A Sufficient Condition for Oriented Graphs to be Hamiltonian *

Pan Lin Qiang, Zhang Ke Min
(Department of Mathematics, Nanjing University,
Nanjing, 210093, P. R. of China)

Abstract: In this paper, we prove the following result: Let D be a diconnected oriented graph of order n. If $d^+(u)+d^+(v) \geq n-2$ for any pair u,v of nonadjacent vertices such that $N^+(u) \cap N^+(v) \neq \emptyset$ and $d^-(u)+d^-(v) \geq n-2$ for any pair u,v of nonadjacent vertices such that $N^-(u) \cap N^-(v) \neq \emptyset$, then D contains a directed Hamiltonian cycle.

We use the terminology and notation of [1]. $D = (V(D), A(D))$ will denote an oriented graph on n vertices. A digraph is diconnected if there is a directed path from u to v for any two vertices u and v. Define $|uv| = 1$ when $uv \in A(D)$ and $|uv| = 0$ when $uv \notin A(D)$. If $v \in V(D)$ and $S \subseteq V(D)$, we denote the set of arcs from v to S (resp. from S to v) by (v, S) (resp. (S, v)). Furthermore, we define $d_S^+(v) = |(v, S)|$,$d_S^-(v) = |(S, v)|$. Define $N^+(u) = \{v|v \in V(D), uv \in A(D)\}$, $N^-(u) = \{v|v \in V(D), vu \in A(D)\}$. If $S \subseteq V(D)$, an S-path is a directed path of length at least two having exactly its origin and terminus in S.

Theorem *Let D be a diconnected oriented graph of order n. If $d^+(u)+d^+(v) \geq n-2$ for any pair u,v of nonadjacent vertices such that $N^+(u) \cap N^+(v) \neq \emptyset$ and $d^-(u)+d^-(v) \geq n-2$ for any pair*

*The project supported by NSFC.

u, v of nonadjacent vertices such that $N^-(u) \cap N^-(v) \neq \emptyset$, then D contains a directed Hamiltonian cycle.

Proof: Suppose that D satisfies the condition of the theorem, but does not contain a directed Hamiltonian cycle. Let S be a longest directed cycle in D.

We first prove that there is a S-path in D. Suppose there is no S-path in D. Then since D is diconnected and S is a proper subset of V, D contains a directed cycle S' having precisely one vertex, say u, in S. Let $S = x_0x_1x_2\cdots x_ax_0$, $S' = x_0y_1y_2\cdots y_bx_0$, $A = \{x_1, x_2, \cdots, x_a\}$, $B = \{y_1, y_2, \cdots, y_b\}$ and $F = V(D) - (A \cup B \cup \{x_0\})$. Obviousely, $|A| = a$, $|B| = b$ and $|F| = n - (a + b) - 1 = f$. Since there is no S-path in D, x_a, y_b and x_1, y_1 are pairs of nonadjacent vertices such that $N^+(x_a) \cap N^+(y_b) \neq \emptyset$ and $N^-(x_1) \cap N^-(y_1) \neq \emptyset$. Since D contains no S-path, we have

$$d_B^+(x_a) = d_B^-(x_1) = d_A^+(y_b) = d_A^-(y_1) = 0 \tag{1}$$

and there is not a path of form x_avy_1 or y_bvx_1, where $v \in F$. Hence,

$$|x_av| + |vy_1| \leq 1, \quad |y_bv| + |vx_1| \leq 1$$

for each $v \in F$. Furthermore,

$$\begin{aligned} & d_F^+(x_a) + d_F^-(y_1) + d_F^+(y_b) + d_F^-(x_1) \\ \leq & \sum_{v \in F} [(|x_av| + |vy_1|) + (|y_bv| + |vx_1|)] \\ \leq & 2f. \end{aligned} \tag{2}$$

Clearly,

$$d_A^+(x_a) \leq a - 2, \ d_A^-(x_1) \leq a - 2, \ d_B^+(y_b) \leq b - 2, \ d_B^-(y_1) \leq b - 2. \tag{3}$$

Combining (1), (2) and (3), we get

$$\begin{aligned} & d^+(x_a) + d^+(y_b) + d^-(x_1) + d^-(y_1) \\ \leq & 2(a - 2) + 2(b - 2) + 2f + |x_ax_0| + |y_bx_0| + |x_0x_1| + |x_0y_1| \\ = & 2(a + b + f + 1) - 6 \\ = & 2n - 6. \end{aligned} \tag{4}$$

The above inequality implies $d^+(x_a) + d^+(y_b) \le n - 3$ or $d^-(x_1) + d^-(y_1) \le n - 3$, but this contradicts the hypothesis of the theorem.

Therefore D contains a S-path P, say $x_\alpha z_1 z_2 \cdots z_\beta x_0$, where $x_0, x_\alpha \in S$. Let $P_1 = x_0 x_1 x_2 \cdots x_\alpha, P_2 = x_\alpha x_{\alpha+1} \cdots x_a x_0$ be the directed path on the cycle S. Let the path P be chosen so that α is maximum. Because of the maximality of $S, \alpha \ne a$. Let $A = \{x_0, x_1, \cdots, x_\alpha\}, B = \{z_1, z_2, \cdots, z_\beta\}, C = \{x_{\alpha+1}, x_{\alpha+2}, \cdots, x_a\}$ and $F = V - (A \cup B \cup C)$. Obviously, $|A| = \alpha + 1, |B| = \beta, |C| = a - \alpha$ and $|F| = n - (|A| + |B| + |C|) = n - a - \beta - 1 = t$. Because of the maximality of α and $\alpha \ne a$, z_β, x_α and $z_1, x_{\alpha+1}$ are pairs of nonadjacent vertices such that $N^+(z_\beta) \cap N^+(x_\alpha) \ne \emptyset$, and $N^-(x_{\alpha+1}) \cap N^-(z_1) \ne \emptyset$ respectively. By the same reason, we have

$$d_C^+(z_\beta) + d_C^-(z_1) + d_B^+(x_{\alpha+1}) + d_B^-(x_a) = 0 \tag{5}$$

If there exist x_i, $x_{i+1} \in A$ $(i = 0, 1, 2, \cdots, \alpha - 1)$ such that $x_i x_{\alpha+1}$, $x_a x_{i+1} \in A(D)$, then

$$x_i x_{\alpha+1} x_{\alpha+2} \cdots x_a x_{i+1} x_{i+2} \cdots x_\alpha z_1 z_2 \cdots z_\beta x_0 x_1 x_2 \cdots x_i$$

is a cycle longer than S. This cotradiction shows that

$$|x_i x_{\alpha+1}| + |x_a x_{i+1}| \le 1 \; (i = 0, 1, \cdots, \alpha - 1)$$

Hence,

$$\begin{aligned} d_A^+(x_a) + d_A^-(x_{\alpha+1}) &= \sum_{i=0}^{\alpha} (|x_a x_{i+1}| + |x_i x_{\alpha+1}|) \\ &= \sum_{i=0}^{\alpha-1} (|x_a x_{i+1}| + |x_i x_{\alpha+1}|) + |x_a x_0| + |x_\alpha x_{\alpha+1}| \\ &\le \alpha + 2 \end{aligned} \tag{6}$$

Similarly, we have

$$d_A^+(z_\beta) + d_A^-(z_1) \le \alpha + 2 \tag{7}$$

By the maximality of S, there exists no vertex $v \in F$ such that $x_a v$, $v z_1 \in A(D)$ or $z_\beta v$, $v x_{\alpha+1} \in A(D)$. Hence,

$$|x_a v| + |v z_1| \le 1, \quad |z_\beta v| + |v x_{\alpha+1}| \le 1$$

for every $v \in F$. Furthermore,

$$d_F^+(x_a) + d_F^-(z_1) \le t, \quad d_F^+(z_\beta) + d_F^-(x_{\alpha+1}) \le t \tag{8}$$

Obviously,

$$d_B^+(z_\beta) \le \beta-2, d_B^-(z_1) \le \beta-2, d_C^+(x_a) \le a-\alpha-2, d_C^-(x_{\alpha+1}) \le a-\alpha-2 \tag{9}$$

Combining (4)-(8), we get

$$\begin{aligned} & d^+(z_\beta) + d^+(x_a) + d^-(z_1) + d^-(x_{\alpha+1}) \\ \le\ & 2(\alpha+2) + 2(\beta-2) + 2(a-\alpha-2) + 2t \\ =\ & 2(a+\beta+1+t) - 6 \\ =\ & 2n - 6 \end{aligned} \tag{10}$$

The above inequlity implies that $d^+(z_\beta)+d^+(x_a) \le n-3$, or $d^-(z_1)+d^-(x_{\alpha+1}) \le n-3$. But this contradicts the hypothesis of the theorem. This completes the proof of the theorem.

Remark: Let D be an oriented graph of order n ($n \ge 5$). In [3], it is shown that if for any two vertices x, y, $xy \notin A$ implies $d^+(x)+d^-(y) \ge n-2$, then D is Hamiltonian. The following example shows that in some sense, our theorem is stronger than the above.

Example: Let C be a directed cycle with $n(\ge 5)$ vertices denoted by $x_1x_2\cdots x_nx_1$. For any $i \in \{1, 2, \cdots, n\}$, we have $x_{i+1}x_i \notin A$, but $d^+(x_{i+1})+d^-(x_i) = 2 < n-2$. C does not satisfy the conditions of the theorem of [3]. $N^+(x)\cap N^+(y) = \emptyset$ and $N^-(x)\cap N^-(y) = \emptyset$ for any pair x, y of vertices, so C satisfies the condition of our Theorem.

The bound presented in our theorem is sharp in the following sense. Let k be positive integers and $n = 2k$. Let D be a digraph with n vertices. See Figure 1. The digraph D is diconnected and nonhamiltonian. Vertices v_{2k}, v_{2k-2} are the only pair which are disjoint and have common out-neighbour or common in-neighbour. Furthermore, $d^-(v_{2k}) + d^-(v_{2k-2}) = (k-2) + (k-1) = n-3$.

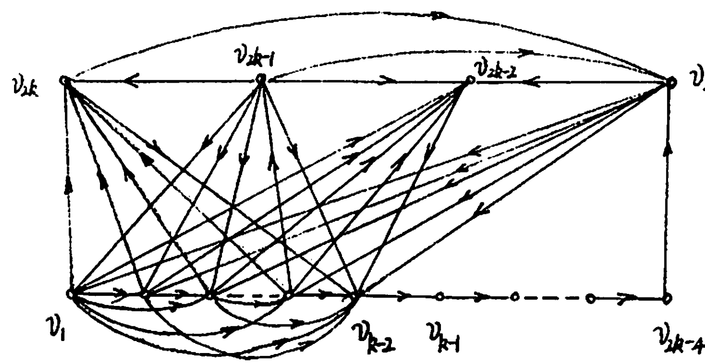

Figure 1

References

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