpoints of the path together with the $6m+2$ vertices occurring in the $3m+1$ copies of K_2 cover the vertices of G (once each).*

(3) *Let $w = 6m+2$, $m \geq 1$, and let $G = \langle \mathbb{Z}_w, \{1,2,3\} \rangle$. Then $G = F + K_4 + 4mK_3 + (3m-1)K_2$ where F is a 1-factor of G and the K_4 and the $3m-1$ copies of K_2 are pairwise vertex disjoint (and hence cover all the vertices of G).*

(4) *Let $w = 6m+2$, $m \geq 1$, and let $G = \langle \mathbb{Z}_w, \{1,2\} \rangle$. Then $G = F_1 + F_2 + C_5 + (2m-1)K_3$ where F_1 and F_2 are 1-factors of G and where C_5 is a 5-cycle.*

(5) *Let $w = 6m+2$, $m \geq 1$, and let $G = \langle \mathbb{Z}_w, \{1,2,3\} \rangle$. Then $G = (4m+2)K_3 + C_{6m}$.*

(6) *Let $w = 6m+4$, $m \geq 1$, and let $G = \langle \mathbb{Z}_w, \{1,2\} \rangle$. Then $G = (2m+2)K_3 + C_{6m+2}$.*

Proof:

(1) Let the triangles be $\{3i+4, 3i+5, 3i+6\}$ with $i = 0,1,\ldots,2m-1$. Let the K_4 have vertices $\{0,1,2,3\}$ and let the $3m$ copies of K_2 be $\{6i+4, 6i+7\}$, $\{6i+5, 6i+8\}$, $\{6i+6, 6i+9\}$ with $i = 0,1,\ldots,m-1$. Let F_1 consist of the edges $\{3i+3, 3i+4\}$, $\{6j+2, 6j+5\}$, $\{0, w-2\}$, $\{1, w-1\}$, F_2 consist of the edges $\{3i+2, 3i+4\}$, $\{6j+3, 6j+6\}$, $\{0, w-1\}$, $\{1, w-2\}$ and F_3 consist of the edges $\{3i+3, 3i+5\}$, $\{6j+1, 6j+4\}$, $\{0, w-3\}$, $\{2, w-1\}$ with $i = 0,1,\ldots,2m-1$ and $j = 0,1,\ldots,m-1$.

(2) Let the triangles be $\{3i, 3i+2, 3i+3\}$ and $\{3i+1, 3i+2, 3i+4\}$ with $i = 0, 1, \ldots, 2m$. Let the 1-factor F consist of the edges $\{0, 1\}$, $\{2, w-1\}$, $\{3i+3, 3i+4\}$, $\{6j+5, 6j+8\}$ with $i = 0, 1, \ldots, 2m-1$ and $j = 0, 1, \ldots, m-1$. Let the path P_3 consist of the edges $\{0, w-1\}$, $\{w-1, 1\}$, $\{1, w-2\}$ and let the $3m+1$ copies of K_2 be $\{3i+1, 3i+3\}$ and $\{6j+2, 6j+5\}$ with $i = 0, 1, \ldots, 2m$ and $j = 0, 1, \ldots, m-1$.

(3) Let the triangles be $\{3i+2, 3i+4, 3i+5\}$ and $\{3i+3, 3i+4, 3i+6\}$ with $i = 0, 1, \ldots, 2m-1$. Let the 1-factor F consist of the edges $\{6i+1, 6i+4\}$, $\{3j+3, 3j+5\}$ with $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, 2m-1$. Let the K_4 have vertices $\{0, 1, 2, 3\}$ and let the $3m-1$ copies of K_2 be $\{3i+5, 3i+6\}$ and $\{6j+4, 6j+7\}$ with $i = 0, 1, \ldots, 2m-2$ and $j = 0, 1, \ldots, m-1$.

(4) Let the 5-cycle be $(0, 2, 1, w-1, w-2)$ and let the triangles be $\{3i+3, 3i+4, 3i+5\}$ with $i = 0, 1, \ldots, 2m-2$. Let the 1-factors F_1 and F_2, consist of the edges $\{0, 1\}$, $\{6i+2, 6i+3\}$, $\{6i+4, 6i+6\}$, $\{6i+5, 6i+7\}$ and $\{0, w-1\}$, $\{6i+5, 6i+6\}$, $\{6i+1, 6i+3\}$, $\{6i+2, 6i+4\}$ respectively with $i = 0, 1, \ldots, m-1$.

(5) Let the triangles be $\{0, 1, w-1\}$, $\{w/2-1, w/2, w/2+2\}$, $\{w/2, w/2+1, w/2+3\}$, $\{3i, 3i+2, 3i+3\}$, $\{3i+1, 3i+2, 3i+4\}$, $\{3i+w/2+1, 3i+w/2+2, 3i+w/2+4\}$, and $\{3j+w/2+3, 3j+w/2+5, 3j+w/2+6\}$, with $i = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, m-2$. Let the edges of C_{6m} be $\{w/2-1, w/2+1\}$, $\{w/2+2, w/2+3\}$, $\{3i+w/2+4, 3i+w/2+6\}$, $\{3i+3, 3i+4\}$, $\{3i+w/2+3, 3i+w/2+4\}$, $\{3j+1, 3j+3\}$, $\{3j+w/2+2, 3j+w/2+5\}$, and $\{3k+w-1, 3k+2\}$, with $i = 0, 1, \ldots, m-2$, $j = 0, 1, \ldots, m-1$ and $k = 0, 1, \ldots, m$.

(6) Let the triangles be $\{3i, 3i+1, 3i+2\}$ and $\{3i+w/2, 3i+w/2+1, 3i+w/2+2\}$ with $i = 0, 1, \ldots, m$. Let the edges of C_{6m+2} be $\{3i+1, 3i+3\}$, $\{3i+w/2+1, 3i+w/2+3\}$, $\{3j+2, 3j+3\}$, and $\{3j+w/2+2, 3j+w/2+3\}$ with $i = 0, 1, \ldots, 2m$ and $j = 0, 1, \ldots, m-1$.

Lemma 2.2. *For each of the following graphs G, we have $G = K_3 + K_3 + \cdots + K_3$.*

(1) $G = \langle \mathbb{Z}_{16}, \{4, 5, 7\} \rangle$, $G = \langle \mathbb{Z}_w, \{4, 5, \ldots, 6t+1\} \setminus \{3t-1\} \rangle$ *with* $w = 12t+4$ *and* $t \geq 2$;

(2) $G = \langle \mathbb{Z}_w, \{4, 5, \ldots, 6t+4\} \setminus \{3t+4\} \rangle$ *with* $w = 12t+10$ *and* $t \geq 0$;

(3) $G = \langle \mathbb{Z}_w, \{4, 5, \ldots, 6t\} \setminus \{3t+1, 3t+2, 5t+1\} \rangle$ *with* $w = 12t+2$ *and* $t \geq 1$;

(4) $G = \langle \mathbb{Z}_w, \{4,5,\ldots,6t+3\} \setminus \{3t+1, 3t+4, 5t+3\}\rangle$ *with* $w = 12t+8$ *and* $t \geq 1$;

(5) $G = \langle \mathbb{Z}_w, \{3,4,\ldots,6t\} \setminus \{5t-1\}\rangle$ *with* $w = 12t+2$ *and* $t \geq 1$;

(6) $G = \langle \mathbb{Z}_w, \{3,4,\ldots,6t+3\} \setminus \{5t+3\}\rangle$ *with* $w = 12t+8$ *and* $t \geq 0$.

Proof: The following difference triples developed modulo w provide the required decomposition.

(1) For $G = \langle \mathbb{Z}_{16}, \{4,5,7\}\rangle$ the difference triple is $(4,5,7)$. For $t = 2$ let the difference triples be $(4,8,12)$, $(6,9,13)$ and $(7,10,11)$. For $t \geq 3$ let the difference triples be
$(4,5t-1,5t+3),(6,5t-2,5t+4),\ldots,(2t,4t+1,6t+1)$,
$(5,3t-2,3t+3),(7,3t-3,3t+4),\ldots,(2t-3,2t+2,4t-1)$,
$(2t-1,3t+1,5t),(2t+1,3t,5t+1),(3t+2,4t,5t+2)$.

(2) For $t = 0$ there is nothing to do and for $t = 1$ let the difference triples be $(4,6,10)$ and $(5,8,9)$. For $t \geq 2$ let the difference triples be
$(4,5t+2,5t+6),(6,5t+1,5t+7),\ldots,(2t,4t+4,6t+4)$,
$(5,3t,3t+5),(7,3t-1,3t+6),\ldots,(2t-1,2t+3,4t+2)$,
$(2t+1,3t+3,5t+4),(2t+2,3t+1,5t+3),(3t+2,4t+3,5t+5)$.

(3) For $t = 1$ there is nothing to do and for $t = 2$ let the difference triples be $(4,6,10)$ and $(5,9,12)$. For $t \geq 3$ let the difference triples be
$(4,5t-2,5t+2),(6,5t-3,5t+3),\ldots,(2t-2,4t+1,6t-1)$,
$(5,3t-2,3t+3),(7,3t-3,3t+4),\ldots,(2t-3,2t+2,4t-1)$,
$(2t-1,3t,5t-1),(2t,4t,6t),(2t+1,3t-1,5t)$.

(4) For $t = 1$ the difference triple is $(5,6,9)$. For $t \geq 2$ let the difference triples be
$(4,5t+1,5t+5),(6,5t,5t+6),\ldots,(2t-2,4t+4,6t+2)$,
$(5,3t,3t+5),(7,3t-1,3t+6),\ldots,(2t-1,2t+3,4t+2)$,
$(2t,3t+2,5t+2),(2t+1,3t+3,5t+4),(2t+2,4t+3,6t+3)$.

(5) For $t = 1$ the difference triple is $(3,5,6)$. For $t \geq 2$ let the difference triples be
$(3,3t-1,3t+2),(5,3t-2,3t+3),\ldots,(2t-3,2t+2,4t-1)$,
$(4,5t-2,5t+2),(6,5t-3,5t+3),\ldots,(2t-2,4t+1,6t-1)$,
$(2t-1,3t+1,5t),(2t,4t,6t),(2t+1,3t,5t+1)$.

(6) For $t = 0$ there is nothing to do and for $t = 1$ let the difference triples be $(3,4,7)$ and $(5,6,9)$. For $t \geq 2$ let the difference triples be
$(3,3t+1,3t+4),(5,3t,3t+5),\ldots,(2t-1,2t+3,4t+2)$,
$(4,5t+1,5t+5),(6,5t,5t+6),\ldots,(2t-2,4t+4,6t+2)$,
$(2t,3t+2,5t+2),(2t+1,3t+3,5t+4),(2t+2,4t+3,6t+3)$.

Lemma 2.3. *Let* $u \equiv 1 \pmod 6$ *and* $v \equiv 5 \pmod 6$ *with* $v \geq 2u$. *Then* $K_v - K_u = K_5 + K_3 + K_3 + \cdots + K_3$.

Proof: If $u = 1$ then the result follows from [8], so assume $u \geq 7$. Let $w = v - u = 6m+4$ and let $r = (u-1)/6$. It is straightforward, using Lemma 2.1 (1) and adjoining four further vertices, to see that $K_4^c \vee \langle \mathbb{Z}_w, \{1,2,3\} \rangle = K_5 + K_3 + K_3 + \cdots + K_3$. We complete the proof by showing that $K_{u-4}^c \vee \langle \mathbb{Z}_w, \{4,5,\ldots,3m+2\} \rangle = K_3 + K_3 + \cdots + K_3$. Note that since $7 \leq u \leq w$ we have $1 \leq r \leq m$. We choose $m-r$ of the $m-1$ difference triples given in Lemma 2.2 (1) or (2) and use these to generate K_3's. The half difference $3m+2$, together with the remaining $3m+1-3-3(m-r) = 3r-2$ differences in $\{4,5,\ldots,3m+1\}$, yield $6r-3 = u-4$ 1-factors by the Stern and Lenz result [11]. Thus we can pair off these 1-factors with the vertices of K_{u-4}^c to obtain the required decomposition.

Lemma 2.4. *Let* $u \equiv 5 \pmod 6$ *and* $v \equiv 3 \pmod 6$ *with* $v \geq 2u$. *Then* $K_v - K_u = C_5 + K_3 + K_3 + \cdots + K_3$.

Proof: Let $w = v - u = 6m+4$ and let $r = (u+1)/6$. It is straightforward, using Lemma 2.1 (2) and adjoining two further vertices, to see that $K_2^c \vee \langle \mathbb{Z}_w, \{1,2,3\} \rangle = C_5 + K_3 + K_3 + \cdots + K_3$. We complete the proof by showing that $K_{u-2}^c \vee \langle \mathbb{Z}_w, \{4,5,\ldots,3m+2\} \rangle = K_3 + K_3 + \cdots + K_3$. Note that since $5 \leq u \leq w$ we have $1 \leq r \leq m$. We choose $m-r$ of the $m-1$ difference triples given in Lemma 2.2 (1) or (2) and use these to generate K_3's. The half difference $3m+2$, together with the remaining $3m+1-3-3(m-r) = 3r-2$ differences in $\{4,5,\ldots,3m+1\}$, yield $6r-3 = u-2$ 1-factors by the Stern and Lenz result [11]. Thus we can pair off these 1-factors with the vertices of K_{u-2}^c to obtain the required decomposition.

Lemma 2.5. *Let* $u \equiv 3 \pmod 6$ *and* $v \equiv 5 \pmod 6$ *with* $v \geq 2u$. *Then* $K_v - K_u = K_5 + K_3 + K_3 + \cdots + K_3$.

Proof: If $u = 3$ then the result follows from [8], so assume $u \geq 9$. Let $w = v - u = 6m+2$ and let $r = (u+3)/6$. It is straightforward, using Lemma 2.1 (3) and adjoining two further vertices, to see that $K_2^c \vee \langle \mathbb{Z}_w, \{1,2,3\} \rangle = K_5 + K_3 + K_3 + \cdots + K_3$. We complete the proof by showing that $K_{u-2}^c \vee \langle \mathbb{Z}_w, \{4,5,\ldots,3m+1\} \rangle = K_3 + K_3 + \ldots + K_3$. Note that since $9 \leq u \leq w$ we have $2 \leq r \leq m$. We choose $m-r$ of the $m-2$ difference triples given in Lemma 2.2 (3) or (4) and use these to generate K_3's. The half difference $3m+1$, together with the remaining $3m-3-3(m-r) = 3r-3$ differences in $\{4,5,\ldots,3m\}$, yield $6r-5 = u-2$ 1-factors by the Stern and Lenz result [11]. Thus we can pair off these 1-factors with the vertices of K_{u-2}^c to obtain the required decomposition.

Lemma 2.6. *Let* $u \equiv 5 \pmod 6$ *and* $v \equiv 1 \pmod 6$ *with* $v \geq 2u$. *Then* $K_v - K_u = C_5 + K_3 + K_3 + \cdots + K_3$.

Proof: Let $w = v - u = 6m+2$ and let $r = (u+1)/6$. It is straightforward, using Lemma 2.1 (4) and adjoining two further vertices, to see that $K_2^c \vee \langle \mathbb{Z}_w, \{1,2\}\rangle = C_5 + K_3 + K_3 + \cdots + K_3$. We complete the proof by showing that $K_{u-2}^c \vee \langle \mathbb{Z}_w, \{3,4,\ldots,3m+1\}\rangle = K_3 + K_3 + \cdots + K_3$. Note that since $5 \le u \le w$ we have $1 \le r \le m$. We choose $m - r$ of the $m - 1$ difference triples given in Lemma 2.2 (5) or (6) and use these to generate K_3's. The half difference $3m+1$, together with the remaining $3m-2-3(m-r) = 3r-2$ differences in $\{3,4,\ldots,3m\}$, yield $6r-3 = u-2$ 1-factors by the Stern and Lenz result [11]. Thus we can pair off these 1-factors with the vertices of K_{u-2}^c to obtain the required decomposition.

Lemma 2.7. *Let $u \equiv 1 \pmod 6$, $u \ge 7$, and $v \equiv 5 \pmod 6$ with $v \ge 2u$. Then $K_v - K_u = C_4 + K_3 + K_3 + \cdots + K_3$ where the C_4 has two vertices in the hole and two vertices not in the hole.*

Proof: Let $w = v - u = 6m+4$ and let $r = (u-1)/6$. It is straightforward, using Lemma 2.1 (6) and adjoining two further vertices, to see that $K_2^c \vee \langle \mathbb{Z}_w, \{1,2\}\rangle = C_4 + K_3 + K_3 + \cdots + K_3$. We complete the proof by showing that $K_{u-2}^c \vee \langle \mathbb{Z}_w, \{3,4,5,\ldots,3m+2\}\rangle = K_3 + K_3 + \cdots + K_3$. Note that since $7 \le u \le w$ we have $1 \le r \le m$. We choose $m - r$ of the $m - 1$ difference triples given in Lemma 2.2 (1) or (2) and use these to generate K_3's. The half difference $3m+2$, together with the remaining $3m+1-2-3(m-r) = 3r-1$ differences in $\{3,4,5,\ldots,3m+1\}$, yield $6r-1 = u-2$ 1-factors by the Stern and Lenz result [11]. Thus we can pair off these 1-factors with the vertices of K_{u-2}^c to obtain the required decomposition.

Lemma 2.8. *Let $u \equiv 3 \pmod 6$ and $v \equiv 5 \pmod 6$ with $v \ge 2u$. Then $K_v - K_u = C_4 + K_3 + K_3 + \cdots + K_3$ where the C_4 has two vertices in the hole and two vertices not in the hole.*

Proof: If $u = 3$ then the result follows from [8], so assume $u \ge 9$. Let $w = v - u = 6m+2$ and let $r = (u+3)/6$. It is straightforward, using Lemma 2.1 (5) and adjoining two further vertices, to see that $K_2^c \vee \langle \mathbb{Z}_w, \{1,2,3\}\rangle = K_4 + K_3 + K_3 + \cdots + K_3$. We complete the proof by showing that $K_{u-2}^c \vee \langle \mathbb{Z}_w, \{4,5,\ldots,3m+1\}\rangle = K_3 + K_3 + \cdots + K_3$. Note that since $9 \le u \le w$ we have $2 \le r \le m$. We choose $m - r$ of the $m - 2$ difference triples given in Lemma 2.2 (3) or (4) and use these to generate K_3's. The half difference $3m+1$, together with the remaining $3m-3-3(m-r) = 3r-3$ differences in $\{4,5,\ldots,3m\}$, yield $6r-5 = u-2$ 1-factors by the Stern and Lenz result [11]. Thus we can pair off these 1-factors with the vertices of K_{u-2}^c to obtain the required decomposition.

The decompositions given in Lemmas 2.3 and 2.5 are not required for our $MPT(K_v - K_u)$ constructions but are of interest because they are equivalent to pairwise balanced designs. Lemmas 2.3 and 2.5 prove that for all $u \equiv 1$ or $3 \pmod 6$ and all $v \equiv 5 \pmod 6$, $v \ge 2u$, there exists a pairwise balanced design (of index 1) on v points with one block of size u, one block of size 5, and the remaining blocks of size 3. These pairwise balanced designs are needed for the constructions in [1].

3 Maximum packings

The results of Section 2 allow us to construct maximum packings of $K_v - K_u$ for each of the remaining congruence classes of u and v.

3.1 $v \leq 2u$

The following Lemma leads to an easy way of constructing maximum packings of $K_v - K_u$ when the size u of the hole is large enough relative to v.

Lemma 3.1. *If all the edges with both vertices not in the hole occur in the triples of a $PTS(K_v - K_u)$, then the PTS is a maximum packing.*

Proof: Since every triple must contain at least one edge with both vertices outside the hole the result follows.

Lemma 3.2. *For $v \leq 2u+1$ and $v-u$ even, there is a maximum packing of $K_v - K_u$ having leave L as shown in Figure 1.*

Proof: Take a 1-factorization of K_{v-u} and pair off the 1-factors with the hole vertices. Since $v \leq 2u+1$, the number of 1-factors $v-u-1$ is at most u.

Lemma 3.3. *For $v \leq 2u$ and $v-u$ odd, there is a maximum packing of $K_v - K_u$ having leave L as shown in Figure 1.*

Proof: Take a near 1-factorization of K_{v-u} and pair off the 1-factors with the hole vertices. Since $v \leq 2u$, the number of near 1-factors $v-u$ is at most u.

3.2 $u \equiv 5 \pmod 6$

The cases $v \equiv 1$ or $3 \pmod 6$ follow immediately from Lemmas 2.6 and 2.4 and so we have the following lemma.

Lemma 3.4. *For $(u,v) \equiv (5,1)$ or $(5,3) \pmod 6$ and $v \geq 2u$, there is a maximum packing of $K_v - K_u$ having leave L as shown in Figure 1.*

As the following lemma shows, the cases $v \equiv 0, 2$ and $4 \pmod 6$ follow easily from the cases $v \equiv 1, 3$ and $5 \pmod 6$.

Lemma 3.5. *For $(u,v) \equiv (5,0), (5,2)$ or $(5,4) \pmod 6$ and $v \geq 2u$, there is a maximum packing of $K_v - K_u$ having leave L as shown in Figure 1.*

Proof: We have constructed an $MPT(K_{v+1}-K_u)$ $(V\cup\{x\},B)$ in Corollary 1.1 or Lemma 3.4 and we can suppose that x is a vertex not in the hole and, when $u\equiv 1$ or 3 (mod 6), x is in the C_5. Hence, $(V,B\backslash X)$ where X is the set of triples in B which contain x, is the required $MPT(K_v-K_u)$.

3.3 $u\equiv 0,2 \pmod 6$

Lemma 3.6. *For $(u,v)\equiv(0,1),(0,3),(2,1)$ or $(2,3) \pmod 6$ and $v\ge 2u$, there is a maximum packing of K_v-K_u having leave L as shown in Figure 1.*

Proof: By Theorem 1.1, an $MPT(u)$ (U,A,L_1) can be embedded in an $MPT(v)$ (V,B,L_2). Then $(V,B\setminus(A\cup X))$ where X is the set of triples in B which contain an edge of L_1, is the required $MPT(K_v-K_u)$

Lemma 3.7. *For $(u,v)\equiv(0,5)$ or $(2,5) \pmod 6$ with $v\ge 2u$ and $v\ne 2u+1$ if $u\equiv 2 \pmod 6$, there is a maximum packing of K_v-K_u having leave L as shown in Figure 1.*

Proof: By Lemmas 2.7 and 2.8, there is an $MPT(K_v-K_{u+1})$ (V,B,L) with hole U such thatL is a 4-cycle (a,b,c,d) with $a,c\in U$ and with $b,d\notin U$. Let K_v-K_u have vertex set V and hole $U\setminus\{a\}$. Then $(V,B\cup\{a,b,c\})$ is the required $MPT(K_v-K_u)$.

The proof of the following lemma is due to D. Hoffman.

Lemma 3.8. *For $u\equiv 2 \pmod 6$ and $v=2u+1$, there is a maximum packing of K_v-K_u having leave L as shown in Figure 1.*

Proof: Take a complete graph on the $v-u=6m+3$ vertices. Remove a set of m pairwise vertex disjoint triangles, and remove one more edge, vertex disjoint from the m removed triangles. The resulting graph is a complete multipartite graph, with m parts of size 3, one part of size 2, and $3m+1$ parts of size 1. This graph has a proper edge coloring with $6m+2$ colors (see [7]). Identify these colors with the $6m+2$ vertices in the hole, forming the remaining triangles.

3.4 $u\equiv 4 \pmod 6$

Lemma 3.9. *For $(u,v)\equiv(4,0),(4,2) \pmod 6$ and $v\ge 2u+1$, there is a maximum packing of K_v-K_u with triangles having leave L as showm in Figure 1.*

Proof: For these congruence classes, $v\ge 2u+2$ so $v+1\ge 2(u+1)+1$. Hence there is an $MPT(K_{v+1}-K_{u+1})$ (V,B,L) with hole $U\cup\{x\}$ where L is a 5-cycle (a,b,c,d,e). When $(u,v)\equiv(5,3) \pmod 6$, we can assume that $a=x$

(and that b, c, d and e are not in U). In this case $(V \setminus \{a\}, B \setminus X)$, where X is the set of triples in B which contain a, is the required $MPT(K_v - K_u)$. When $(u, v) \equiv (5, 1) \pmod 6$, we can assume (see the proof of Lemma 2.1 (4)) that a, b, c, d and e are not in U and that $\{a, c, x\} \in B$. In this case $(V \setminus \{x\}, (B \cup \{a, b, c\}) \setminus X)$, where X is the set of triples in B which contain x, is the required $MPT(K_v - K_u)$.

Lemma 3.10. *For $(u, v) \equiv (4, 5) \pmod 6$ and $v \geq 2u$, there is a maximum packing of $K_v - K_u$ with triangles having leave L as showm in Figure 1.*

Proof: By Corollary 1.1 there is an $MPT(K_v - K_{u+1})$ (V, B, L) with hole $U \cup \{x\}$. Then (V, B) with hole U is the required $MPT(K_v - K_u)$.

Lemma 3.11. *For $(u, v) \equiv (4, 1)$ and $(4, 3) \pmod 6$ with $v \geq 2u$ and $v \neq 2u + 1$ if $u \equiv 4 \pmod 6$, there is a maximum packing of $K_v - K_u$ with triangles having leave L as shown in Figure 1.*

Proof: By Theorem 1.1, an $MPT(u)$ (U, A, L_1) can be embedded in an $MPT(v)$ (V, B, L_2). Then $(V, B \setminus (A \cup X))$ where X is the set of triples in B which contain an edge of L_1, is the required $MPT(K_v - K_u)$.

Lemma 3.12. *For $u \equiv 4 \pmod 6$ and $v = 2u + 1$, there is a maximum packing of $K_v - K_u$ with triangles having leave L as shown in Figure 1.*

Proof: Take a complete graph on the $v - u = 6m + 5$ vertices. Remove a set of m pairwise vertex disjoint triangles, and remove two more edges, vertex disjoint from the m removed triangles. The resulting graph is a complete multipartite graph, with m parts of size 3, two parts of size 2, and $3m + 1$ parts of size 1. This graph has a proper edge coloring with $6m + 4$ colors (see [7]). Identify these colors with the $6m + 4$ vertices in the hole, forming the remaining triangles.

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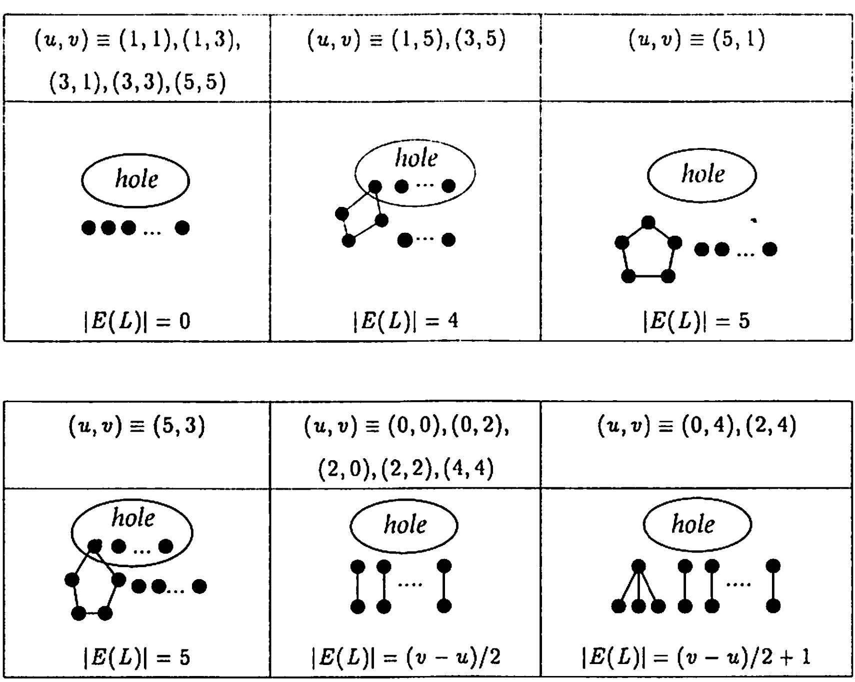

Figure 1

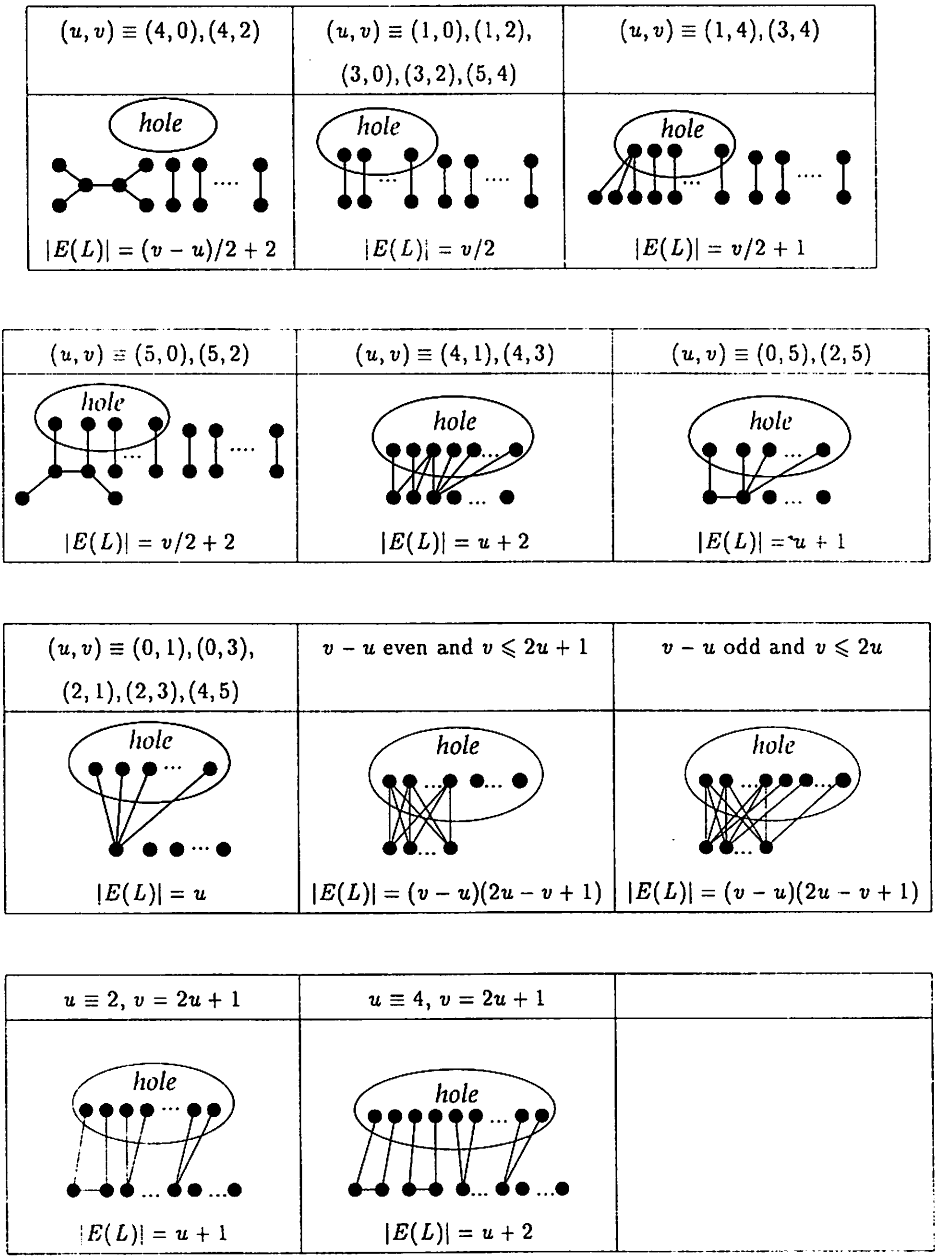

Figure 1 (continued)