Critical sets in products of latin squares

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This paper is dedicated to the memory of Derrick Breach.

Abstract

This paper is about critical sets in latin squares and the weaker concept of partial latin squares with unique completion. This work involves taking two known partial latin squares with unique completion, or critical sets in latin squares, and using a product construction to produce new partial latin squares with unique completion, or new critical sets in larger latin squares.

1 Definitions and background results

Throughout this paper the concepts of partial latin squares with unique completion and critical sets of latin squares are used, discussed and extended. In this section these terms are defined and the existing results are presented.

Definition 1.1 *A latin square L of order n is an $n \times n$ array with entries chosen from a set $\mathcal{N}$, of size n, such that each element of $\mathcal{N}$ occurs precisely once in each row and in each column.*

Definition 1.2 *If $\mathcal{N} = \{0, 1, \ldots, n-1\}$ and the rows and columns of the latin square are indexed from 0 to $n-1$, then a back circulant latin square has the integer $i + j$ (mod n) in cell (i, j). A back circulant latin square, of order n, corresponds to the cyclic group C_n. Hence C_n will be used to represent such a latin square.*

For convenience, a latin square will sometimes be represented as a set of ordered triples $(i, j; k)$, this is read to mean that element k occurs in cell (i, j) of the latin square.

Using this notation, a back circulant latin square can be represented by the set $\{(i, j; i + j) \mid 0 \leq i, j \leq n-1\}$, where addition is taken modulo n.

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Definition 1.3 *Let L be a latin square of order n. If $n - s$ rows of L can be deleted, and $n - s$ columns of L can be deleted to leave s^2 elements of L which form a latin square S of order s then S is a latin subsquare (or simply a subsquare) of L.*

Definition 1.4 *A partial latin square P of order n is an $n \times n$ array with entries chosen from a set $\mathcal{N}$, of size n, such that each element of $\mathcal{N}$ occurs at most once in each row and column.*

So P may contain a number of empty cells and a triple $(i, j; k) \in P$ if and only if the (i, j) position of this partial latin square has entry k.

In the following example, and in all examples in this paper which involve partial latin squares, an entry of $-$ in the array is used to indicate that the cell is empty.

Example 1.5 *Let* $P = \{(0,0;1),(0,1;2),(1,2;0),(2,0;0),(2,2;2)\}$.

$$P = \begin{bmatrix} 1 & 2 & - \\ - & - & 0 \\ 0 & - & 2 \end{bmatrix}$$

Since each of the entries 0,1 and 2 occur at most once in any row or column, P is a partial latin square. However, there is no latin square of order 3 which contains P.

As the above example demonstrates, a partial latin square need not be contained in any latin square of the same order as itself. This point becomes important later on when partial latin squares are used in Definition 1.23.

On the other hand, some partial latin squares are contained in many latin squares of the same order. The partial latin squares of greatest interest in this paper are those which are contained in precisely one latin square of that order.

It is possible that not all the elements of the set $\mathcal{N}$, from which the entries of P are chosen, are used in the partial latin square P. It is for this reason that the set $\mathcal{N}$ is listed in full for discussions of unique completion.

Definition 1.6 *A partial latin square P of order n, is said to be uniquely completable (or P completes uniquely to L, or P has (UC)) if for the given set of possible entries, $\mathcal{N}$, there is one and only one latin square, L, of order n which has element k in position (i, j) for each $(i, j; k) \in P$.*

The following concept was introduced by Nelder [10].

Definition 1.7 *A critical set in a latin square L is a partial latin square which has a unique completion to L and all proper subsets of the partial latin square complete to at least two distinct latin squares. Formally, a critical set, in a latin square L of order n, is a set $C = \{(i, j; k) \mid i, j \in \{0, 1, \ldots, n-1\}$ and $k \in \mathcal{N}\}$ such that,*

1. *L is the only latin square of order n which has element k in position (i, j), for each $(i, j; k) \in C$;*

2. *no proper subset of C satisfies 1.*

Example 1.8 *Consider C_6, the back circulant latin square of order 6, and let $\mathcal{E}_6 = \{(0,0;0),(0,1;1),(0,2;2),(1,0;1),(1,1;2),(2,0;2),(4,5;3),(5,4;3),(5,5;4)\}$. The set $\mathcal{E}_6$ is a critical set in C_6. The latin square C_6 and the critical set $\mathcal{E}_6$ are as follows.*

$$C_6 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 5 & 0 & 1 & 2 & 3 & 4 \end{bmatrix} \qquad \mathcal{E}_6 = \begin{bmatrix} 0 & 1 & 2 & - & - & - \\ 1 & 2 & - & - & - & - \\ 2 & - & - & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & - & 3 \\ - & - & - & - & 3 & 4 \end{bmatrix}$$

Colbourn, Colbourn and Stinson [2] make the observation that although the recognition of critical sets in "special cases" (where the unique completion of the partial latin square is relatively easy to verify) is "straightforward", it is "not the case in general". They prove that "deciding whether a partial latin square has more than one completion is NP–complete, *even if* one completion is given as part of the problem description".

The definitions below follow the terminology of groupoids as used in Dénes and Keedwell [6], p. 23.

Definition 1.9 *Two latin squares L and M (both of order n) are said to be isotopic if there exists an ordered triple (ϕ, ψ, χ) of one-to-one mappings such that ϕ, ψ and χ map the rows, columns and entries, respectively, of L onto M.*

Then $M = \{(r\phi, c\psi; e\chi) \mid (r, c; e) \in L\}$.

That is, two latin squares are isotopic if one can be transformed into the other by rearranging rows, rearranging columns and renaming entries.

Definition 1.10 *The two latin squares are said to be isomorphic if the one-to-one mappings ϕ, ψ and χ are equal.*

Definition 1.11 *Two partial latin squares (or critical sets) $\mathcal{P}$ and P are said to be isotopic if there exists an ordered triple (ϕ, ψ, χ) of one-to-one mappings which maps the elements $(a, b; c)$ of $\mathcal{P}$ onto the elements $(x, y; z)$ of P.*

Definition 1.12 *As for latin squares, two partial latin squares are said to be isomorphic if the one-to-one mappings ϕ, ψ and χ are equal.*

Lemma 1.13 *(Donovan et al. [8]) If C is a critical set in the latin square L and (ϕ, ψ, χ) is an isotopism from C to $\mathcal{C}$ then $\mathcal{C}$ is a critical set in the latin square $\mathcal{L}$ and $\mathcal{L}$ is isotopic to L.*

It follows that a partial latin square isotopic to a partial latin square with unique completion also has unique completion.

Definition 1.14 *Let P be a partial latin square and L be a latin square to which P completes. An element p of the partial latin square P is a–essential if there is an $a \times a$ subsquare S of the latin square L such that $(P \setminus \{p\}) \cap S$ does not have (UC) in S.*

Let P be a partial latin square which completes uniquely to a latin square L, then an element p of the partial latin square P is 2–essential if there is a 2×2 subsquare S of the latin square L such that $(P \setminus \{p\}) \cap S$ does not have (UC) in S. Since a partial latin square of order 2 with at least one entry is always uniquely completable, $(P \setminus \{p\}) \cap S$ does not have (UC) if and only if $(P \setminus \{p\}) \cap S = \emptyset$. Hence for an element p to be 2–essential there must be a latin subsquare S of order 2 such that $S \cap P = \{p\}$.

Example 1.15 *Consider a partial latin square P with unique completion to C_4, the back circulant latin square of order 4. Let*

$$P = \begin{bmatrix} 0 & 1 & 2 & - \\ 1 & 2 & - & - \\ - & - & 0 & - \\ 3 & - & 1 & - \end{bmatrix} \qquad \text{and } C_4 = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$$

The partial latin square P is not a critical set since some of the elements may be removed to form a new partial latin square with unique completion. For example, the element $(0,2;2)$ can be removed from P, that is to say, $P \setminus \{(0,2;2)\}$ has (UC).

However, some elements cannot be removed if the resulting partial latin square is to have unique completion. Both the elements $(0,1;1)$ and $(1,1;2)$ are 2–essential. To see that $(0,1;1)$ is 2–essential consider $S = \{(0,1;1),(0,3;3),(2,1;3),(2,3;1)\}$, a subsquare of order 2 of C_4. Now $S \cap P = \{(0,1;1)\}$, so $(P \setminus \{(0,1;1)\}) \cap S = \emptyset$ which obviously cannot have (UC) in S. Similarly there is another subsquare of order 2 of which $(1,1;2)$ is the only element in P. Hence this element is also 2–essential.

Definition 1.16 *A partial latin square P is a–critical if it has unique completion to the latin square L and every element p of P is a–essential.*

If a partial latin square is a–critical then the partial latin square is a critical set. (That is to say, a–critical sets form a special subclass of the class of critical sets.)

A partial latin square P is 2–critical if it has unique completion to the latin square L and every element p of P is 2–essential. If a partial latin square is 2–critical then it is a critical set. Curran and van Rees used this method to show that certain sets are critical sets in their paper [5].

Nelder [10] uses scs(m) to denote the size of the smallest of the critical sets of the latin squares of order m and lcs(m) to denote the size of the largest of the critical sets of the latin squares of order m.

In all of the subsequent sections of this paper, partial latin squares and critical sets are examined which complete to latin squares constructed from smaller latin squares. The following is as defined in Street and Street [13], p. 31.

Definition 1.17 *Let M and N be latin squares of order m and n respectively with entries from the sets $\{0, 1, \ldots, m-1\}$ and $\{0, 1, \ldots, n-1\}$ respectively. Define N^r to be the array obtained from N by adding rn to each entry of N, for $r = 0, 1, \ldots, m-1$. (Then $N^r = rnJ + N$ where J is the matrix whose entries are all 1's.) The direct product of M with N is L, the latin square of order mn constructed by replacing the entry r in M by the array N^r. One writes $L = M \times N$.*

The direct product of C_2 with a latin square L of order n is:

$$L^* = \{(a, b; c), (a+n, b; c+n), (a, b+n; c+n), (a+n, b+n; c) \mid (a, b; c) \in L\}.$$

Or, in block matrix form:

$$L^* = \begin{bmatrix} L^0 & L^1 \\ L^1 & L^0 \end{bmatrix}$$

Since the array L^0 is identical to the array L, the superscript is omitted from now on.

Example 1.18 *The direct product of C_2 with C_4, that is $C_2 \times C_4$, is the latin square:*

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 0 & 5 & 6 & 7 & 4 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 3 & 0 & 1 & 2 & 7 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 5 & 6 & 7 & 4 & 1 & 2 & 3 & 0 \\ 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ 7 & 4 & 5 & 6 & 3 & 0 & 1 & 2 \end{bmatrix}$$

One may establish results about partial latin squares with unique completion for the latin squares arising from direct products of other latin squares. These results follow from information about the partial latin squares with unique completion and the critical sets in the component latin squares.

The next two results were originally stated by Stinson and van Rees as one lemma.

Lemma 1.19 *(Stinson and van Rees* [12]*) For* L*, a latin square of order* n*, and* C*, a critical set of* L*, define the partial latin square* P^* *to be*

$$P^* = L \cup \{(x+n, y; z+n), (x, y+n; z+n), (x+n, y+n; z) \mid (x, y; z) \in C\}$$

or in block matrix form:

$$P^* = \begin{bmatrix} L & C^1 \\ C^1 & C \end{bmatrix}$$

where C^1 *is the appropriate image of* C *in* L^1*. Then* P^* *completes uniquely to the latin square* L^* *of order* $2n$ *which is the direct product of* C_2 *with* L*.*

Lemma 1.20 *(Stinson and van Rees* [12]*) For* L*, a latin square of order* n*, and* C*, a critical set of* L*, define the partial latin square* E^* *to be*

$$E^* = (L \backslash C) \cup \{(x+n, y; z+n), (x, y+n; z+n), (x+n, y+n; z) \mid (x, y; z) \in C\}$$

or in block matrix form:

$$E^* = \begin{bmatrix} L \setminus C & C^1 \\ C^1 & C \end{bmatrix}$$

Then there is a critical set $\mathcal{C}$ *for* $L^* = C_2 \times L$*, the latin square of order* $2n$*, such that:*

$$E^* \subseteq \mathcal{C} \subseteq P^*$$

where P^* *is as in Lemma 1.19.*

To prove this, Stinson and van Rees showed that every element of E^* is essential.

Lemma 1.21 *(Stinson and van Rees* [12]*) Let* L *be a latin square of order* n*, let* C *be a 2–critical set of* L*, and let* L^* *denote the latin square of order* $2n$ *which is the direct product of* C_2 *with* L*. Then the partial latin square* P^* *is 2–critical in the latin square* L^**, where*

$$P^* = \begin{bmatrix} L & C^1 \\ C^1 & C \end{bmatrix}$$

and C^1 *is the appropriate image of* C *in* L^1*. (Note that* P^* *is exactly as in Lemma 1.19.)*

This was the first method for constructing critical sets from smaller critical sets. This paper and the paper by Cooper, Donovan and Gower [3] extend the work of Stinson and van Rees.

Since there is a critical set given by Curran and van Rees for C_n, the back circulant latin square of order n, which is 2–critical, the above result gives a construction for critical sets in $C_2 \times C_n$ for n even. Here $\mathcal{E}_n$ is used to denote a critical set isotopic to that of Curran and van Rees. Use $\mathcal{E}_{2,n}$ to denote the critical set of $C_2 \times C_n$ obtained from this construction for n even, then:

$$\mathcal{E}_{2,n} = C_n \cup \{(a, b+n; c+n), (a+n, b; c+n), (a+n, b+n; c) \mid (a, b; c) \in \mathcal{E}_n\}.$$

Example 1.22 *Below is $\mathcal{E}_{2,4}$ which is a critical set in the latin square $C_2 \times C_4$ of Example 1.18.*

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & - & - \\ 1 & 2 & 3 & 0 & 5 & - & - & - \\ 2 & 3 & 0 & 1 & - & - & - & - \\ 3 & 0 & 1 & 2 & - & - & - & 6 \\ 4 & 5 & - & - & 0 & 1 & - & - \\ 5 & - & - & - & 1 & - & - & - \\ - & - & - & - & - & - & - & - \\ - & - & - & 6 & - & - & - & 2 \end{bmatrix}$$

The following definition is used by Cooper, Donovan and Seberry [4].

Definition 1.23 *A critical set C of order n is said to be a strongly critical set of a latin square L, based on the set $\mathcal{N}$, if there exists a set $\{P_1, P_2, \ldots, P_f\}$ of partial latin squares of order n with $f = n^2 - |C|$ such that:*

1. $C = P_1 \subset P_2 \subset \cdots \subset P_f \subset L$;
2. *for any i, $2 \le i \le f$, where $P_i = P_{i-1} \cup \{(r_{i-1}, s_{i-1}; t_{i-1})\}$, the set $P_{i-1} \cup \{(r_{i-1}, s_{i-1}; t')\}$ is not a partial latin square for any $t' \in \mathcal{N} \setminus \{t\}$.*

Example 1.24 *Consider L, the back circulant latin square of order 3 and the critical set C which completes to L, both of which are given below.*

$$L = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & - & 2 \\ - & - & - \\ 2 & - & - \end{bmatrix}$$

The entries for L and C are obviously members of the set $\mathcal{N} = \{0, 1, 2\}$. Put $P_1 = C$, $P_2 = P_1 \cup \{(0,1;1)\}$, $P_3 = P_2 \cup \{(1,0;1)\}$, $P_4 = P_3 \cup \{(1,2;0)\}$, $P_5 = P_4 \cup \{(2,1;0)\}$ and $P_6 = P_5 \cup \{(1,1;2)\}$; then $L = P_6 \cup \{(2,2;1)\}$. So there exists a set $\{P_1, \ldots P_6\}$ of partial latin squares which satisfy the conditions given in Definition 1.23, hence it can be seen that C is a strongly critical set.

Note that the set of partial latin squares, $\{P_1, P_2, \ldots, P_f\}$, satisfying the requirements of Definition 1.23, given in this example is not unique. This is generally the case.

The vast majority of critical sets which can be found in the literature are actually strongly critical sets but it is not true that all critical sets are strongly critical. It is helpful to consider an example of a critical set which is not strongly critical.

Example 1.25 *The following set C is a critical set which is not strongly critical,* [7].

$$C = \begin{bmatrix} 5 & - & 3 & 4 & - & - \\ 2 & - & 1 & - & 6 & - \\ 3 & 1 & 2 & - & - & - \\ 4 & - & - & 1 & 2 & - \\ - & 6 & - & - & 3 & - \\ - & - & - & - & - & - \end{bmatrix}$$

Obviously $\mathcal{N}$ *is* $\{1,2,3,4,5,6\}$. *Consider the cell* $(4,0)$ *which is empty. The union of the entries in the column and in the row in which this cell sits is* $\{5,2,3,4\} \cup \{6,3\} = \mathcal{N} \setminus \{1\}$. *Hence* $C \cup \{(4,0;t)\}$ *for any* $t \in \mathcal{N} \setminus \{1\}$ *is not a partial latin square. Thus when completing C to a latin square the entry in the cell (4,0) is "forced" to be 1. This in turn "forces" the entry in the cell (5,0) to be 6.*

By similar arguments the top row can be filled in. The following partial latin square, C′, is that derived from C when these elements are added to the left–most column and the top row.

$$C' = \begin{bmatrix} 5 & 2 & 3 & 4 & 1 & 6 \\ 2 & - & 1 & - & 6 & - \\ 3 & 1 & 2 & - & - & - \\ 4 & - & - & 1 & 2 & - \\ 1 & 6 & - & - & 3 & - \\ 6 & - & - & - & - & - \end{bmatrix}$$

It is easily checked that is as far as one can proceed with this style of argument. However, the following type of argument can be used to show that the partial latin square has unique completion.

Consider the third column of C′, in particular the empty cells; (3,2), (4,2) and (5,2). Suppose that the (3,2) cell is filled with the entry 5, then the result is a partial latin square, however, this new partial latin square cannot be completed to a latin square. If there was a latin square (of order 6) containing $C' \cup \{(3,2;5)\}$ *then the entry 6 would have to occur somewhere in this column. Now, a 6 could not be placed in the (4,2) cell as there is already a 6 in this row, nor could a 6 be placed in the (5,2) cell because, as discussed above, the cell (5,0) must contain a 6 and only one 6 is permitted in any row. Hence it can be seen that there is nowhere to place a 6 if the 5 goes in the cell (3,2).*

Since the notion of placing a 5 in the (3,2) cell has been dismissed, the only remaining option is to place a 6 there.

In order to determine the admissible entry for some other empty cells, similar arguments to the one just used must be invoked.

Bate and van Rees, [1], capture the essence of the second type of argument used in the previous example in the following definition.

Definition 1.26 *A critical set C of order n is said to be a semi-strong critical set of a latin square L, based on the set $\mathcal{N}$, if there exists a set $\{P_1, P_2, \ldots, P_f\}$ of partial latin squares of order n with $f = n^2 - |C|$ such that:*

1. $C = P_1 \subset P_2 \subset \cdots \subset P_f \subset L$;
2. *for any i, $2 \leq i \leq f$, where $P_i = P_{i-1} \cup \{(r_{i-1}, s_{i-1}; t_{i-1})\}$, one of the sets $P_{i-1} \cup \{(r_{i-1}, s_{i-1}; t')\}$, $P_{i-1} \cup \{(r_{i-1}, s'; t_{i-1})\}$ or $P_{i-1} \cup \{(r', s_{i-1}; t_{i-1})\}$ is not a partial latin square for any $t' \in \mathcal{N} \setminus \{t\}$, $s' \in \{0, 1, \ldots, n-1\} \setminus \{s\}$, or $r' \in \{0, 1, \ldots, n-1\} \setminus \{r\}$ respectively.*

Then C in Example 1.25 is a semi-strong critical set.

Definition 1.27 *If P is a partial latin square of order n with unique completion to a latin square L and there exists a set $\{P_1, P_2, \ldots, P_f\}$ of partial latin squares of order n with $f = n^2 - |P|$ such that:*

1. $P = P_1 \subset P_2 \subset \cdots \subset P_f \subset L$;
2. *for any i, $2 \leq i \leq f$, where $P_i = P_{i-1} \cup \{(r_{i-1}, s_{i-1}; t_{i-1})\}$, the set $P_{i-1} \cup \{(r_{i-1}, s_{i-1}; t')\}$ is not a partial latin square for any $t' \in \mathcal{N} \setminus \{t\}$*

then P is said to be strongly uniquely completable.

If P is strongly uniquely completable, then P is a strongly critical set or some subset of P is a strongly critical set.

It is an obvious generalisation of the definition above to define a semi-strong uniquely completable set.

2 A product construction

This work stems from an interest in extending the definitions and results of Stinson and van Rees, published in [12], which were given in Section 1.

2.1 Products of partial latin squares

The following definition has a similar flavour to Definition 1.17.

Definition 2.1 *Let P be a partial latin square of order m with entries taken from the set $\{0, 1, \ldots, m-1\}$ such that P completes uniquely to the latin square M and let Q be a partial latin square, but of order n, with entries taken from the set $\{0, 1, \ldots, n-1\}$ such that Q completes uniquely to N.*

Let Q^r be the array obtained from Q by adding rn to the entry in each non-empty cell of Q, for $r = 0, 1, \ldots, m-1$. Similarly let N^r be the array obtained from N by adding rn to the entry in each cell of N, for $r = 0, 1, \ldots, m-1$.

Define the completable product of P with Q, written $P \times Q$, to be the partial latin square R of order mn which is the array obtained by replacing each entry r of P with the array N^r and each entry r of $M \setminus P$ with the array Q^r.

The completable product of P with Q is contained in the direct product of M with N.

Then Curran and van Rees' work is for completable products $P \times Q$ with P a partial latin square with unique completion to M where

$$P = \begin{bmatrix} 0 & - \\ - & - \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(such as in Example 1.22).

Remark 2.2 *Let $(a, b; c)$ be an element of the latin square M and let $(\alpha, \beta; \gamma)$ be an element of the latin square N. Then it is a consequence of Definition 1.17 that the latin square $L = M \times N$ contains the element $(an + \alpha, bn + \beta; cn + \gamma)$. (Here $a, b, c \in \{0, 1, \ldots, m-1\}$ and $\alpha, \beta, \gamma \in \{0, 1, \ldots, n-1\}$.) If P is a partial latin square with unique completion to M and Q is a partial latin square with unique completion to N then the following statements can be made about the completable product $R = P \times Q$:*

1. *if $(a, b; c) \in P$ then $(an + \alpha, bn + \beta; cn + \gamma) \in R$;*
2. *if $(a, b; c) \notin P$ but $(\alpha, \beta; \gamma) \in Q$ then $(an + \alpha, bn + \beta; cn + \gamma)$ is still an element of the partial latin square R;*
3. *if $(a, b; c) \notin P$ and $(\alpha, \beta; \gamma) \notin Q$ then $(an + \alpha, bn + \beta; cn + \gamma) \notin R$.*

Since the direct product of latin squares is not commutative, the completable product of partial latin squares is not commutative either. However the following results are very useful even though they are weaker than commutativity.

Lemma 2.3 *The direct product $L = M \times N$ of two latin squares M and N is isomorphic to the direct product $L' = N \times M$.*

Proof Without loss of generality suppose that M is a latin square of order m whose entries are those of the set $\{0, 1, \ldots, m-1\}$ and that N is a latin square of order n whose entries belong to the set $\{0, 1, \ldots, n-1\}$.

If $(a, b; c)$ is an element of the latin square M and if $(x, y; z)$ is an element of N, then there will be an element $(a.n + x,\ b.n + y;\ c.n + z)$ in L and an element $(x.m + a,\ y.m + b;\ z.m + c)$ in L'.

Then the one-to-one mapping $i.n + j \to j.m + i$ can be applied to the rows, columns and entries of L to map the elements of L to those of L'. □

Lemma 2.4 *Let P and Q be partial latin squares with unique completion to the latin square M and N respectively. The completable products $P \times Q$ and $Q \times P$ are isomorphic.*

Proof The same mappings are used as in the proof above. It is only necessary to show that each empty cell of the completable product $P \times Q$ is mapped to an empty cell of the completable product $Q \times P$.

In Remark 2.2, it is pointed out that the only empty cells of the completable product $P \times Q$ are those of the form $(a.n + x, b.n + y)$ where there is an element $(a, b; c)$ of the latin square M which is not an element of P and there is an element $(x, y; z)$ of N which is not an element of Q.

In this case the element $(x.m + a, y.m + b; z.m + c)$ of $L' = N \times M$ is not an element of $Q \times P$ so the cell $(x.m + a, y.m + b)$ is empty. □

2.2 The unique completion of products of partial latin squares

The following results about completable products generalise the work of Stinson and van Rees beyond a doubling construction. In [9] a trebling construction was given but the following theorem is much more general.

Theorem 2.5 *Let P be a partial latin square which is strongly uniquely completable to the latin square M of order m and let Q be a partial latin square with unique completion to the latin square N of order n. Then the partial latin square $R = P \times Q$ has unique completion to L, the direct product of M and N.*

Proof Since P is strongly uniquely completable there is a set of partial latin squares, $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_f\}$, with $f = m^2 - |P|$, satisfying the conditions of Definition 1.27. Then $\mathcal{P}_i = \mathcal{P}_{i-1} \cup \{(r_{i-1}, s_{i-1}; t_{i-1})\}$ and it is known that for each t' in the set $\{0, 1, \ldots, m-1\} \setminus \{t_{i-1}\}$, t' is either in row r_{i-1} of $\mathcal{P}_{i-1}$ or in column s_{i-1} of $\mathcal{P}_{i-1}$. That is to say there is a cell $(r_i, x; t') \in \mathcal{P}_i$, for some x, $x \neq s_i$, or there is a cell $(x, s_i; t') \in \mathcal{P}_i$, for some x, $x \neq r_i$.

Recall that $\mathcal{P}_1 = P$ by the definition and define $\mathcal{R}_1 = \mathcal{P}_1 \times Q = P \times Q = R$.

Now $\mathcal{P}_2 = \mathcal{P}_1 \cup \{(r_1, s_1; t_1)\}$ which means that in the positions of $\mathcal{R}_1 = R$ with rows indexed from $r_1 n$ to $(r_1 + 1)n - 1$ and columns indexed from $s_1 n$ to $(s_1 + 1)n - 1$ there is a copy of Q^{t_1}. Also, from the discussion above it can be concluded that for each $t' \in \{0, 1, \ldots, m-1\} \setminus \{t_1\}$ there is a copy of $N^{t'}$ which is positioned such that it is either in the rows indexed from $r_1 n$ to $(r_1 + 1)n - 1$ or in the columns indexed from $s_1 n$ to $(s_1 + 1)n - 1$. So that each element of $\{0 + t'n, 1 + t'n, \ldots, n - 1 + t'n\}$ occurs exactly once in each of these rows or it occurs exactly once in each of these columns. Hence the only elements of $\mathcal{N}_L = \{0, 1, \ldots, mn - 1\}$ which have not already occurred in the rows indexed from $r_1 n$ to $(r_1 + 1)n - 1$ or in the columns indexed from $s_1 n$ to $(s_1 + 1)n - 1$

are $0 + t_1 n,\ 1 + t_1 n,\ \ldots,\ n - 1 + t_1 n$. So these are the only entries which may be placed in the cells which are defined by the intersection of these rows and columns, but these cells already contain a copy of Q^{t_1} which has unique completion to N^{t_1} when restricted to this set of entries, so this $n \times n$ subarray of $R = \mathcal{R}_1$ is forced to complete to N^{t_1}. This produces a new partial latin square $\mathcal{R}_2 = \mathcal{P}_2 \times Q$.

Similarly, a sequence of partial latin squares, $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_f$ is obtained with $\mathcal{R}_f = \mathcal{P}_f \times Q$. Since $\mathcal{P}_f$ has only one entry less than the latin square M, the partial latin square $\mathcal{R}_f$ has only one subarray containing a copy of Q^i for some i. There is only one possible completion from this point and it leads to the latin square $L = M \times N$. □

Corollary 2.6 *The result still holds if Q, instead of P, is the partial latin square which is strongly uniquely completable.*

Proof By Lemma 2.4 and Lemma 1.13 the completable product of P with Q must have unique completion. □

If both of the partial latin squares P and Q are strongly uniquely completable then their completable product is strongly uniquely completable. In the case where one of P or Q is not strongly uniquely completable (or if both are not) then the completable product $P \times Q$ is not strongly uniquely completable either.

Theorem 2.7 *Let P be a partial latin square which is uniquely completable to the latin square M of order m and let Q be a partial latin square with unique completion to the latin square N of order n. Suppose that at least one of the partial latin squares P and Q is semi-strong uniquely completable. Then the partial latin square $R = P \times Q$ has unique completion to L, the direct product of M and N.*

Proof Without loss of generality suppose that P is semi-strong uniquely completable. (If it weren't then Q would be and the following argument could have P and Q interchanged and by Lemma 2.4 the two products are isomorphic.) Then there is a set of partial latin squares, $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_f\}$, with $f = m^2 - |P|$, satisfying Conditions 1 and 2 of Definition 1.26. However, P is not necessarily a critical set so it might not meet all the requirements of that definition, but it will meet the enumerated conditions.

As in Theorem 2.5, the idea is to develop a sequence, $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_f$, of partial latin squares with $\mathcal{R}_1 = \mathcal{P}_1 \times Q = P \times Q = R$.

If $\mathcal{P}_i = \mathcal{P}_{i-1} \cup \{(r_{i-1}, s_{i-1}; t_{i-1})\}$ it is because either

1. t' is in row r_{i-1} or column s_{i-1} already for all $t' \in \mathcal{N}_M \setminus \{t_{i-1}\}$;

2. the cell (r_{i-1}, s') is not empty or the entry t_{i-1} already appears somewhere in column s' for all $s' \in \{0, 1, \ldots, n-1\} \setminus \{s_{i-1}\}$;

3. the cell (r', s_{i-1}) is not empty or the entry t_{i-1} already appears somewhere in row r' for all $r' \in \{0, 1, \ldots, n-1\} \setminus \{r_{i-1}\}$.

In case (1), by the arguement of Theorem 2.5, a copy of $N^{t_{i-1}}$ must sit in the cells of $P \times Q$ with rows indexed by $r_{i-1}n$ to $(r_{i-1}+1)n-1$ and columns indexed by $s_{i-1}n$ to $(s_{i-1}+1)n-1$.

In case (2), the entries $0 + t_{i-1}n, 1 + t_{i-1}n, \ldots$ must appear in each of the rows $r_{i-1}n$ to $(r_{i-1}+1)n-1$. Some already appear in the copy of $Q^{t_{i-1}}$ sitting in columns $s_{i-1}n$ to $(s_{i-1}+1)n-1$. Why must all the hitherto unused entries also sit in these columns? From the consequences of the definition of semi-strong uniquely completable detailed above, it can be seen that for each set of columns $s'n$ to $(s'+1)n-1$ there is either a copy of N^x for some x (that is, the cells are full and nothing can be put there) or there is a copy of $N^{t_{i-1}}$ in some other rows of that column which prevents any of those entries from being used elsewhere in those columns. Hence the only place those entries may be used in these rows is in columns $s_{i-1}n$ to $(s_{i-1}+1)n-1$. Furthermore, there is only one possible way to arrange them in those rows and columns given that there is a copy of $Q^{t_{i-1}}$ there already.

The arguement for case (3) is similar to that for case (2).

Hence the sequence $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_f$, of partial latin squares can be forced and the partial latin square $P \times Q$ completes uniquely. □

The product of any two semi-strong uniquely completable partial latin squares is also semi-strong uniquely completable. (Note that the strongly uniquely completable partial latin squares are special cases of the semi-strong ones.)

These results are not as general as one might like since they require that at least one of the partial latin squares be semi-strong uniquely completable. It may be possible to prove a similar result without that condition but a greater understanding of the critical sets which are not strongly or semi-strong critical must be gained first. Although this result is restrictive, it is sufficient for the later work in this paper.

Lemma 2.8 *The direct product of latin squares is associative. (That is, if L, M and N are latin squares then $L \times (M \times N) = (L \times M) \times N$.)*

Proof Let L be a latin square of order l, M be a latin square of order m and N be a latin square of order n. Let S be the latin square formed by the direct products of L with M and N such that $S = L \times (M \times N)$ and let $T = (L \times M) \times N$.

Let l_{xy} denote the entry in cell (x, y) of L, let m_{wv} denote the entry in cell (w, v) of M, and let n_{uz} denote the entry in cell (u, z) of N.

Consider the direct product of M with N. The cell (w, v) of M and the cell (u, z) of N combine to determine the cell $(w.n + u, v.n + z)$ of $M \times N$. The entries m_{wv} of M and n_{uz} of N yield an entry of $m_{wv}.n + n_{uz}$ in this cell of $M \times N$. Now this cell of $M \times N$ and the cell (x, y) of L combine to determine the cell $(x.mn + (w.n + u), y.mn + (v.n + z))$ of S. The entries l_{xy} of L and $m_{wv}.n + n_{uz}$ of $M \times N$ give rise to an entry of $l_{xy}.mn + (m_{wv}.n + n_{uz})$ in this cell of S.

Now consider the direct product of L with M. The cells (x, y) of L and (w, v) of M combine to determine the cell $(x.m+w, y.m+v)$ of their direct product. The entries of these cells of L and M are l_{xy} and m_{wv} respectively, hence the entry in this cell of $L \times M$ is $l_{xy}.m+m_{wv}$. Now this cell of $L \times M$ and the cell (u, z) of N combine to determine the cell $((x.m+w).n+u, (y.m+v).n+z)$ of T. The entries $l_{xy}.m + m_{wv}$ of $L \times M$ and n_{uz} of N yield an entry of $(l_{xy}.m + m_{wv}).n + n_{uz}$ in this cell of T.

It is easy to see that the cell $(x.mn + (w.n + u), y.mn + (v.n + z))$ of S is the same cell as the cell $((x.m+w).n+u, (y.m+v).n+z)$ of T. Now it is necessary to show that these cells of S and T have the same entries. It follows from properties of the integers that $l_{xy}.mn + (m_{wv}.n + n_{uz}) = (l_{xy}.m + m_{wv}).n + n_{uz}$ so these cells have the same entries. □

Let L be a latin square of order l, M be a latin square of order m and N be a latin square of order n. Let P, Q and R be partial latin squares which complete uniquely to L, M and N respectively.

Let U be the completable product $P \times (Q \times R)$ and let V be the completable product $(P \times Q) \times R$. Since the completable product of two partial latin squares is only defined if both partial latin squares have unique completion, the completable product $Q \times R$ must have unique completion for U to be defined. So far the only results about the unique completion of completable products are Theorems 2.5 and 2.7 so it can be proven that $Q \times R$ has unique completion only if either Q or R is semi-strong or strongly uniquely completable. Then U is well defined if at least one of Q or R is semi-strong or strongly uniquely completable (it may be possible that there are weaker condidtions which guarantee that U is well defined but they are not known). Similarly V is well defined if at least one of P or Q is semi-strong or strongly uniquely completable.

Theorem 2.9 *Let L be a latin square of order l, M be a latin square of order m and N be a latin square of order n. Let P, Q and R be partial latin squares which complete uniquely to L, M and N respectively.*

Where the completable product of the three partial latin squares P, Q and R is well defined, it is an associative operation.

Proof Let S denote the latin square formed by the direct products of L with M and N such that $S = L \times (M \times N) = (L \times M) \times N$. Obviously both $U = P \times (Q \times R)$

and $V = (P \times Q) \times R$ are partial latin squares which can be completed to S. The nonempty cells of U and V must both contain the same entries or they could not both complete to the same latin square S. Hence to show that $U = V$ it suffices to show that cell (i, j) of U is empty if and only if cell (i, j) of V is empty.

In general a cell of U has the form $(\alpha m.n + (\beta n + \gamma), \delta m.n + (\epsilon n + \zeta))$ for $\alpha, \delta \in \{0, 1, \ldots l - 1\}, \beta, \epsilon \in \{0, 1, \ldots m - 1\}$ and $\gamma, \zeta \in \{0, 1, \ldots n - 1\}$.

As explained in Remark 2.2, such a cell of U is empty if and only if the cell (α, δ) of P is empty and the cell $(\beta n + \gamma, \epsilon n + \zeta)$ of $Q \times R$ is empty too. Now, the cell $(\beta n + \gamma, \epsilon n + \zeta)$ of $Q \times R$ is empty if and only if both the cell (β, ϵ) of Q and the cell (γ, ζ) of R are empty.

Hence the cell $(\alpha m.n + (\beta n + \gamma), \delta m.n + (\epsilon n + \zeta))$ of U is empty if and only if the cell (α, δ) of P is empty, the cell (β, ϵ) of Q is empty and the cell (γ, ζ) of R is empty. This happens if and only if the cell $(\alpha m + \beta, \delta m + \epsilon)$ of $P \times Q$ is empty and the cell (γ, ζ) of R is empty.

Now, $(\alpha m + \beta, \delta m + \epsilon)$ of $P \times Q$ is empty and the cell (γ, ζ) of R is empty if and only if the cell $((\alpha m + \beta)n + \gamma, (\delta m + \epsilon)n + \zeta)$ of V is empty. Hence the completable product of partial latin squares is associative. □

2.3 Critical sets from products of partial latin squares

The following theorem is an extension of the result of Stinson and van Rees stated in Lemma 1.21.

Theorem 2.10 *If P and Q are both 2–critical sets and at least one of them is semi-strong or strongly critical, then the partial latin square $R = P \times Q$ is also 2–critical.*

Proof By Theorem 2.5 or 2.7 the partial latin square $R = P \times Q$ has unique completion.

Let P be a partial latin square of order m which completes uniquely to the latin square M and let Q be a partial latin square of order n which completes uniquely to the latin square N.

It must be shown that each element of R is 2–essential. That is to say, for every element $c \in R$ there is a 2×2 subsquare S of $L = M \times N$ such that $(R \setminus \{c\}) \cap S$ does not have (UC) in S. Recall that a partial latin square of order 2 needs only one entry to be uniquely completable. That is to say, $(R \setminus \{c\}) \cap S$ does not have (UC) if and only if $(R \setminus \{c\}) \cap S = \emptyset$.

Without loss of generality let the set of entries for the latin square M be $\mathcal{N}_M = \{0, 1, \ldots, m - 1\}$.

First consider an element of R chosen from one of the $n \times n$ subarrays which is a copy of $Q^i, i \in \{0, 1, \ldots, m\}$. Consider also the array N^i of $L = M \times N$ to which the element belongs. Since each Q^i is a 2–critical set in the corresponding

N^i, there must be a subsquare of order 2 in N^i such that the element of Q^i is 2–essential.

The elements of R which lie in the $n \times n$ subarrays which are copies of N^i, must be considered in two parts, those belonging to $N^i \setminus Q^i$ and those elements of N^i which are also in Q^i. An element in one of these $n \times n$ subarrays will have the form $(\alpha n + x, \beta n + y; in + z)$ for some $\alpha, \beta \in \{0, 1, \ldots, m-1\}$ and for some $x, y, z \in \{0, 1, \ldots, n-1\}$ such that P has an element $(\alpha, \beta; i)$ and there is an element $(x, y; z) \in N$.

Consider an element of an $n \times n$ subarray of R which is one of the elements of $N^i \setminus Q^i$, $(i \in \{0, 1, \ldots, m-1\})$. Since this element of R was chosen to belong to $N^i \setminus Q^i$, $(x, y; z) \notin Q$. The element $(\alpha, \beta; i)$ is 2–essential in P so there must be a subsquare of M of order 2 which intersects P in the element $(\alpha, \beta; i)$ alone. Suppose this subsquare is $S = \{(\alpha, \beta; i), (\alpha, \gamma; j), (\delta, \beta; j), (\delta, \gamma; i)\}$. In $L = M \times N$ there is a subsquare $S' = \{(\alpha n + x, \beta n + y; in + z), (\alpha n + x, \gamma n + y; jn + z), (\delta n + x, \beta n + y; jn + z), (\delta n + x, \gamma n + y; in + z)\}$. Since $S \cap P = \{(\alpha, \beta; i)\}$ and $(x, y; z) \notin Q$ it follows from Remark 2.2 that $S' \cap R = \{(\alpha n + x, \beta n + y; in + z)\}$.

Hence any element $(\alpha n + x, \beta n + y; ni + z)$ of $N^i \setminus Q^i$ is 2–essential.

Now consider an element of R which sits in a copy of N^i and is also an element of Q^i. Since this element is in a copy of N^i the element $(\alpha, \beta; i) \in P$ and because this element is also in Q^i, the element $(w, x; z) \in Q$. Now, both P and Q are 2–critical so there must be a subsquare S in M and a subsquare T in N, with S and T both of order 2, such that $S \cap P = \{(\alpha, \beta; i)\}$ and $T \cap Q = \{(w, x; z)\}$.

Suppose $S = \{(\alpha, \beta; i), (\alpha, \gamma; j), (\delta, \beta; j), (\delta, \gamma; i)\}$ for some γ, δ in the set $\{0, \ldots, m-1\}$ and $T = \{(w, x; z), (w, v; y), (u, x; y), (u, v; z)\}$ for some u, v in the set $\{0, \ldots, n-1\}$. Then there is a subsquare, Λ, of order 2 of L such that $\Lambda = \{(\alpha n + w, \beta n + x; ni + z), (\alpha n + w, \gamma n + v; nj + y), (\delta n + u, \beta n + x; nj + y), (\delta n + u, \gamma n + v; ni + z)\}$.

Consider the element $(\delta n + u, \beta n + x; nj + y)$ of Λ. It is not an element of R, as explained in Remark 2.2, since neither $(\delta, \beta; j) \in P$ nor $(u, x; y) \in Q$. Similarly for $(\alpha n + w, \gamma n + v; nj + y)$ and $(\delta n + u, \gamma n + v; ni + z)$. Hence $\Lambda \cap R = \{(\alpha n + w, \beta n + x; ni + z)\}$ and this element is 2–essential.

Thus it is shown that any element of R, in a copy of N^i such that it also belongs to Q^i, is 2-essential.

Hence every entry of R is 2–essential and so R is 2–critical. □

Up to isotopy there is only one latin square of order 3 so one may consider C_3 and the critical sets in this latin square without any loss of generality.

It is stated in Curran and van Rees [5] that scs(3)=2 and lcs(3)=3.

Lemma 2.11 *Up to isotopy there is only one critical set of size 2 and one critical set of size 3 in C_3, the back circulant latin square of order 3. (Both of these critical sets are strongly critical. See Example 1.24.) They are:*

$\mathcal{C} = \{(0, 0; 0), (1, 1; 2)\}$, *and*

$\mathcal{C}' = \{(0,0;0),(0,1;1),(1,0;1)\}$.

Under certain conditions the completable product of two 3–critical partial latin squares can be shown to be 3–critical.

Example 2.12 *Choose both M and N to be the back circulant latin square of order 3. That is,*

$$M = N = C_3 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Let P be the 3–critical set $\{(0,0;0),(0,1;1),(1,0;1)\}$ (of the second type discussed above in Lemma 2.11) which completes uniquely to $M = C_3$ and let Q be the 3–critical set $\{(0,0;0),(1,1;2)\}$ (of the first type in Lemma 2.11), which completes to $N = C_3$. Then the completable product $R = P \times Q$ is:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & - & - \\ 1 & 2 & 0 & 4 & 5 & 3 & - & 8 & - \\ 2 & 0 & 1 & 5 & 3 & 4 & - & - & - \\ & & & & & & & & \\ 3 & 4 & 5 & 6 & - & - & 0 & - & - \\ 4 & 5 & 3 & - & 8 & - & - & 2 & - \\ 5 & 3 & 4 & - & - & - & - & - & - \\ & & & & & & & & \\ 6 & - & - & 0 & - & - & 3 & - & - \\ - & 8 & - & - & 2 & - & - & 5 & - \\ - & - & - & - & - & - & - & - & - \end{bmatrix}$$

All the elements of R are 3–essential. However, some of the elements are not contained in any 3×3 subsquare of $L = M \times N$ which meets R in a critical set. Instead each of these elements is in a subsquare which makes the element 3–essential because it intersects R in a partial latin square with unique completion in which the element is essential, but not all of the other elements are essential. For instance there is no subsquare of order 3 which contains the element $(0,0;0)$ and meets R in a critical set. This element is only 3–essential because of the subsquare of $L = M \times N$ of order 3 whose elements are $\{(0,0;0),(0,5;5),(0,7;7),(5,0;5),(5,5;7),(5,7;0),(7,0;7),(7,5;0),(7,7;5)\}$. This subsquare intersects R in the partial latin square $\mathcal{P} = \{(0,0;0),(0,5;5),(5,0;5),(7,7;5)\}$ which is not a critical set of the subsquare (for example, $\mathcal{P} \setminus \{(7,7;5)\}$ has unique completion), yet the partial latin square $\mathcal{P} \setminus \{(0,0;0)\}$ does not have (UC). Hence $(0,0;0)$ is 3–essential.

Since this has unique completion and all the elements are 3–essential, this is a critical set of size 39.

Theorem 2.13 *Let P be a critical set of size 2 in a latin square of order 3. (Then P is isotopic to the first critical set in the last lemma.) Let Q be a critical set of the latin square N, of order n. If Q is 3–critical then the completable product $R = P \times Q$ is a 3–critical set.*

Proof Without loss of generality let $P = \{(0,0;0),(1,1;2)\}$ which completes to C_3.

By Theorem 2.5, R completes uniquely to the latin square $L = C_3 \times N$.

The partial latin square R can be partitioned into nine $n \times n$ subarrays, two of which contain copies of N^i and the remaining seven of which contain copies of Q^i, for $i \in \{0,1,2\}$.

An element of R is of the general form $(x.n + a, y.n + b; z.n + c)$ where $a, b, c \in \{0, 1, \ldots, n-1\}$ and $x, y, z \in \{0, 1, 2\}$ for some $(a, b; c) \in N$ and $(x, y; z) \in C_3$.

If the element of R sits in one of the copies of Q^i then $(x, y; z) \in (C_3 \setminus P) = \{(0,1;1), (0,2;2), (1,0;1), (1,2;0), (2,0;2), (2,1;0), (2,2;1)\}$ and $(a, b; c) \in Q$. (See Remark 2.2.) Since Q is 3–critical, it follows from the definition that there exists a subsquare of order 3, S_N, of N which meets Q in such a way that the element $(a, b; c)$ is essential. That is, $S_N \cap Q$ has unique completion to S_N but $(S_N \cap Q) \setminus \{(a, b; c)\}$ does not complete uniquely. Suppose that the subsquare, S_N, of the latin square N is as shown below. The headline and sideline are used to indicate the columns and rows (respectively) in which the entries sit.

	b	e	h
a	c	k	f
d	k	f	c
g	f	c	k

where $d, e, f, g, h, k \in \{0, 1, \ldots, n-1\}$. (There is no information about the ordering of the rows a, d and g, nor about the ordering of the columns b, e and h, nor of the entries c, f and k.)

There is a subsquare, S_L, of L given by:

	$y.n + b$	$y.n + e$	$y.n + h$
$x.n + a$	$z.n + c$	$z.n + k$	$z.n + f$
$x.n + d$	$z.n + k$	$z.n + f$	$z.n + c$
$x.n + g$	$z.n + f$	$z.n + c$	$z.n + k$

(Note that the elements of S_L are all contained in the same $n \times n$ subarray of the latin square L.) As explained in Remark 2.2, since $(x, y; z) \notin P$, the only elements of S_L which are in R are those which are derived from elements of S_N which are also elements of Q. Hence the partial latin square of order 3, $S_L \cap R$, is isomorphic to $S_N \cap Q$ and since $(a, b; c)$ is 3–essential in Q the element $(x.n + a, y.n + b; z.n + c)$ is 3–essential in R.

If the element of R is not, as described above, in a copy of Q^i, then it is in one of the copies of N^i. In this case $(x, y; z) \in P$, that is, $(x, y; z) \in \{(0, 0; 0), (1, 1; 2)\}$.

Suppose $(a, b; c)$ is an element of $N \setminus Q$ then there is a subsquare of L which meets R in the two elements $(0.n+a, 0.n+b; 0.n+c)$ and $(1.n+a, 1.n+b; 2.n+c)$ only. The elements of such a subsquare are shown below.

	$0.n + b$	$\dots$	$1.n + b$	$\dots$	$2.n + b$
$0.n + a$	$0.n + c$		$1.n + c$		$2.n + c$
$\vdots$					
$1.n + a$	$1.n + c$		$2.n + c$		$0.n + c$
$\vdots$					
$2.n + a$	$2.n + c$		$0.n + c$		$1.n + c$

(There is one element here from each of the nine copies of N^i which make up the latin square $C_3 \times N$.) Hence $(x.n + a, y.n + b; z.n + c)$ is 3–essential.

If, however, $(a, b; c)$ is an element of Q as well as an element of N then all the elements of the subsquare above are elements of R and this subsquare does not make $(x.n + a, y.n + b; z.n + c)$ 3–essential in R.

Recall that there is a subsquare, S_N, of N of order 3 which meets Q in such a way that the element $(a, b; c)$ is essential. That is, $S_N \cap Q$ has unique completion to S_N, but $(S_N \cap Q) \setminus \{(a, b; c)\}$ does not complete uniquely. Then either $(a, b; c)$ is the only element in row a of $S_N \cap Q$ and all other elements of $S_N \cap Q$ are in one other row of Q (and S_N), or $(a, b; c)$ is the only element in column b of $S_N \cap Q$ and all other elements of $S_N \cap Q$ are in one other column of Q (and S_N), or, $(a, b; c)$ is the only element with entry c in $S_N \cap Q$ and all other elements of $S_N \cap Q$ have one other entry.

Suppose the first case, that is, $(a, b; c)$ is the only element in row a of $S_N \cap Q$, and without loss of generality, suppose that all the other elements of $S_N \cap Q$ are in the row d of Q, then $S_N \cap Q \subseteq \{(a, b; c), (d, b; k), (d, e; f), (d, h; c)\}$.

Now consider the subsquare of $L = M \times N$ which is shown below.

	$0.n+b$	$\ldots$	$1.n+e$	$\ldots$	$2.n+h$
$0.n+a$	$0.n+c$		$1.n+k$		$2.n+f$
$\vdots$					
$1.n+d$	$1.n+k$		$2.n+f$		$0.n+c$
$\vdots$					
$2.n+g$	$2.n+f$		$0.n+c$		$1.n+k$

(There is one element here from each of the nine copies of N^i which make up the latin square $C_3 \times N$.) This subsquare of order 3 meets R in a subset of the four elements $(0.n+a, 0.n+b; 0.n+c), (1.n+d, 0.n+b; 1.n+k)$, $(1.n+d, 1.n+e; 2.n+f), (1.n+d, 2.n+h; 0.n+c)$. (The first and third of these will definitely be in R because they are formed from the products of elements of C_3 which are also elements of P. The second and fourth may be in R but this will depend on which elements of S_N are also elements of Q.) Hence the element $(x.n+a, y.n+b; z.n+c)$ is 3–essential as it is the only element in its row in this subsquare which is also in R and there is another row of the subsquare which has no elements in R.

Similarly, if $(a, b; c)$ is the only element in column b of $S_N \cap Q$ or if it is the only element with entry c in $S_N \cap Q$ then there is a subsquare of L which meets R in such a way that $(x.n+a, y.n+b; z.n+c)$ is 3–essential in R. □

Corollary 2.14 *Let P be:*

$$\begin{bmatrix} 0 & - & - \\ - & - & - \\ - & & 2 \end{bmatrix}$$

then the completable product of P with itself i times is 3–critical for all $i \in \mathbb{N}$.

(This partial latin square occurs in Gower [9] but for the purposes of that work it was only necessary to know that it has unique completion.)

Stinson and van Rees used the construction described in Lemma 1.21 to show that for $m = 2^l$ for some l, $\text{lcs}(m) \geq 1 - (3/4)^l$. The corollary above gives the corresponding result for m a power of 3. That is, for $m = 3^l$ for some integer l, $\text{lcs}(m) \geq 1 - (7/9)^l$. Which means that in the limit as $m \to \infty$, $\text{lcs}(m)/m^2 \to 1$ for m a power of 3 just as it does for m a power of 2.

Corollary 2.15 *The result of Theorem 2.13 also holds for $Q \times P$.*

Corollary 2.16 *Furthermore, let Q be:*

$$\begin{bmatrix} 0 & 2 & - \\ 2 & - & - \\ - & - & - \end{bmatrix}$$

then $Q \times P^i$ (where P is as in Corollary 2.14) is 3–critical for all $i \in \mathbb{N}$.

However, not all completable products of 3–critical sets are 3–critical.

Example 2.17 *Let*

$$M = N = C_3 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$P = \{(0,0;0),(1,0;1),(1,2;0)\}$ *and* $Q = \{(0,2;2),(2,0;2),(2,2;1)\}$.
Then the completable product, $P \times Q$ is:

$$\begin{bmatrix} 0 & 1 & 2 & - & & 5 & - & - & 8 \\ 1 & 2 & 0 & - & & - & - & - & - \\ 2 & 0 & 1 & 5 & - & 4 & 8 & - & 7 \\ & & & & & & & & \\ 3 & 4 & 5 & - & - & 8 & 0 & 1 & 2 \\ 4 & 5 & 3 & - & - & - & 1 & 2 & 0 \\ 5 & 3 & 4 & 8 & - & 7 & 2 & 0 & 1 \\ & & & & & & & & \\ - & - & 8 & - & - & 2 & - & - & 5 \\ - & - & - & - & - & - & - & - & - \\ 8 & - & 7 & 2 & - & 1 & 5 & - & 4 \end{bmatrix}$$

Consider the element $(2,2;1)$. *It is an element of four* 3×3 *subsquares of order 3 of $M \times N$. One of those subsquares is a copy of N which is the top left hand* 3×3 *subarray of the direct product. Another is the subsquare based on the entries 1, 4 and 7 which appear in the cells (2,2), (2,5), (2,8), (5,2), (5,5), (5,8), (8,2), (8,5) and (8,8). Another is the subsquare based on the entries 1,3 and 8 which appear in the cells (2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4) and (6,6). The last subsquare is based on the entries 1,5 and 6 which appear in the cells (2,2), (2,3), (2,7), (3,2), (3,3), (3,7), (7,2), (7,3) and (7,7).*

All the elements of the first two subsquares of $M \times N$ are also elements of $P \times Q$, hence $(2,2;1)$ *cannot be an essential element for the completion of these subsquares.*

Let us turn our attention to the third subsquare, it intersects $P \times Q$ in a set of five elements, namely (2,2;1), (2,6;8), (4,2;3), (4,6;1) and (6,2;8). The element (2,2;1) is not essential in this subsquare as the two elements (2,6;8) and (4,2;3) are sufficient to form a uniquely completable partial latin square of this subsquare, as are the two elements (6,2;8) and (4,6;1).

Now consider the fourth subsquare, the elements which this subsquare has in common with $P \times Q$ are; (2,2;1), (2,3;5), (3,2;5) and (3,7;1). The element (2,2;1) is not essential in this subsquare either as the two elements (2,3;5) and (3,7;1) are sufficient to form a partial latin square with unique completion in this subsquare.

As none of the four subsquares containing the element (2,2;1) meets $P \times Q$ in such a way that (2,2;1) is essential for the unique completion, this element is not 3–essential.

Since not all of the elements of $P \times Q$ are 3–essential, $P \times Q$ is not 3–critical. Furthermore, $P \times Q$ is not even a critical set. Besides the element (2,2;1), the other elements which are not 3–essential are (2,0;2), (3,2;5), (5,0;5), (3,8;2) and (5,8;1). All six of these elements which are not 3–essential can be removed from this partial latin square to give a new partial latin square whose size is 39 which is a (strongly) 3–critical set. (All the remaining elements are essential and it was checked by computer [11] *that the set was uniquely completable.)*

This critical set of size 39 is isotopic to the one used by Stinson and van Rees [12] *to show that $lcs(9) \geq 39$. It is not isotopic to the one of the form $Q \times P^i$ for $i = 1$ as described in Corollary 2.16 which is isotopic to the 3–critical set in Example 2.12.*

Theorem 2.18 *The completable product of two 3–critical sets of order 3, each of which is of size 3, is neither 3–critical nor critical.*

Proof By Example 2.17 and Lemma 1.13 it follows that in any product of two 3–critical sets of size 3 there are six elements isotopic to those of Example 2.17, none of which are essential and without which the partial latin square is a critical set. □

At this stage results have been given for some of the properties of partial latin squares arising from the completable products of 2–critical sets and those arising from the completable products of some 3–critical sets. The next obvious result to hope for is one involving the completable product of some 4–critical sets.

Up to isomorphism there are two latin squares of order 4 [6], all the minimal critical sets of one of these two latin squares are 2–critical. The other one, the back circulant latin square of order 4, has a minimal critical set which is not 2–critical. This minimal critical set has 5 elements and it is the sensible starting point for a search for results involving 4–critical sets. It is used in the next example.

Example 2.19 *Let*

$$M = N = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \quad \text{and } P = Q = \begin{bmatrix} 0 & 1 & - & - \\ - & - & - & 2 \\ - & 3 & - & - \\ - & - & 1 & - \end{bmatrix}$$

Then $P = Q$ is a strongly critical set. Four of the elements in this critical set are 2–essential but the fifth is not, it is 4–essential. The element $(0,1;1)$ is the element which is not 2–essential.

The completable product, R, of P with Q is:

$$
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & - & - & 12 & 13 & - & - \\
1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 & - & - & - & 10 & - & - & - & 14 \\
2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 & - & 11 & - & - & - & 15 & - & - \\
3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 & - & - & 9 & - & - & - & 13 & - \\
4 & 5 & - & - & 0 & 1 & - & - & 12 & 13 & - & - & 8 & 9 & 10 & 11 \\
- & - & - & 6 & - & - & - & 2 & - & - & - & 14 & 9 & 8 & 11 & 10 \\
- & 7 & - & - & - & 3 & - & - & - & 15 & - & - & 10 & 11 & 8 & 9 \\
- & - & 5 & - & - & - & 1 & - & - & - & 13 & - & 11 & 10 & 9 & 8 \\
8 & 9 & - & - & 12 & 13 & 14 & 15 & 0 & 1 & - & - & 4 & 5 & - & - \\
- & - & - & 10 & 13 & 12 & 15 & 14 & - & - & - & 2 & - & - & - & 6 \\
- & 11 & - & - & 14 & 15 & 12 & 13 & - & 3 & - & - & - & 7 & - & - \\
- & - & 9 & - & 15 & 14 & 13 & 12 & - & - & 1 & - & - & - & 5 & - \\
12 & 13 & - & - & 8 & 9 & - & - & 4 & 5 & 6 & 7 & 0 & 1 & - & - \\
- & - & - & 14 & - & - & - & 10 & 5 & 4 & 7 & 6 & - & - & - & 2 \\
- & 15 & - & - & - & 11 & - & - & 6 & 7 & 4 & 5 & - & 3 & - & - \\
- & - & 13 & - & - & - & 9 & - & 7 & 6 & 5 & 4 & - & - & 1 & -
\end{bmatrix}
$$

This partial latin square is not a 4–critical set. It is not even a critical set.

The partial latin squares $R \setminus \{(0,1;1)\}$, $R \setminus \{(1,7;6)\}$, $R \setminus \{(8,5;13)\}$ *and the partial latin square* $R \setminus \{(12,9;5)\}$ *all have (UC). In fact, all four of these elements can be removed at once and the remaining partial latin square still has (UC)* [11].

Once these four elements are removed the resulting partial latin square is a critical set.

There are some other elements of R which are not essential. Those elements are (0,4;4), (2,5;7), (3,6;5) and (4,13;9).

Since this minimal critical set of order 4 did not produce a larger 4–critical set, the search for further results about the completable products of a–critical sets giving rise to new, larger a–critical sets does not look so promising.

The following is an extension of the result of Stinson and van Rees given in Lemma 1.19.

Theorem 2.20 *If P is a critical set in the latin square M of order m with entries taken from the set* $\{0, 1, \ldots, m-1\}$ *and Q is a critical set in the latin square N of order n, with entries taken from the set* $\{0, 1, \ldots, n-1\}$ *then another partial latin square of order mn can be defined.*

As before, let Q^r *be the array obtained from Q by adding rn to the entry in each cell of Q (which is not empty), for* $r = 0, 1, \ldots, m-1$. *Similarly let* $(N \setminus Q)^r$ *be the array obtained from N by adding rn to the entry in each cell of* $(N \setminus Q)$, *for* $r = 0, 1, \ldots, m-1$.

Now define $P \circ Q$, to be the partial latin square of order mn which is the array obtained by replacing each entry r of P with the array $(N \setminus Q)^r$ and each entry r of $M \setminus P$ with the array Q^r. (Clearly, $P \circ Q$ is a subset of $P \times Q$.)

If at least one of P and Q is either semi-strong or strongly critical then there is a critical set C in the latin square $M \times N$ such that

$$P \circ Q \subseteq C \subseteq P \times Q.$$

Proof It is known that $P \times Q$ has unique completion so it follows that some subset of $P \times Q$ is a critical set. (It will not necessarily be a proper subset as demonstrated in some of the earlier work in this section.) So all that needs to be shown is that the elements of $P \circ Q$ are essential.

The elements of $P \circ Q$ are of two types, those which lie in subarrays containing copies of Q^i and those lying in subarrays containing copies of $N^i \setminus Q^i$.

Consider an element of $P \circ Q$ which is in a subarray containing a copy of Q^i. Since all the elements of Q are essential for the unique completion of Q to N, it follows that all the elements of this type are essential in $P \circ Q$.

Consider an element of $P \circ Q$ which is in a subarray containing a copy of $N^i \setminus Q^i$. Then the element is of the form $(gn + a, hn + b; in + c)$ where $g, h, i \in \{0, 1, \ldots, m-1\}$ and $a, b, c \in \{0, 1, \ldots, n-1\}$ such that $(h, k; i) \in P$ and $(a, b; c) \in N$ but $(a, b; c) \notin Q$.

Now consider the subsquare of $M \times N$ whose elements are given by the set $\{(jn + a, kn + b, M_{j,k}n + c) \mid (j, k; M_{j,k}) \in M\}$. Some of these elements are members of $P \circ Q$. Those elements form a partial latin square isotopic to P. Since all the elements of P are essential in the unique completion of P to M, then the element $(gn + a, hn + b; in + c)$ is essential in $P \circ Q$. □

It might be possible to extend these results to include completable products of two partial latin squares where neither of the component partial latin squares is semi-strong or strongly uniquely completable. Perhaps more needs to be known about the properties of partial latin squares which are uniquely completable but are not semi-strong or strongly uniquely completable before the results shown here can be generalised. The following conjecture is offered as a generalisation to Theorem 2.5.

Conjecture 2.21 *The completable product of any two partial latin squares with unique completion also has unique completion.*

Acknowledgements

Many thanks are due to both Diane Donovan and Sheila Oates–Williams for their help with the content of this paper. I would also like to acknowledge the support of the Mathematics Department at the University of Queensland where a large part of this work was carried out.

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