A Note on k-Rotation Graphs

Erich Prisner *

Mathematisches Seminar
Universität Hamburg

For an integer $k \geq 1$ the k-rotation graph $\mathcal{R}_k(G)$ of a graph $G = (V, E)$ has the set of all connected k-edge subgraphs of G as vertex set. Two such vertices $H_1 \neq H_2$ are adjacent in $\mathcal{R}_k(G)$ whenever there are labelings $e_1, \ldots e_k$ and $d_1, \ldots d_k$ of the edges of H_1 respectively H_2 such that e_i and d_i have exactly one common vertex, for every $i = 1, \ldots k$ [2]. Surely the 1-rotation graph is just the ordinary line graph.

It is very easy to see that the line graph of every connected graph must be again connected. Since $L^2(G)$ is a spanning subgraph of $\mathcal{R}_2(G)$ [2], the same is true for the 2-rotation graph. For higher k, the k'th iterated line graph $L^k(G)$ is not necessarily a subgraph of $\mathcal{R}_k(G)$, so this approach ends here. However, in this note we show:

Theorem 1 *For every $k \geq 1$ the k-rotation graph of every connected graph is again connected.*

We need the following

Lemma 2 *For every $k \geq 1$, if two connected k-edge graphs H_1 and H_2 have $k-1$ common edges and nonempty intersection, then the corresponding vertices in $\mathcal{R}_k(H_1 \cup H_2)$ are adjacent.*

Proof: We prove the result by induction over k. The case $k = 2$ is rather obvious. Let now $k > 2$, let the result be true for all smaller integers greater than 2, and let H_1 and H_2 be as above. We denote by e_1 and e_2 the edges in $H_1 - H_2$ and $H_2 - H_1$ respectively. Since H_1 and H_2 are connected with nonempty intersection, $H_1 \cup H_2$ is connected, whence we find some path $x_0, x_1, x_2, \ldots, x_{\ell-1}, x_\ell$ with $e_1 = x_0x_1$ and $e_2 = x_{\ell-1}x_\ell$. Let P_0 denote the vertex set of this path. Now we define recursively P_i as the set of vertices of $V(H_1 \cup H_2) \setminus \bigcup_{j<i} P_j$ adjacent to some vertex in $\bigcup_{j<i} P_j$. It is easy to see that every vertex in P_i must be adjacent to some vertex in

*Mathematisches Seminar, Universität Hamburg, Bundesstr. 55, 20146 Hamburg, Germany, e-mail: ms6a013@math.uni-hamburg.de

ARS COMBINATORIA 55(2000), pp. 289-292

P_{i-1}, for $i \geq 1$. Since $H_1 \cup H_2$ is connected, $P_0 \cup P_1 \cup \cdots \dot{\cup} P_t$ is a partition of its vertex set for some integer $t \geq 0$. We distinguish the following cases:

Case 1: $t \leq 1$ and $H_1 \cup H_2$ is a tree. Then every edge outside our basic path is incident with exactly one vertex of the path, but not with x_0 or x_ℓ (since H_1 and H_2 are connected). We can find a linear ordering $e_1 = d_1, d_2, \ldots, d_{k+1} = e_2$ of the edges of $H_1 \cup H_2$ such that every pair d_i, d_{i+1} has some common vertex. This ordering is found by starting with e_1, then adding all edges outside our basic path and incident to x_1, then adding x_1x_2 and all edges outside the path and incident to x_2, and so on. It is obvious how this labelling induces the necessary labellings of H_1 and H_2.

Case 2: $t \geq 1$ and there is some vertex $v \in P_t$ with at least two neighbors in $P_t \cup P_{t-1}$. Let F denote the set of edges incident with v. For $i = 1, 2$, we obtain H_i' from H_i by deleting all edges in F, and then all isolated vertices. Now H_1' and H_2' are connected, with $k - |F|$ edges, $k - |F| - 1$ common edges, and nonempty intersection. We apply induction hypothesis to obtain feasable labellings of H_1' and H_2'. Since $|F| \geq 2$, it is easy to extend them to feasable labellings of H_1 and H_2.

Case 3: $t \geq 2$ and the vertices in P_t are independent. We choose some $u \in P_t$ and some neighbor v in P_{t-1}. The degree of v is at least 2. As in Case 2, denote by F the set of all edges incident with v, and obtain H_i' from $H_i - F$ by deleting all isolated vertices. The rest of the proof is exactly as in Case 2.

If $t = 0$ then Case 1 holds. If $t = 1$ then either $H_1 \cup H_2$ is a tree, or Case 2 holds. Surely Case 2 or 3 (maybe both) holds for $t \geq 2$. □

Note that the nonempty intersection of $H_1 \cap H_2$ follows for $k \geq 2$. The case $k = 1$ is only included for technical reasons — we needed it in the induction.

Proof of Theorem 1 We use the convention that connected k-edge subgraphs of G are denoted by capitals X, Y, Z, whereas the corresponding vertices of $\mathcal{R}_k(G)$ are denoted by corresponding small letters x, y, z.

By Lemma 2, distinct vertices x, y of $\mathcal{R}_k(G)$ are adjacent provided $|E(X \cap Y)| = k - 1$. We define a measure of similiarity on the vertices of $\mathcal{R}_k(G)$: Let for $x, y \in \mathcal{R}_k(G)$, $d(X, Y)$ denote the length of some shortest X-Y path in G and $c(X, Y)$ the maximum edge number of a component of $X \cap Y$. Then we define

$$s(x, y) := d(X, Y) + k - c(X, Y).$$

These numbers are nonnegative integers (since $c(X, Y) \leq k, d(X, Y) \geq 0$), and equal to 0 if and only if $x = y$. Moreover $s(x, y) = 1$ implies $c(X, Y) =$

$k-1$, that is, $xy \in E(\mathcal{R}_k(G))$ by Lemma 1. Thus it suffices to show that for every $x, y \in V(\mathcal{R}_k(G))$ with $s(x,y) \geq 2$ there is some vertex z of $\mathcal{R}_k(G)$ with $|E(X \cap Z)| = k-1$ and $s(z,y) < s(x,y)$. We distinguish two cases:

In case $X \cap Y = \emptyset$, choose some shortest X-Y path, and let e be that edge of the path having one of its vertices in X. Then there is some edge a in $E(X)$ such that $E(X) \cup \{e\} \setminus \{a\}$ generates some connected subgraph Z of G. For, we can choose a as any edge in an end block (that is, an end vertex of the block-cutvertex tree) of X that does not intersect e. But if all edges in the end blocks of X intersect e, then $X = K_{1,k}$ and we can choose any edge as a. Obviously $|E(X \cap Z)| = k-1$, but since $d(Z,Y) = d(X,Y) - 1$, we also get $s(z,y) < s(x,y)$.

In case $X \cap Y \neq \emptyset$, we have $d(X,Y) = 0$. Let e be any edge of $E(Y) \setminus E(X)$ touching the component of $X \cap Y$ with largest edge number $c(X,Y)$. As above, there must be some edge a of X such that $E(X) \cup \{e\} \setminus \{a\}$ generates some connected graph Z in G. Again $|E(X \cap Z)| = k-1$, and again $s(z,y) < s(x,y)$, since $c(Z,Y) > c(X,Y)$ and $d(Z,Y) = 0$. □

Actually we have proven a little more. The *facet graph* of a set of k-element sets has this set as vertex set, and two distinct sets form adjacent vertices whenever they have $k-1$ common elements, see [5] or [4]. We take the edge sets of the connected k-edge subgraphs of G as our set, and form the facet graph $\mathcal{F}_k(G)$ thereof. Then Lemma 2 assures that $\mathcal{F}_k(G)$ is a subgraph of $\mathcal{R}_k(G)$, and Theorem 1 proves that $\mathcal{F}_k(G)$ is connected for connected G.

Let us define one more graph-operator. For $k \geq 1$, and a graph $G = (V, E)$, let $\Phi_k(G)$ denote the graph with all connected induced k-vertex subgraphs of G as vertices, where such $H_1 \neq H_2$ are adjacent whenever there are labellings $x_1, \ldots x_k$ and $y_1, \ldots, y_k$ of $V(H_1)$ and $V(H_2)$ such that every pair x_i, y_i is distinct and adjacent. Now $\mathcal{R}_k(G) = \Phi_k(L(G))$ for every $k \geq 1$. Surely $\Phi_1(G) = G$, and $\Phi_2(G)$ is called the edge graph of G in [1].

Now our Theorem would follow from an affirmative answer of the following problem:

Question: Is $\Phi_k(G)$ is connected for every connected graph G and every $k \geq 1$?

A statement corresponding to Lemma 2 is not true in this case, as can be seen by the graph in Figure 1: $G - x$ and $G - y$ are connected 7-vertex subgraphs of G with 6 common vertices, nevertheless the corresponding vertices in $\Phi_7(G)$ are not adjacent.

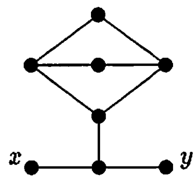

Figure 1

Another open question is whether the k-rotation graph of every n-connected graph were n-connected. This holds for $k = 1$ [3] and $k = 2$, since $L^2(G)$ is a spanning subgraph of $\mathcal{R}_2(G)$, as mentioned above.

References

[1] H.-J. Bandelt, M. Farber, P. Hell, Absolute reflexive retracts and absolute bipartite retracts, *Discrete Appl. Math.* 44 (1993) 9-20.

[2] G. Chartrand, H. Hevia, E.B. Jarrett, Planarity of double-rotation graphs, *Congressus Numerantium* 76 (1990) 183-192.

[3] G. Chartrand, M.J. Stewart, The connectivity of line graphs, *Math. Ann.* 182 (1969) 170-174.

[4] M.S. Jacobson, A.E. Kézdy, J. Lehel, Intersection graphs associated with uniform hypergraphs, in: *Surveys in Graph Theory* (G. Chartrand, M. Jacobson eds.) *Congressus Numerantium* 116 (1996) 173-192.

[5] V.B. Le, E. Prisner, Facet graphs of pure simplicial complexes and related concepts, Hamburger Beiträge zur Mathematik aus dem Mathematischen Seminar, Heft 19 (1992).