K-Dominating Sets of Cardinal Products of Paths

Antoaneta Klobucar
Ekonomski fakultet
HR-31000 Osijek
Croatia
email: aneta@oliver.efos.hr

Norbert Seifter
Department of Mathematics
Montanuniversität Leoben
A-8700 Leoben
Austria
email: seifter@unileoben.ac.at

ABSTRACT. In this paper we determine the k-domination numbers of the cardinal products $P_2 \times P_n, \ldots, P_{2k+1} \times P_n$ for all integers $k \geq 2$, $n \geq 3$.

1 Introduction and terminology.

For any graph G we denote the vertex-set and edge-set of G by $V(G)$ and $E(G)$, respectively. A subset $D \subset V(G)$ is called a *k-dominating set*, $k \geq 1$, if for every vertex y not in D, there exists at least one vertex $x \in D$, such that $d(x, y) \leq k$. For convenience we also say that D *k-dominates* G. The *k-domination number* $\gamma_k(G)$ is the cardinality of a smallest k-dominating set.

The cardinal product of two graphs is a graph with $V(G \times H) = V(G) \times V(H)$ and $((g_1, h_1), (g_2, h_2)) \in E(G \times H)$, if and only if $(g_1, g_2) \in E(G)$ and $(h_1, h_2) \in E(H)$.

The problem of determining the domination numbers of graphs first occurs in a paper of de Jaenisch [1]. He wanted to find the minimal number of queens on a chessboard, such that every square is either occupied by a queen or can be reached by a queen with a single move.

The study of domination numbers of products of graphs was initiated by Vizing [14]. He conjectured that the domination number of the cartesian product of two graphs is always greater than or equal to the product of the domination numbers of the two factors; a conjecture which is still unproven. For cardinal products of graphs this conjecture does not hold as was shown in [11].

Starting in the eighties the domination numbers of cartesian products were intensively investigated (see e. g. [2], [3], [4], [6], [7], [10]). In the mean time also some papers on domination numbers of cardinal products of graphs were published. We refer the interested reader to [5], [8], [9], [11], [12], [13].

Many of the papers written on domination numbers of cartesian products of graphs mainly deal with the problem of determining the domination numbers of products of paths and cycles, respectively. Following this approach, the domination numbers of $P_2 \times P_n, \ldots P_6 \times P_n$ were determined in [12] and [13]. In this paper we turn our attention to k-domination numbers of cardinal products of paths and determine those number for $P_2 \times P_n, \ldots, P_{2k+1} \times P_n$ for all integers $k \geq 2, n \geq 3$.

For completeness we first present some obvious results.

Proposition 1 *If C_n is the cycle of order n, $n \geq 3, k \geq 1$, then*

$$\gamma_k(C_n) = \lceil \frac{n}{2k+1} \rceil.$$

Also, if P_n is the path of order n, then

$$\gamma_k(P_n) = \lceil \frac{n}{2k+1} \rceil.$$

For the path P_2 and any odd cycle $C_{2n+1}, n \geq 1$

$$\gamma_k(P_2 \times C_{2n+1}) = \lceil \frac{4n+2}{2k+1} \rceil$$

holds.

Proof: The first two assertions are completely obvious. The third one holds since the cardinal product of P_2 and C_{2n+1} is isomorphic to C_{4n+2} and because of our first assertion. □

To fix terminology for the proofs of our results we need some more definitions.

Let $1, ..., m$ and $1, ..., n$ be the vertices of P_m and P_n, respectively. Then the vertices of $P_m \times P_n$ are denoted by (i, j) where $i = 1, ..., m$ and $j = 1, ..., n$

The cardinal product $P_m \times P_n$, $m, n \geq 2$, always consists of two components. Let $C_1(C_2)$ denote the component of $P_m \times P_n$ that contains all the vertices (i, j) with $i + j$ even (respectively, odd).

If both m and n are odd, these components are not isomorphic. If at least o these two numbers is even, the components are isomorphic.

For a fixed r, $1 \le r \le n$, the set $(P_m)_r = \{(i,r)|i = 1,...,m\}$ is called a *column* of $P_m \times P_n$. The set $(P_n)_r = \{(r,j)|j = 1,...,n\}$ is called a *row* of $P_m \times P_n$. A column (row) of one of the components of $P_m \times P_n$ then consists only of those vertices contained in the respective component. A set $B = \{(P_m)_r, (P_m)_{r+1}, ..., (P_m)_{r+l}, |l \ge 0, r \ge 1, r+l \le n\}$, of columns is called a *block* of size $m \times (l+1)$ of $P_m \times P_n$. If another block B_1 contains the column $(P_m)_{r-1}$ or the column $(P_m)_{r+l+1}$, then we say that B_1 is *adjacent* to B. A block B is called *internal*, if it is adjacent to two other blocks; it is called external if it is only adjacent to one block.

Theorem 1 *For* $n \ge 3, k \ge 2$

$$\gamma_k(P_2 \times P_n) = ... = \gamma_k(P_{2k} \times P_n) = 2 \cdot \lceil \frac{n}{2k+1} \rceil.$$

Proof: We first consider the graph $P_{2k} \times P_n$ and partition it into $2k \times (2k+1)$ blocks. There are $\lfloor \frac{n}{2k+1} \rfloor$ such blocks. Since $2k$ is even, $P_{2k} \times P_n$ has two isomorphic components. Therefore we consider only one component, say C_1.

If $n \le 2k$ then our result obviously holds. If $n > 2k$ we set $m = \lfloor \frac{n}{2k+1} \rfloor$ Let m be odd. Then the vertices $\{(k+1, k+1+2l(2k+1)), l = 0, ..., \lfloor \frac{m}{2} \rfloor\}$ k-dominate all vertices on the first, third,..., m-th block, and if $m \ge 3$ the vertices $\{(k, 3k+2+2l(2k+1)), l = 0, ..., \lfloor \frac{m}{2} \rfloor - 1\}$ k-dominate all vertices on the second, fourth,...,(m-1)-st block of the component C_1 if $n \equiv 0(mod(2k+1))$. If $n \equiv (k+1)(mod(2k+1)), ..., n \equiv 2k(mod(2k+1))$ then we have to add the vertex $(k, 3k+2+2\lfloor \frac{m}{2} \rfloor(2k+1))$. For all other n we add the vertex (k,n) or the vertex $(k+1,n)$ to the above sets to k-dominate all vertices of C_1. Hence we have found a set D_1 with $m+1 = \lceil \frac{n}{2k+1} \rceil$ vertices which k-dominates C_1 if m is odd.

If m is even the set $\{(k+1, k+1+2l(2k+1)), l = 0, ..., \lfloor \frac{m}{2} \rfloor - 1\} \cup \{(k, 3k+2+2l(2k+1)), l = 0, ..., \lfloor \frac{m}{2} \rfloor - 1\}$ has the property that it k-dominates all vertices of the m blocks of size $2k \times (2k+1)$. Again, as above depending on n, we either have to add the vertex $(k+1, k+1+2\lfloor \frac{m}{2} \rfloor(2k+1))$ or one of the vertices (k,n) or $(k+1,n)$ to obtain a dominating set D_1 of C_1.

Since C_1 and C_2 are isomorphic we have thus found a dominating set D of $P_{2k} \times P_n$ which contains $2 \cdot \lceil \frac{n}{2k+1} \rceil$ vertices.

The fact that this set is minimal follows from the following observation: Let $Q_n = ((1,1),(2,2),(1,3),\dots(s,n))$, $s \in \{1,2\}$, denote the induced subpath of length n of C_1 which only contains vertices of the first and second row. Furthermore we assume that there exists a k-dominating set D' of $P_{2k} \times P_n$ with $|D'| < |D|$. For $i = 1,2$, let $D'_i = D' \cap V(C_i)$. Then D'_i k-dominates C_i. At least one of the sets D'_1 or D'_2, say D'_1, contains less than $\lceil \frac{n}{2k+1} \rceil$ vertices. Then of course the vertices of D_1' must also k-dominate all vertices of Q_n. Let $(P_{2k})_{i_1}, \ldots, (P_{2k})_{i_l}, l < |D|$, denote the columns which contain vertices of $|D_1'|$. Let $(j, i_d) \in D_1'$, $d \in \{1, \ldots, l\}$ and let $M \subset V(Q_n)$ denote the set of those vertices of Q_n which are k-dominated by (j, i_d). Then it immediately follows from the structure of $P_{2k} \times P_n$ that the vertex of $(P_{2k})_{i_d} \cap Q_n$ also k-dominates at least the vertices of M. This implies that those vertices of Q_n which are contained in the same colums as the vertices of D_1' also k-dominate Q_n. But this contradicts the second assertion of Proposition 1 since we assumed that $|D_1'| < \lceil \frac{n}{2k+1} \rceil$.

We now consider $P_r \times P_n$ for some $r, 2 \le r < 2k$. Again we partition this component into $r \times (2k+1)$ blocks. If $r \ge k+1$, then the same sets as above dominate C_1 and C_2. If $r \le k$, then the vertices for a k-dominating set of C_1 can be chosen from the same columns as above but from the $(r-1)$-st row and r-th row, respectively. The minimality again follows with the same argument as above. Hence if at least one of the numbers r or n is even, our result holds immediately.

Let both, r and n, be odd now. Since $r \le 2k-1$ it is also obvious that we can find a set which dominates C_2 and contains one vertex of each $r \times (2k+1)$ block, and one additional vertex if $n \not\equiv 0(mod(2k+1))$. Since the minimality argument does not depend on the parity of r and n, our result again follows. (For C_2 the path Q_n is given by $Q_n = ((2,1),(1,2),(2,3),\ldots,(s,n))$, $s \in \{1,2\}$.) □

Remark. If $n \equiv 0(mod(2k+1))$ or $n \equiv k+1(mod(2k+1)), \ldots n \equiv 2k(mod(2k+1))$ the minimality of the k-dominating sets described in the above proof follows from the simple observation that each vertex of $P_{2k} \times P_n$ is k-dominated by exactly one vertex of these sets!

Theorem 2 *Let $k \ge 2$, $n \ge 3$. Then*

$$\gamma_k(P_{2k+1} \times P_n) = \begin{cases} 2 \cdot \lfloor \frac{n}{2k} \rfloor + 1 & if\ n \equiv 1(mod 2k) \\ 2 \cdot \lceil \frac{n}{2k} \rceil & otherwise \end{cases}$$

Proof: If $n \le 2k+1$ the result again obviously holds. Also if $2k+1 < n \le 4k$ it is easy to see that we need at least two vertices to k-dominate each of the components of $P_{2k+1} \times P_n$.

If $n \equiv 0(mod 2k), n \equiv 2k-1(mod 2k), \ldots, n \equiv k+1(mod 2k)$ then the set $S_1 = \{(k+1, k+2lk+1) | l = 0, 1, \ldots, \lceil \frac{n}{2k} \rceil - 1\}$ k-dominates C_1 and the set $\{(k+1, k+2lk) | l = 0, 1, \ldots, \lceil \frac{n}{2k} \rceil - 1\}$ k-dominates C_2. Hence $S = S_1 \cup S_2$, $|S| = 2 \cdot \lceil \frac{n}{2k} \rceil$, k-dominates $P_{2k+1} \times P_n$.

If $n \equiv k(mod 2k), n \equiv k-1(mod 2k), \ldots, n \equiv 2(mod 2k)$ we set $S_1' = \{(k+1, k+2lk+1) | l = 0, 1, \ldots, \lfloor \frac{n}{2k} \rfloor\}$ and $S_2' = \{(k+1, k+2lk) | l = 0, 1, \ldots, \lfloor \frac{n}{2k} \rfloor - 1\}$. Depending upon n we define $S_1 = S_1{}' \cup (k+1, n)$ or $S_1 = S_1'$ as well as $S_2 = S_2' \cup (k+1, n)$ or $S_2 = S_2' \cup (k+1, n-1)$. Again $S = S_1 \cup S_2$ k-dominates our graph and has cardinality $2 \cdot \lceil \frac{n}{2k} \rceil$.

If $n \equiv 1(mod2k)$ we take $S_1 = \{(k+1, k+2lk+1)|l = 0,1,...,\lfloor\frac{n}{2k}\rfloor - 1\}$ to k-dominate C_1. Let $S_2' = \{(k+1, k+2lk)|l = 0,1,...,\lfloor\frac{n}{2k}\rfloor - 1\}$. Then $S_2 = S_2' \cup (k+1, n-1)$ k-dominates C_2 if k is even and $S_2 = S_2' \cup (k+1, n)$ k-dominates C_2 if k is odd. Hence $S = S_1 \cup S_2$, $|S| = 2\cdot\lfloor\frac{n}{2k}\rfloor + 1$ k-dominates $P_{2k+1} \times P_n$.

In the sequel we show that S always is a minimal k-dominating set.

If n is even, then it is again clear that we only have to consider one component of $P_{2k+1} \times P_n$. But as it will turn out, if n is odd, our lemmas also hold for the component C_2. Hence we first only consider the component C_1, not depending on the parity of n.

For our proof of minimality we partition the graph $P_{2k+1} \times P_n$ into $(2k+1)\times 2k$ blocks. If such a block is external we denote it by R. If $n \not\equiv 0(mod2k)$ then we also have an external block R', which is not a $(2k+1) \times 2k$ block. By D_1 we denote a k-dominating set of C_1 in the sequel.

Lemma 1 *There is no k-dominating set D_1 such that $|D_1 \cap R| = 0$.*

Proof: Without loss of generality we assume that R is the first block in the graph $P_{2k+1} \times P_n$ (it contains (1,1)). By vertices from the adjacent block, we can at most k-dominate vertices of $(P_{2k+1})_{k+1}, ..., (P_{2k+1})_{2k}$. To k-dominate the remaining vertices we need at least one vertex which is contained in R. □

Lemma 2 *Let $|D_1 \cap B_i| = 0$ for some internal block B_i, and let $n > 4k$. If k is odd, then $|D_1 \cap B_{i-1}| \geq 1$, and $|D_1 \cap B_{i+1}| \geq 2$. If B_{i-1} (B_{i+1}) is external then $|D_1 \cap B_{i-1}| \geq 2$ ($|D_1 \cap B_{i+1}| \geq 3$). If B_{i+1} is external but not a $(2k+1) \times 2k$ block, then it contains at least 2 vertices.*

Proof: Let $B_i = \{(P_{2k+1})_j, ..., (P_{2k+1})_{j+2k-1}\}, j \geq 2k+1$, be some internal block. If $|D_1 \cap B_i| = 0$, all vertices of B_i must be k-dominated by vertices from adjacent blocks. Then at least one vertex of $(P_{2k+1})_{j-1}$ (of B_{i-1}) is in D_1, and at least one vertex of $(P_{2k+1})_{j+2k}$ (of B_{i+1}) is in D_1.

If k is odd, the vertex (k+1,j-1) is contained in C_1. This vertex k-dominates all vertices of the columns $(P_{2k+1})_j, ..., (P_{2k+1})_{j+k-1}$. So $|D_1 \cap B_{i-1}| = 1$ can only hold if $(k+1, j-1) \in D_1$. This vertex k-dominates the columns $(P_{2k+1})_{j-1}, ..., (P_{2k+1})_{j-k-1}$ on B_{i-1}, too. If B_{i-1} is internal we thus only obtain that $|D_1 \cap B_{i-1}| \geq 1$. If B_{i-1} is external, the vertices of $(P_{2k+1})_1, ..., (P_{2k+1})_{j-2}$ must be k-dominated by a vertex which is contained in B_{i-1}. Therefore $|D_1 \cap B_{i-1}| \geq 2$ holds in this case.

The vertex $(k+1, j+2k)$ is not contained in C_1. None of the other vertices of B_{i+1} k-dominates all vertices of the columns $(P_{2k+1})_{j+k}, \dots, (P_{2k+1})_{j+2k-1}$ in C_1. Therefore D_1 must contain at least 2 vertices of $(P_{2k+1})_{j+2k}$. But of course these two vertices can not k-dominate all vertices of B_{i+1}; hence $|D_1 \cap B_{i+1}| \geq 3$ if B_{i+1} is external.

If $B_{i+1} = R'$ is external but not a $(2k+1) \times 2k$ block, then it may happen that the two vertices of $D_1 \cap (P_{2k+1})_{j+2k}$ also k-dominate all vertices of R'. Therefore we only obtain $|R' \cap D_1| \geq 2$ in this case. □

Lemma 3 *Let $|D_1 \cap B_i| = 0$ for some internal block B_i, and let $n > 4k$. If k is even, then $|D_1 \cap B_{i+1}| \geq 1$, and $|D_1 \cap B_{i-1}| \geq 2$. If B_{i+1} (B_{i-1}) is external then $|D_1 \cap B_{i+1}| \geq 2$ ($|D_1 \cap B_{i-1}| \geq 3$). If $B_{i+1} = R'$ is external but not a $(2k+1) \times 2k$ block, then it contains at least 1 vertex.*

Proof: Analogously to the proof of Lemma 2. □

Lemma 4 *Let $n \not\equiv 0(mod 2k)$, $n \not\equiv 1(mod 2k)$. Then there exists an external block R' with x, $2 \leq x \leq 2k-1$, columns and the following assertions hold:*

(1) If R' contains no vertex of D_1, then B_m, $m = \lfloor \frac{n}{2k} \rfloor$ contains at least two vertices of D_1 if k is even.

(2) If k is odd and $|R' \cap D_1| = 0$, then $|B_m \cap D_1| \geq 1$ holds. If $|B_m \cap D_1| = 1$, then there exists at least one block B_r, $1 \leq r \leq m-1$, such that $|B_r \cap D_1| \geq 2$ and $|D_1 \cap B_l| \geq 1$ holds for all $B_l, r < l < m-1$. If in addition $|B_{r-1} \cap D_1| = 0$ holds, then $|B_r \cap D_1| \geq 3$.

Proof: If k is even, then C_1 does not contain the vertex $(k+1, n-x)$. Therefore our first assertion obviously holds.

Let k be odd, $|D_1 \cap R'| = 0$ and $|D_1 \cap B_m| = 1$.. Then $B_m \cap D_1 = \{(k+1, n-x)\}$ must hold. So the first $k-1$ columns of B_m must be k-dominated by a vertex of B_{m-1}. If B_{m-1} also contains only one vertex of D_1, then this vertex must again be contained in the column adjacent to B_m etc. Hence $|B_l \cap D_1| = 1$ must hold for all B_l, $r < l < m$. Also, a block B_r with $|B_r \cap D_1| > 1$ must exist since at least B_1 must contain a second vertex of D_1 because the vertex $(k+1, 2k)$ alone cannot k-dominate all vertices of B_1.

It follows analogously to the considerations in the proof of Lemma 2 that the first column of B_r contains at least two vertices of D_1 if B_{r-1} contains no vertex of D_1. Since D_1 also contains a vertex of the last column of B_r, $|B_r \cap D_1| \geq 3$ holds. □

In the sequel we finish the proof for even n by showing that $|D_1| \geq |S_1|$ holds for every dominating set D_1 of C_1.

Let first $n \equiv 0(mod 2k)$. The set S_1 we determined above has the property that it contains exactly one vertex of each $(2k+1) \times 2k$ block of C_1. Let B_s, $s \in M \subset \{1, \dots, \frac{n}{2k}\}$, denote those $(2k+1) \times 2k$ blocks of C_1 which contain no vertex of D_1. Then, depending on the parity of k either all blocks B_{s+1} or all blocks B_{s-1} contain at least two vertices of D_1. This immediately implies that $|D_1| \geq |S_1|$.

Let $n \not\equiv 0(mod 2k)$. We know from the above that S_1 also contains exactly one vertex of the external block R'. If R' now also contains at least one vertex of D_1 our result again follows from Lemmas 1, 2 and 3. If $|R' \cap D_1| = 0$ and k is even, then $|D_1| \geq |S_1|$ follows from Lemma 3 and the first assertion of Lemma 4. If k is odd, Lemma 2 together with assertion (2) of Lemma 4 implies that $|D_1| \geq |S_1|$.

For odd n the above lemmas, with minor modifications, also hold for C_2. We list those results without proofs, since they are proven completely in the same way as for C_1. The set D_2 now denotes a k-dominating set of C_2.

Lemma 5 *Let $|D_2 \cap B_i| = 0$ for some internal block B_i, and let $n > 4k$. If k is odd, then $|D_2 \cap B_{i+1}| \geq 1$, and $|D_2 \cap B_{i-1}| \geq 2$. If B_{i+1} (B_{i-1}) is external then $|D_2 \cap B_{i+1}| \geq 2$ ($|D_2 \cap B_{i-1}| \geq 3$). If $B_{i+1} = R'$ is external but not a $(2k+1) \times 2k$ block, then it contains at least 1 vertex.*

Lemma 6 *Let $|D_2 \cap B_i| = 0$ for some internal block B_i, and let $n > 4k$. If k is even, then $|D_2 \cap B_{i-1}| \geq 1$, and $|D_2 \cap B_{i+1}| \geq 2$. If B_{i-1} (B_{i+1}) is external then $|D_2 \cap B_{i-1}| \geq 2$ ($|D_2 \cap B_{i+1}| \geq 3$). If B_{i+1} is external but not a $(2k+1) \times 2k$ block, then it contains at least 2 vertices.*

Lemma 7 *Let $n \not\equiv 0(mod 2k)$, $n \not\equiv 1(mod 2k)$. Then there exists an external block R' with x, $2 \leq x \leq 2k-1$, columns and the following assertions hold:*

(1) If R' contains no vertex of D_2, then $B_m, m = \lfloor \frac{n}{2k} \rfloor$ contains at least two vertices of D_2 if k is odd.

(2) If k is even and $|R' \cap D_2| = 0$, then $|B_m \cap D_2| \geq 1$ holds. If $|B_m \cap D_2| = 1$, then there exists at least one block $B_r, 1 \leq r \leq m-1$, such that $|B_r \cap D_2| \geq 2$ and $|D_2 \cap B_l| \geq 1$ holds for all $B_l, r < l < m-1$. If in addition $|B_{r-1} \cap D_2| = 0$ holds, then $|B_r \cap D_2| \geq 3$.

If n is odd but $n \not\equiv 1(mod 2k)$ then of course we can now argue as above for C_1 and C_2 that $|D_1| \geq |S_1|$ and $|D_2| \geq |S_2|$ hold.

Finally we consider the case that $n \equiv 1(mod 2k)$.

a) Let k first be even. Clearly R' now consists of one column. Assume that on C_1 this block contains no vertex of a k-dominating set D_1. From above we know that S_1 also contains no vertex of R'. Let $B_s, s \in M \subset \{1, \dots, \lfloor \frac{n}{2k} \rfloor\}$, again denote those $(2k+1) \times 2k$ blocks of C_1 which contain no vertex of D_1. By Lemma 3 all blocks B_{s-1} now contain at least two vertices of D_1 which implies that $|D_1| \geq |S_1|$.

The fact that $|D_2| \geq |S_2|$ holds for any k-dominating set D_2 of C_2 follows since assertion (2) of Lemma 7 also holds in this case. This can be seen immediately if we consider the proof of Lemma 4, assertion (2)!

b) Let k now be odd. For C_1 it is again clear that S_1 is minimal by Lemma 2 and the fact that the block R' also contains no vertex of S_1.

To show that S_2 is also minimal we need another lemma.

Lemma 8 *Lemma 7, assertion (2), also holds if k is odd and $n \equiv 1 (mod 2k)$.*

Proof: If k is odd the vertex $(k+1, n-2)$ k-dominates all vertices of R'. But this vertex does not k-dominate the vertices of the first $k-2$ columns of B_m. Therefore we can proceed as in the proof of Lemma 4, assertion (2), to show that our claim holds. □

If the blocks $B_s, s \in M \subset \{1, 2, \dots m\}$, are now again those $(2k+1) \times 2k$ blocks which contain no vertex of D_2, then Lemma 5 implies blocks B_{s-1} contain at least two vertices of D_2. This, together with Lemma 8 again shows that S_2 is a minimal k-dominating set of C_2. □

References

[1] C.F. de Jaenisch, *Applications de l'Analyse Mathematique an Jenudes Echecs*, Petrograd, 1862.

[2] M. El-Zahar, C.M.Pareek, Domination Number of Products of Graphs, *Ars Combin.* **31** (1991), 223–227.

[3] R.J. Faudree, R.H. Schelp, The Domination Number for the Product of Graphs, *Congr. Numer.* **79** (1990), 29–33.

[4] D.C. Fisher, The domination number of complete grid graphs, *J. Graph Theory*, to appear.

[5] S. Gravier, A. Khelladi, On the dominating number of cross product of graphs, *Discrete Math.* **145** (1995), 273–277.

[6] M.S. Jacobson, L.F. Kinch, On the Domination Number of Products of Graphs I, *Ars Combin.* **18** (1983), 33–44.

[7] M.S. Jacobson, L.F. Kinch, On the Domination Number of the Products of Graphs II: Trees, *J. Graph Theory* **10** (1986), 97–106.

[8] P.K. Jha, S. Klavžar, Independance and matching in direct-product Graphs, preprint 1995.

[9] P.K. Jha, S. Klavzar, B. Zmazek, Isomorphic components of Kronecker product of bipartite graphs, preprint 1994.

[10] S. Klavzar, N. Seifter, Dominating Cartesian products of cycles, *Discrete Appl. Math.* **59** (1995), 129–136.

[11] S. Klavzar, B. Zmazek, On a Vizing-like conjecture for direct product graphs, preprint 1995.

[12] A. Klobucar, Domination numbers of cardinal products of graphs, preprint 1996.

[13] A. Klobucar, The domination numbers of the cardinal products $P_6 \times P_n$, preprint 1996.

[14] V.G. Vizing, The cartesian product of graphs, *Vychisl. Sistemy* **9** (1963), 30–43.