RELATIONSHIPS BETWEEN VERTEX-NEIGHBOR-INTEGRITY AND OTHER PARAMETERS

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Abstract. Let G be a graph. A vertex subversion strategy of G, S, is a set of vertices in G whose closed neighborhood is deleted from G. The survival-subgraph is denoted by G/S. The vertex-neighbor-integrity of G, VNI(G), is defined to be $VNI(G) = \min_{S \subseteq V(G)} \{|S| + \omega(G/S)\}$, where S is any vertex subversion strategy of G, and $\omega(G/S)$ is the maximum order of the components of G/S. In this paper, we discuss the relationship between the vertex-neighbor-integrity and some well-known graphic parameters.

I. Introduction

Integrity was introduced by Barefoot, Entringer, and Swart as an alternative measure of the vulnerability of graphs to disruption caused by the removal of vertices. [1,2] Goddard and Swart established the bounds for integrity in terms of independence number, vertex covering number, connectivity, and chromatic number. [9]

A spy network can be modeled by a graph whose vertices represent the stations and whose edges represent the lines of communication. If a station is destroyed, the adjacent stations will be betrayed so that the betrayed stations become useless to network as a whole. [10] Therefore instead of considering the integrity of a communication graph, in [6,7] we discussed the vertex-neighbor-integrity of graphs — disruption caused by the removal of vertices and all of their adjacent vertices. Incidentally, the edge-neighbor-connectivity and the edge-neighbor-integrity of graphs were discussed in [4,5,8].

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Let G=(V,E) be a graph and u be a vertex in G. $N(u) = \{v \in V(G) | v \neq u,\ v$ and u are adjacent$\}$ is the ***open neighborhood*** of u, and $N[u] = \{u\} \cup N(u)$ denotes the ***closed neighborhood*** of u. A vertex u in G is said to be ***subverted*** if the closed neighborhood $N[u]$ is deleted from G. A set of vertices $S = \{u_1, u_2, ..., u_m\}$ is called a ***vertex subversion strategy*** of G if each of the vertices in S has been subverted from G. Let G/S be the ***survival-subgraph*** left after each vertex of S has been subverted from G. The ***vertex-neighbor-integrity*** of a graph G, VNI(G), is defined to be

$$\mathrm{VNI}(G) = \min_{S \subseteq V(G)} \{|S| + \omega(G/S)\},$$

where S is any vertex subversion strategy of G, and $\omega(G/S)$ is the maximum order of the components of G/S.

$\lceil x \rceil$ is the smallest integer greater than or equal to x. $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

The values of VNI of graphs can be very small or very large, see the following examples:

Example 1.1: $K_{1,n-1}$, where $n \geq 2$, is a star. By the definition of VNI, it is clear that $\mathrm{VNI}(K_{1,n-1}) = 1$.

Example 1.2: P_n, where $n \geq 2$, is a path with n vertices. We have shown that $\mathrm{VNI}(P_n) = \lceil 2\sqrt{n+3} \rceil - 4$. [6]

In Section III, we find the lower and upper bounds of VNI for all graphs related to some well-known graphic parameters. Moreover, we discuss some properties of the graphs with VNI equal to some of those graphic parameters. For the completeness of the paper, we present the related graphic parameters and some basic properties in Section II.

II. Related Graphic Parameters and Properties

In this section, we present the related graphic parameters and some basic properties. All other undefined terminology and notations are taken from [3].

Let G=(V,E) be a graph. The integrity of G, I(G), is defined to be

$$I(G) = \min_{S \subseteq V(G)} \{|S| + \omega(G - S)\},$$

where $\omega(G-S)$ is the maximum order of the components of G−S. A subset S' of V is called an *I-set* of G if $I(G)=|S'| + \omega(G - S')$.

Let S be a subset of V. S is called a *vertex cut strategy* of G if the survival-subgraph G/S is disconnected, or is a clique, or is $\emptyset$. The *neighbor-connectivity* of G, K(G), is defined to be the minimum size of all vertex cut strategies S of G. [10] A subset S^* of V is called a *VNI-set* of G if $VNI(G) = |S^*| + \omega(G/S^*)$. Then we have the following theorem:

Theorem 2.1: For any connected incomplete graph G with order ≥ 3, every VNI-set of G is a vertex cut strategy of G.

Proof: Let S^* be a VNI-set of G, and assume that S^* is not a vertex cut strategy of G. Then G/S^* is a nontrivial connected graph, and is not a clique. Let v_0 be a vertex of $V(G/S^*)$. Then

$$\begin{aligned} VNI(G) &= |S^*| + \omega(G/S^*) \\ &= |S^*| + |V(G/S^*)| \\ &> |S^*| + |\{v_0\}| + \omega((G/S^*)/\{v_0\}) \\ &= |S^* \cup \{v_0\}| + \omega(G/(S^* \cup \{v_0\})) \\ &\geq VNI(G), \quad \text{since } S^* \cup \{v_0\} \subseteq V(G). \end{aligned}$$

A contradiction. Therefore S^* is a vertex cut strategy of G. QED.

A subset C of V is called a *covering* of G if every edge of G has at least one end in C. A covering C is a *minimum covering* if G has no covering C' with $|C'| < |C|$. The *covering number* of G, $\alpha_0(G)$, is the number of vertices in a minimum covering of G.

A subset I of V is called an *independent set* of G if no two vertices of I are adjacent in G. An independent set I is *maximum* if G has no independent set I' with $|I'| > |I|$. The *independence number* of G, $\beta_0(G)$, is the number of vertices in a maximum independent set of G. The following results will be used in Section III.

Lemma 2.2: A set $I \subseteq V$ is an independent set of G if and only if $V-I$ is a covering of G. [3]

Lemma 2.3: For any graph G, $\alpha_0(G)+\beta_0(G) = |V(G)|$. [3]

A subset M of E is called a *matching* in G if no two edges of M are adjacent in G. A matching M is *maximum* if G has no matching M' with $|M'| > |M|$. Let $\beta_1(G)$ be the number of edges in a maximum matching in G. The following results will be used in Section III.

Lemma 2.4: For any graph G, $\beta_1(G) \leq \alpha_0(G)$. [3]

Lemma 2.5: In a bipartite graph G, $\beta_1(G) = \alpha_0(G)$. [3]

A subset L of E is called an *edge covering* of G if each vertex of V(G) is an end of some edge in L. An edge covering L is a *minimum edge covering* if G has no edge covering L' with $|L'| < |L|$. The *edge covering number* of G, $\alpha_1(G)$, is the number of edges in a minimum edge covering of G.

III. Lower and Upper Bounds

We have proved in Theorem 2.1 that for any connected incomplete graph G with order ≥ 3, every VNI-set of G is a vertex cut strategy of G, and hence every VNI-set has cardinality at least K(G). If a graph G is disconnected or is a complete graph, then the neighbor-connectivity of G, $K(G) = 0$, and VNI(G) is still a positive integer. Therefore it is easy to show that K(G) is a lower bound of VNI(G).

Theorem 3.1: For any graph G, $K(G) \leq VNI(G)$.

Proof: If G is disconnected or is a complete graph, then $K(G) = 0$ and VNI(G) is a positive integer, hence $K(G) \leq VNI(G)$.

If $|V(G)| \leq 2$, then G is complete or disconnected, and hence $K(G) \leq VNI(G)$.

If G is connected and $|V(G)| \geq 3$, then let S^* be a VNI-set of G, and $VNI(G) = |S^*|+\omega(G/S^*)$. By Theorem 2.1, S^* is a vertex cut strategy of G, hence $|S^*| \geq K(G)$, and then $VNI(G) \geq K(G) + \omega(G/S^*) \geq K(G)$. QED.

Theorem 3.2: For any connected graph G with order ≥ 2, if VNI(G) = $\mathbf{K}$(G), then every VNI-set S^* of G is a minimum vertex cut strategy of G and $G/S^* = \emptyset$.

Proof: Since VNI(G) = $\mathbf{K}$(G), G is incomplete. Let S^* be a VNI-set of G, then $VNI(G) = |S^*| + \omega(G/S^*) = \mathbf{K}(G)$, and $|S^*| \leq \mathbf{K}(G)$. By Theorem 2.1, $|S^*| \geq \mathbf{K}(G)$, so $|S^*| = \mathbf{K}(G)$ and $\omega(G/S^*) = 0$. Therefore S^* is a minimum vertex cut strategy of G and $G/S^* = \emptyset$. QED.

The following example is a graph whose value of VNI equals the neighbor-connectivity $\mathbf{K}$.

Example 3.1: The graph G is shown in Figure 3.1. VNI(G) = $\mathbf{K}$(G) = 2. $S = \{u, v\}$ is a unique VNI-set of G and is also a minimum vertex cut strategy of G.

G:

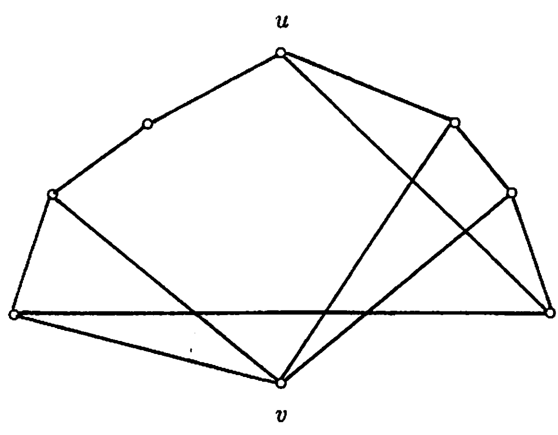

Figure 3.1

For any graph G, the integrity I(G), the edge covering number $\alpha_1(G)$, and the independence number $\beta_0(G)$ are upper bounds of VNI(G). For any graph G without any isolated vertices, the covering number $\alpha_0(G)$ and the size of a maximum matching $\beta_1(G)$ are upper bounds of VNI(G).

Theorem 3.3: For any graph G, $VNI(G) \leq I(G)$.

Proof: Let $S'=\{u_1, u_2, ..., u_m\}$ be an I-set of G, so $I(G)=|S'| + \omega(G - S')$. $G/S'=G-N[S']\subseteq G-S'$, where $N[S']=\cup_{i=1}^{m} N[u_i]$, so $\omega(G/S') = \omega(G - N[S']) \leq \omega(G - S')$. Thus

$$VNI(G) = \min_{S\subseteq V(G)} \{|S| + \omega(G/S)\}$$

$$\leq |S'| + \omega(G/S')$$

$$\leq |S'| + \omega(G - S') = I(G).$$

QED.

Let S' be an I-set of G, $< S' >$ be the induced subgraph of G by S', and $r = \Delta(< S' >)$ be the maximum degree of $< S' >$. Then we can improve the upper bound:

Theorem 3.4: $VNI(G) \leq I(G) - r$.

Proof: As described above, S' is an I-set of G and $< S' >$ is the induced subgraph of G by S'. Let v be a vertex in $< S' >$ with the maximum degree. i.e., $deg_{<S'>}(v) = \Delta(< S' >) = r$. Let v be adjacent with $u_1, u_2, ..., u_r$ in $< S' >$, and $S^* = S' - \{u_1, u_2, ..., u_r\}$. Then $G/S^* = G - N[S^*] \subseteq G - S'$, where $N[S^*]=\cup_{u\in S^*} N[u]$, so $\omega(G/S^*) \leq \omega(G - S')$. Thus

$$VNI(G) = \min_{S\subseteq V(G)} \{|S| + \omega(G/S)\}$$

$$\leq |S^*| + \omega(G/S^*)$$

$$\leq |S'| + \omega(G - S') - r = I(G) - r.$$

QED.

Theorem 3.5: For any graph G without isolated vertices, $VNI(G) \leq \beta_1(G)$.

Proof: Let $M = \{[u_1, v_1], [u_2, v_2], ..., [u_m, v_m]\}$ be a maximum matching in G, where $m = \beta_1(G)$. Let $V^*=V(G)-\{v_1, v_2, ..., v_m, u_1, u_2, ..., u_m\}$.

Assume that there are two distinct vertices $x, y \in V^*$, such that x is adjacent with u_i and y is adjacent with v_i. Then there is an M-augmenting path (x, u_i, v_i, y) in G, and $M' = (M-[u_i, v_i]) \cup \{[x, u_i], [v_i, y]\}$ is a matching in G with size $|M| + 1$, a contradiction to the maximum matching M. Therefore at most one end of each edge in M is adjacent with some vertices of V^*, or for some edges in M, both ends of each are adjacent with a same vertex of V^*

If at most one end of each edge in M is adjacent with some vertices of V^*, w.l.o.g., we assume that no vertex in V^* is adjacent with any vertex of $\{u_1, u_2, ..., u_m\}$; if for some edges of M, both ends of each are adjacent with a same vertex in V^*, w.l.o.g., we assume that both ends of an edge, v_i and u_i, are adjacent with the same vertex $x_i \in V^*$. Since G has no isolated vertices, in each of the cases, each vertex of V^* is adjacent with some vertices of $\{v_1, v_2, ..., v_m\}$. Now we let $S^* = \{v_1, v_2, ..., v_m\} \subset V(G)$. Then $G/S^* = \emptyset$ and $\omega(G/S^*) = 0$. Thus

$$VNI(G) = \min_{S \subseteq V(G)} \{|S| + \omega(G/S)\}$$

$$\leq |S^*| + \omega(G/S^*)$$

$$= m + 0 = m = \beta_1(G).$$

QED.

Theorem 3.6: For any graph G without isolated vertices, $VNI(G) \leq \alpha_0(G)$.

Proof: By Lemma 2.4, $\beta_1(G) \leq \alpha_0(G)$, and by Theorem 3.5, VNI(G) $\leq \beta_1(G)$, so we have $VNI(G) \leq \alpha_0(G)$. QED.

We can improve the upper bound $\alpha_0(G)$ as described below. Let C be a minimum covering of G, and $< C >$ be the induced subgraph of G by C. If $m = \Delta(< C >)$ is the maximum degree of $< C >$, then we have the following theorem.

Theorem 3.7: $VNI(G) \leq \alpha_0(G) - m + 1$.

Proof: Let v be a vertex in $< C >$ with $\deg_{<C>}(v) = m$, where m is the maximum degree of $< C >$. Let $S = \{v_1, v_2, ..., v_m\}$ be a subset of C and each vertex of S be adjacent with v in $< C >$. Now let $C' = C-S$. Then G/C' is $\emptyset$ or a set of isolated vertices, and $\omega(G/C') \leq 1$. Hence

$$\mathrm{VNI(G)} \leq |\mathrm{C}'| + \omega(\mathrm{G/C}')$$

$$= |\mathrm{C}| - |\mathrm{S}| + \omega(\mathrm{G/C}')$$

$$\leq \alpha_0(\mathrm{G}) - m + 1.$$

QED.

Using the above theorem we can determine $< C >$ when $\mathrm{VNI(G)} = \alpha_0(\mathrm{G})$.

Corollary 3.8: For any graph G, if $\mathrm{VNI(G)} = \alpha_0(\mathrm{G})$, then for any minimum covering C, the induced subgraph $< C > = K_2 \cup \bar{K}_r$, where $r = \alpha_0(G) - 2$, or $< C > = \bar{K}_{\alpha_0(G)}$, a null graph.

Proof: $\mathrm{VNI(G)} = \alpha_0(G) \leq \alpha_0(G) - m + 1$, where m is the maximum degree of $< C >$, so $m = 1$.

Assume that there are at least three vertices in $< C >$ with degree 1, then there are at least four vertices in $< C >$ with degree 1. W.l.o.g., let $\deg_{<C>}(v_1) = \deg_{<C>}(v_2) = \deg_{<C>}(v_3) = \deg_{<C>}(v_4) = 1$, v_1 be adjacent with v_2, and v_3 be adjacent with v_4. Let $C' = C - \{v_1, v_3\}$, then $\mathrm{VNI(G)} \leq |C'| + \omega(G/C') \leq (\alpha_0(G) - 2) + 1 = \alpha_0(G) - 1$, a contradiction to $\mathrm{VNI(G)} = \alpha_0(G)$. Therefore there are at most two vertices in $< C >$ with degree 1. Hence $< C > = K_2 \cup \bar{K}_r$, where $r = \alpha_0(G) - 2$, or $< C > = \bar{K}_{\alpha_0(G)}$. QED.

Please note that if $\mathrm{VNI(G)} = \alpha_0(G)$, C is a minimum covering of G, and the induced subgraph $< C >$ is a null graph $\bar{K}_{\alpha_0(G)}$, then G is a bipartite graph with a bipartition (C, V(G)−C), since by Lemma 2.2 and Lemma 2.3, I = V(G)−C is a maximum independent set of G and there is no edge between any two vertices of I.

Theorem 3.9: For any graph G, $\mathrm{VNI(G)} \leq \beta_0(G)$.

Proof: Let C be a minimum covering of G, then by Lemma 2.2 and Lemma 2.3, V(G)−C=I is a maximum independent set of G, and $|I| = \beta_0(G)$. Each vertex in C is adjacent with some vertices in I, since if u in C is not adjacent with any vertex in I, then $I \cup \{u\}$ is an independent set of G with size $|I| + 1$, a contradiction to a maximum independent set I. Thus $G/I = \emptyset$, and

$$\text{VNI}(G) = \min_{S \subseteq V(G)} \{|S| + \omega(G/S)\}$$

$$\leq |I| + \omega(G/I)$$

$$= \beta_0(G) + 0 = \beta_0(G).$$

QED.

Theorem 3.10: For any graph G, $\text{VNI}(G) \leq \alpha_1(G)$.

Proof: Let L=$\{[u_1, v_1], [u_2, v_2], ..., [u_m, v_m]\}$ be a minimum edge covering of G, $\alpha_1(G) = m$, and $S^*=\{u_1, u_2, ..., u_m\}$. Since the listed vertices in S^*, $u_1, u_2, ..., u_m$, may be the same, $|S^*| \leq m$. Then $G/S^*=\emptyset$ and $\omega(G/S^*) = 0$.

$$\text{VNI}(G) = \min_{S \subseteq V(G)} \{|S| + \omega(G/S)\}$$

$$\leq |S^*| + \omega(G/S^*) \leq m = \alpha_1(G).$$

QED.

For any graph G without isolated vertices, since $\alpha_0(G)$, $\alpha_1(G)$, $\beta_0(G)$, and $\beta_1(G)$ are upper bounds of VNI(G), by Lemma 2.3 and Lemma 2.4, the following corollaries are easily obtained.

Corollary 3.11: For any graph G without isolated vertices, $\text{VNI}(G) \leq \min(\alpha_0(G), \alpha_1(G), \beta_0(G), \beta_1(G)) = \min(\alpha_1(G), \beta_0(G), \beta_1(G))$.

Corollary 3.12: For any graph G without isolated vertices, $\text{VNI}(G) \leq |V(G)|/2$.

Next we discuss the conditions necessarily satisfied by a graph G when VNI(G) equals the upper bound $\alpha_0(G)$.

Theorem 3.13: For any graph G without isolated vertices, if $\text{VNI}(G) = \alpha_0(G)$ then $\alpha_0(G) = \beta_1(G)$.

Proof: $\text{VNI}(G) \leq \beta_1(G)$ and $\beta_1(G) \leq \alpha_0(G)$, so if $\text{VNI}(G) = \alpha_0(G)$ then $\alpha_0(G) = \beta_1(G)$. QED.

Although a necessary condition for a graph G to have a value of VNI that achieves the upper bound $\alpha_0(G)$ is $\alpha_0(G) = \beta_1(G)$, the graph may or may not be a bipartite graph. To construct a graph G with VNI(G) $= \alpha_0(G)$, Corollary 3.8 can be applied. Let $C = \{u_1, u_2, ..., u_{\alpha_0(G)}\}$ be a minimum covering of G. If VNI(G)=α_0(G), then $< C >= K_2 \cup \bar{K}_{\alpha_0(G)-2}$ or $< C >= \bar{K}_{\alpha_0(G)}$. If $< C >= \bar{K}_{\alpha_0(G)}$ then the graph G is a bipartite graph. Hence, to have a non-bipartite graph G, $< C >$ must be $K_2 \cup \bar{K}_{\alpha_0(G)-2}$. See the following examples:

Example 3.2: The graph G_1 is shown in Figure 3.2. VNI(G_1)=$|\{u, v, w\}| + \omega(G_1/\{u, v, w\}) = 3 + 0 = 3$. C=$\{u, v, w\}$ is a minimum covering of G_1 and M=$\{e_1, e_2, e_3\}$ is a maximum matching in G_1. VNI(G_1)=$\alpha_0(G_1) = \beta_1(G_1) = 3$. $< C >= K_2 \cup \bar{K}_1$. G_1 is not a bipartite graph.

G_1:

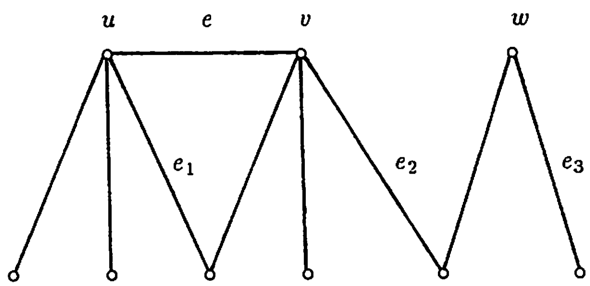

Figure 3.2

Example 3.3: Let $G_2 = G_1 - e$, where G_1 is shown in Figure 3.2. VNI(G_2) $= |\{u, v, w\}| + \omega(G_2/\{u, v, w\}) = 3$. C=$\{u, v, w\}$ is a minimum covering of G_2 and M=$\{e_1, e_2, e_3\}$ is a maximum matching in G_2. VNI(G_2)=$\alpha_0(G_2) = \beta_1(G_2) = 3$. $< C >= \bar{K}_3$. G_2 is a bipartite graph.

We have known that for any graph G without any isolated vertices, $K(G) \leq VNI(G) \leq \beta_1(G) \leq \alpha_0(G)$. If the lower bound K(G) and the upper bound $\alpha_0(G)$ are equal then VNI(G) = K(G) = $\beta_1(G) = \alpha_0(G)$. However there is only one possible VNI-value for such graphs, as described in Theorem 3.14.

Theorem 3.14: For any graph G without any isolated vertices, if K(G) $= \alpha_0(G) \geq 1$ then VNI(G) = K(G) = $\beta_1(G) = \alpha_0(G) = 1$.

Proof: Since K(G)$\geq$1, G is a connected graph, and G has no isolated vertices. It is clear that if $\mathbf{K}(G) = \alpha_0(G)$ then $VNI(G) = \mathbf{K}(G) = \beta_1(G) = \alpha_0(G)$. The remaining part is to show that $VNI(G) = 1$.

Assume that $\alpha_0(G) \geq 2$. Let C be a minimum covering of G, and $< C >$ be the induced subgraph of G by C. Since $VNI(G)=\alpha_0(G)$, by Corollary 3.8, $< C >= K_2 \cup \bar{K}_{\alpha_0(G)-2}$ or $< C >=\bar{K}_{\alpha_0(G)}$. Now we assume that $< C >= K_2 \cup \bar{K}_{\alpha_0(G)-2}$, and two vertices, u and v, are adjacent in $< C >$. Then let $C' = C-\{u\}$. G/C' is a set of isolated vertices or $\emptyset$, so C' is a vertex cut strategy of G, and $\mathbf{K}(G) \leq |C'| = \alpha_0(G)-1$, a contradiction. Therefore $< C >$ is a null subgraph of G. By Lemma 2.2, $V(G)-C$ is an independent set of G, and $G-C$ is also a null subgraph of G. Therefore G is a bipartite graph with a vertex bipartition $(C, V(G)-C)$.

By the assumption, $\alpha_0(G) \geq 2$. Let x and y be two vertices in C. Since G is connected, there is a path P starting with x and ending with y. We let the path $P=(x, w_1, u_1, w_2, u_2, ..., u_{m-1}, w_m, y)$, where $x, u_1, u_2, ..., u_{m-1}, y$ are the vertices in C, and $w_1, w_2, ..., w_m$ are the vertices in $V(G)-C$. Let $S^* = (C-\{x, u_1, u_2, ..., u_{m-1}, y\}) \cup \{w_1, w_2, ..., w_m\}$, then $|S^*| = \alpha_0(G) - (m-1+2) + m = \alpha_0(G) - 1$. Since G/S^* is a set of isolated vertices or $\emptyset$, S^* is a vertex cut strategy of G, and $\mathbf{K}(G) \leq |S^*| = \alpha_0(G) - 1$, a contradiction. Therefore $\alpha_0(G) = 1$, and $VNI(G) = 1$. QED.

Next, we discuss a characterization of graphs with the value of VNI=1.

Theorem 3.15: Let G be a graph of order $n \geq 1$. $VNI(G) = 1$ if and only if G contains a star spanning subgraph or $G = \bar{K}_n$, a null graph of order n.

Proof: If $VNI(G) = 1$, then let S^* be a VNI-set of G, and $VNI(G) = |S^*| + \omega(G/S^*) = 1$. Hence there are two cases:
Case 1: $|S^*| = 1$ and $\omega(G/S^*) = 0$.
Thus $S^* = \{v\}$ and $G/S^* = \emptyset$. Hence each vertex, except v itself, of G is adjacent with v. Therefore G contains a star spanning subgraph.
Case 2: $|S^*| = 0$ and $\omega(G/S^*) = 1$.
Thus $S^* = \emptyset$ and $\omega(G) = \omega(G/S^*) = 1$. Therefore $G = \bar{K}_n$, a null graph of order n.

Conversely, if $G = \bar{K}_n$, then take $S^* = \emptyset$, and $VNI(G) = |S^*| + \omega(G/S^*) = 0 + 1 = 1$. If G contains a star spanning subgraph, then let u be the vertex in G with the degree $n-1$. Let $S^* = \{u\}$, then $G/S^* = \emptyset$. Hence $VNI(G) = |S^*| + \omega(G/S^*) = 1 + 0 = 1$. QED.

Corollary 3.16: Let G be a graph of order n. If $\mathbf{K}(G) = \alpha_0(G) \geq 1$, then $VNI(G) = 1$ and $G = K_{1,n-1}$, a star.

Proof: By Theorem 3.14 and its proof, we have VNI(G) = **K**(G) = β_1(G) = α_0(G) = 1, and G is a connected bipartite graph. By Theorem 3.15, if VNI(G) = 1 and G is connected, then G contains a star spanning subgraph. Let u be the vertex in G with the degree $n-1$. Since G is a bipartite graph, no two vertices of V(G)−{u} are adjacent. Therefore G = $K_{1,n-1}$, a star. QED.

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