Halving Complete 4-Partite Graphs *

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ABSTRACT. We completely determine the spectrum (i.e. set of orders) of complete 4-partite graphs with at most one odd part which are decomposable into two isomorphic factors with a finite diameter. For complete 4-partite graphs with all parts odd we solve the spectrum problem completely for factors with diameter 5. As regards the remaining possible finite diameters, 2,3,4, we present partial results, focusing on decompositions of $K_{n,n,n,m}$ and $K_{n,n,m,m}$ for odd m and n.

1 Introductory notes and definitions

There is an intimate connection between edge-decompositions of complete (or complete multipartite) graphs into subgraphs of a given type on one side, and various combinatorial problems (mostly design-theoretical) on the other side. For instance, the existence of a $(v, k, 1)$-BIBD is equivalent to the existence of an edge-decomposition of the complete graph K_v into k-cliques.

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On a more general note, a k-group-divisible design of type $\{m_1, \ldots, m_r\}$ corresponds to an edge-decomposition of the complete multipartite graph $K_{m_1,\ldots,m_r}$ into k-cliques. Besides cliques, research in this area focuses also on decompositions into other kinds of subgraphs with given properties, e.g. paths, cycles, matchings, etc.

A considerable interest was given to investigating decompositions into spanning subgraphs (i.e. factors) of given diameter. A systematic study of this type of decompositions of *complete* graphs was initiated by the remarkable pioneering work of Bosák, Rosa and Znám [1]. Later, Kotzig and Rosa [7], Tomasta [9], Palumbíny [8], and finally Híc and Palumbíny [6] investigated decompositions of complete graphs into (two or more) *isomorphic* factors of a given diameter. Decompositions of complete *bipartite* graphs into two factors with given diameters were studied by Tomová [10], who determined all possible pairs of diameters of such factors. T. Gangopadhyay [5] extended the above by investigating decompositions of complete *r-partite* graphs ($r \geq 3$) into two factors with given diameters and also determined all possible pairs of diameters of such factors. Decompositions of complete bipartite and tripartite graphs into two *isomorphic* factors have been recently dealt with in [3].

Motivated by the above, the next natural step seems to be the study of *decompositions of complete 4-partite graphs into two isomorphic factors with a given diameter*, which is the objective of the present paper.

In what follows we briefly recall (and introduce) some basic notions and notation convenient for our purposes. According to the standard terminology, a *factor* F of a graph G is a subgraph of G which contains each vertex of G. Two factors F_1 and F_2 of G are said to form a *decomposition* of G if their edge sets partition the edge set of G.

A decomposition of G into factors F_1 and F_2 will be called a *halving* of G if F_1 is isomorphic to F_2. Each isomorphism ϕ: $F_1 \rightarrow F_2$ will then be referred to as *halving isomorphism*, and in such case the factors F_1 and F_2 are simply called *halves* of G. If the halves have (the same) finite diameter d, we will say that G has a *d-halving*.

In a complete 4-partite graph K_{m_1,m_2,m_3,m_4} the four maximal independent sets of vertices of sizes m_1, m_2, m_3, m_4 will be called *parts*. It is obvious that any complete 4-partite graph with a halving must have an even number of edges. This fact combined with a simple counting argument shows that the number of odd parts (i.e. parts with an odd number of vertices) must be 0, 1 or 4. Any graph K_{m_1,m_2,m_3,m_4} with this property will be called *admissible*; we will use the same term for the corresponding quadruple m_1, m_2, m_3, m_4.

As it happens to be, for some instances m_1, m_2, m_3, m_4 and d, the graph K_{m_1,m_2,m_3,m_4} does not admit a d-halving. We refer to such results as negative; as a rule, they usually are hard to prove (see Sections 2 and 3). For instances where we show the existence of d-halvings, we use recursive constructions (Section 4) as well as direct constructions (Section 5). The last section contains a summary of results and open problems.

2 Negative results I: Simple cases

In this section we show that complete 4-partite graphs with three equal odd parts cannot have a halving; this result is based on investigating degree sequences. Further, we show that the graph $K_{1,1,r,r}$ has no 2-halving. We note in advance that the same graph has no 5-halving either, which will be a consequence of a more general theorem in Section 3. (As we shall see later, $K_{1,1,r,r}$ admits a d-halving for $d = 3, 4$.)

We introduce first a few auxiliary concepts. Recall that the *degree sequence* of a graph G with vertices $v_1, v_2, \ldots, v_n$ is the non-increasing sequence $D = (d_1, d_2, \ldots, d_n)$ where $d_i = \deg v_{\phi(i)}$ for a suitable permutation ϕ of the set $\{1, 2, \ldots, n\}$.

In general, two sequences $B = (b_1, b_2, \ldots, b_n)$ and $C = (c_1, c_2, \ldots, c_n)$ of non-negative integers will be called *matchable* if there exists a permutation ψ of the set $\{1, 2, \ldots, n\}$ such that $b_i = c_{\psi(i)}$. A sequence $A = (a_1, a_2, \ldots, a_n)$ is said to have a *halving* if there exist matchable sequences $B = (b_1, b_2, \ldots, b_n)$ and $C = (c_1, c_2, \ldots, c_n)$ such that $a_i = b_i + c_i$ for each $i \in \{1, 2, \ldots, n\}$. Obviously, a graph G has a halving only if the degree sequence of G has a halving.

Theorem 1. *Let n, m be odd numbers with $n \neq m$. Then the graph $K_{n,n,n,m}$ has no halving.*

Proof: Assume the contrary and let $K_{n,n,n,m}$ have a halving. Then its degree sequence has a halving as well. Suppose first that $m > n$ (the proof is almost identical in the opposite case, as we shall see later). Let $p = 2n + m$ and $q = 3n$; since $m > n$ we have $p > q$. Now, the degree sequence of $K_{n,n,n,m}$ is $p, p, \ldots, p, q, q, \ldots, q$ where both p and q are odd and both appear in the sequence an odd number of times, namely p appears $3n$ times and q appears m times. Let $r = 3n$, $s = 3n + m$, and let $A = (a_1, a_2, \ldots, a_s)$ and $B = (b_1, b_2, \ldots, b_s)$ be two matchable sequences which constitute a halving of the degree sequence of our graph, that is, $a_l + b_l = p$ for $l = 1, 2, \ldots, r$ and $a_l + b_l = q$ for $l = r+1, r+2, \ldots, s$. For $i = 0, 1, \ldots, p$ let $\alpha(i)$ and $\beta(i)$ be the number of terms in the subsequences $(a_1, a_2, \ldots, a_r)$ and $(b_1, b_2, \ldots, b_r)$, respectively, which are equal to i. Similarly, for $j = 0, 1, \ldots, q$ let $\alpha'(j)$ and $\beta'(j)$ be the number of terms in the subsequences $(a_{r+1}, a_{r+2}, \ldots, a_s)$ and $(b_{r+1}, b_{r+2}, \ldots, b_s)$ which are equal to j. Obviously $\alpha(i) = \beta(p - i)$ and $\alpha'(j) = \beta'(q - j)$.

Since both p and r are odd and because of the above "symmetry relations", there must be an $i \in \{0, 1, \ldots, p\}$ such that $\alpha(i) > \beta(i)$. Let i_0 be the largest such i and let $k = \alpha(i_0) - \beta(i_0)$. Since the sequences A and B are matchable, there are k more appearances of i_0 in the subsequence $(b_{r+1}, b_{r+2}, \ldots, b_s)$ than in $(a_{r+1}, a_{r+2}, \ldots, a_s)$, i.e., $\beta'(i_0) - \alpha'(i_0) = k$. Then, $\alpha'(q-i_0) - \beta'(q-i_0) = k$, and so there are k more occurences of $q-i_0$ in $(a_{r+1}, a_{r+2}, \ldots, a_s)$ than in $(b_{r+1}, b_{r+2}, \ldots, bs)$. Invoking matchability of A and B again, $q - i_0$ must have in $(b_1, b_2, \ldots, b_r)$ k more appearances than in $(a_1, a_2, \ldots, a_r)$, which yields $\beta(q-i_0) - \alpha(q-i_0) = k$. This is equivalent to $\alpha(i_0 + p - q) - \beta(i_0 + p - q) = k$, and so $\alpha(i_0 + p - q) > \beta(i_0 + p - q)$. From the maximality of i_0 it follows that $i_0 + p - q \leq i_0$, or $p \leq q$, which contradicts our assumption that $m > n$. Therefore $K_{n,n,n,m}$ has no halving.

In the case when $m < n$ (and hence $p < q$) we proceed as above, except that we define i_0 as the smallest i such that $\alpha(i) > \beta(i)$. The same computations as in the preceding paragraph yield $\alpha(i_0+p-q) > \beta(i_0+p-q)$; by minimality of i_0 we have $i_0+p-q \geq i_0$, which gives $p \geq q$, a contradiction. The proof is complete. □

The distance between two vertices u, v in a graph H will be denoted by $\text{dist}_H(u,v)$. The *eccentricity* of a vertex u in H, $\text{ecc}_H\ v$, is the maximum of $\text{dist}_H(u,v)$ taken over all vertices $v \in H$. Note that in this terminology $\text{diam}H$, the diameter of H, is simply the largest eccentricity of a vertex of H.

Theorem 2. *A complete 4-partite graph $K_{1,1,r,r}$ has no 2-halving for any r.*

Proof: If r is even, then $1, 1, r, r$ is not an admissible quadruple. Hence we may assume that $K_{1,1,r,r}$ has a 2-halving for an odd $r \geq 3$ (the case $r = 1$ is trivial). Let $V_1 = \{v_{11}\}$, $V_2 = \{v_{21}, v_{22}, \ldots, v_{2r}\}$, $V_3 = \{v_{31}, v_{32}, \ldots, v_{3r}\}$, $V_4 = \{v_{41}\}$ be the parts of $K_{1,1,r,r}$ and F_1 and F_2 the halves of $K_{1,1,r,r}$ with diameter 2.

We first observe that if F_1 contains a vertex u of degree 1, then the only neighbour of u in F_1, say w, must be (in F_1) adjacent to all other vertices, otherwise $\text{ecc}_{F_1}\ u > 2$. But then w would be an isolated vertex in F_2, which is impossible.

We may assume without loss of generality that the edge $v_{11}v_{41}$ belongs to F_1. Then in F_2 there must be a vertex, say v_{3r}, adjacent to both v_{11} and v_{41}, otherwise $\text{dist}_{F_2}(v_{11}, v_{41}) > 2$. Because v_{3r} is not adjacent in F_1 to any of $v_{11}, v_{31}, v_{32}, \ldots, v_{3r-1}, v_{41}$, the distance $\text{dist}_{F_1}(v_{3r}, v_{2i})$ can never be 2. Hence v_{3r} must be adjacent in F_1 to all $v_{21}, v_{22}, \ldots, v_{2r}$, otherwise $\text{diam}F_1 > 2$. Therefore v_{3r} is of degree 2 in F_2.

Since v_{3r} is in F_1 adjacent neither to v_{11} nor to v_{41}, at least one vertex of V_2, say v_{21}, must be adjacent in F_1 to v_{41}, and another, say v_{2r}, to v_{11}. Furthermore, if v_{ij} is any vertex of $V_2 \cup V_3$ then it must be adjacent in F_2 either to v_{11} or to v_{41} (in the opposite case $\text{dist}_{F_2}(v_{3r}, v_{ij}) > 2$ because v_{3r} has no other neighbours than v_{11} and v_{41}).

Now we are going to show that every vertex v_{ij}, $i = 2, 3$ with the exception of v_{3r} is adjacent in F_2 (and consequently in F_1) to exactly one of v_{11}, v_{41}. Suppose the contrary and let there be a vertex other than v_{3r} adjacent in F_2 to both v_{11} and v_{41}. Then $\deg_{F_2} v_{11} + \deg_{F_2} v_4 \geq 2r + 2$. We distinguish two cases:

(i) $\deg_{F_2} v_{11} = \deg_{F_2} v_{41} = r + 1$. Clearly, F_1 now must contain two vertices of degree $r+1$. Because now $\deg_{F_1} v_{11} = \deg_{F_1} v_{41} = r$, there must be another vertex v_{ij}, $i \neq 1, 4$, of degree $r+1$ in F_1. But then $\deg_{F_2} v_{ij} = 1$, which is impossible.

(ii) One of the vertices v_{11}, v_{41}, say v_{11}, is of degree $r + k \geq r + 2$ in F_2. There is only one vertex which could possibly be of degree $r + k$ in F_1, namely v_{41}, because all other vertices are only of degree $\leq r + 2$ in F_1. But in this case $\deg_{F_2} v_{41} = 2r + 1 - (r + k) = r - k + 1$, which yields $\deg_{F_2} v_{11} + \deg_{F_2} v_{41} = (r+k) + (r-k+1) = 2r+1$. This contradicts our assumption and hence each vertex $v_{21}, v_{22}, \ldots, v_{2r}, v_{31}, v_{32}, \ldots, v_{3r}$ is in F_2 adjacent to just one of v_{11}, v_{41}. Clearly, the same conclusion holds for F_1 as well, because (F_1, F_2) is a halving, that is, F_1 is isomorphic to F_2.

As we have seen above, $\deg_{F_2} v_{3r} = 2$ and so F_1 must also contain a vertex w of degree 2. Suppose for a moment that w is one of v_{11}, v_{41}, say v_{11}. Then $\deg_{F_2} v_{11} = 2r - 1$ and so F_1 contains a vertex of degree $2r - 1$; since $r \geq 2$, this vertex must be v_{41}. Then $\deg_{F_2} v_{41} = 2$ and thus F_2 contains two vertices of degree 2 (and so does F_1). This discussion shows that there must be at least one vertex of degree 2 in F_1 different from both v_{11} and v_{41}. We again have two cases.

Suppose that $\deg_{F_1} v_{2i} = 2$. The vertex v_{2i} is in F_1 adjacent to v_{3r} and one of v_{11}, v_{41}, say v_{11}. Then each vertex v_{3j}, $j \in \{1, 2, \ldots, r-1\}$ is adjacent in F_1 to v_{11}, otherwise $\mathrm{dist}_{F_1}(v_{2i}, v_{3j}) > 2$. So in F_2, v_{11} is adjacent only to v_{3r} and some vertices v_{2t}, $t \neq i$. But none of the vertices v_{2t} is a neighbour of v_{2i} in F_2 and $\mathrm{dist}_{F_2}(v_{11}, v_{2i}) > 2$, which is a contradiction.

Now suppose that $\deg_{F_1} v_{3i} = 2$, $i < r$. Each vertex of V_3 different from v_{3r} is adjacent to exactly one of v_{11}, v_{41} in F_1; hence we may assume without loss of generality that v_{3i} is adjacent in F_1 to v_{11} and some v_{2j}. Now each vertex v_{2s}, $s \neq j$ must be adjacent to v_{11} in F_1, otherwise $\mathrm{dist}_{F_1}(v_{2s}, v_{3i}) > 2$. As we have seen, v_{2r} is adjacent in F_1 to v_{41} and therefore $j = r$. So v_{2r} is in F_2 adjacent to v_{11} while v_{3i} to v_{41}. But neither the edge $v_{2r}v_{3i}$ belongs to F_2 nor v_{2r} and v_{3i} have a common neighbour in F_2, which yields $\mathrm{dist}_{F_2}(v_{2r}, v_{3i}) > 2$. This contradiction completes the proof. □

3 Negative results II: The 5-halving case

This section is devoted to proving that complete 4-partite graphs whose all parts are odd have no 5-halving. It turns out that the number of parts does not play the key role here: In fact, we show that no complete $4t$-partite graph with all parts odd can have a 5-halving. The proof itself requires a subtle analysis of complete multipartite graphs which do have a 5-halving and is split into several auxiliary results.

In what follows we shall work with complete r-partite graphs $K_{m_1,m_2,\ldots,m_r}$ which admit a 5-halving and whose vertices are properly r-coloured, so that we can refer to parts as *colour classes.* For convenience we will refer to colour classes as *blue* (b), *red* (r) and *white* (w); we will not use more than three specific colours at a time in our considerations.

Let (F_1, F_2) be a 5-halving of $K_{m_1,m_2,\ldots,m_r}$, $r \geq 3$, and let z be a vertex of eccentricity 5 in F_1. For $0 \leq i \leq 5$ let U_i be the set of vertices that have distance exactly i from z in the factor F_1. In order to facilitate the analysis, we shall introduce the following notation. A vertex v (of no specified colour) which belongs to the set U_i will be denoted v_i. Similarly, if we want to consider, say, an arbitrary vertex $v \in U_i \cup U_i \cup U_k$, we simply write v_{ijk}. Moreover, if the latter vertex has colour c, then it will often be denoted c_{ijk}. The self-explanatory symbol $\bar{c}$ will stand for a vertex of any colour different from c.

Lemma 1. *Let a complete multipartite graph $K_{m_1,m_2,\ldots,m_r}$, $r \geq 3$, have a 5-halving with factors F_1 and F_2. Let z be a vertex of eccentricity 5 in F_1. Then, all vertices at distance 5 from z in F_1 are of the same colour as z.*

Proof: Let z be a vertex of $K_{m_1,m_2,\ldots,m_r}$ such that $\mathrm{ecc}_{F_1} z = 5$ and let U_i be the set of vertices that have distance exactly i from z in F_1, $0 \leq i \leq 5$. Throughout the proof, we shall use the notational conventions introduced above. Let $z = w_0$, that is, let our vertex $z \in U_0$ be coloured white. In our terminology, we will be done if we prove that all vertices in U_5 are white. In order to do so, from now on we assume the contrary and let the set U_5 contain a (say) blue vertex b_5. We show that in such case $\mathrm{diam} F_2 \leq 4$, which will be incompatible with the facts that $F_1 \simeq F_2$ and $\mathrm{diam} F_1 = 5$.

We first prove that the set $U_3 \cup U_4 \cup U_5$ contains at least one red vertex (i.e., a vertex whose colour is different from blue and white). Suppose this is not the case. Then there is at least one white vertex $w_4 \in U_4$, namely the neighbour of b_5 in F_1. All vertices other than white and blue must then belong to $U_1 \cup U_2$. Consider one of them, say, $r_1 \in U_1$. Now, $\mathrm{dist}_{F_2}(r_1, x_{345}) = 1$ for any vertex x_{345}, because x is either white or blue. For any non-white vertex $\bar{w}_{12}$ we have $\mathrm{dist}_{F_2}(r_1, \bar{w}_{12}) \leq 2$, because F_2 contains a path of the form $r_1 w_4 \bar{w}_{12}$. For each white vertex w_{02} we also have $\mathrm{dist}_{F_2}(r_1, w_{02}) \leq 2$, since F_2 contains a path $r_1 b_5 w_{02}$. Thus, $\mathrm{dist}_{F_2}(r_1, x) \leq 2$ for every vertex x. If there are only blue vertices in U_1, then there is a red vertex $r_2 \in U_2$ and we have $\mathrm{dist}_{F_2}(r_2, x_{45}) = 1$ and $\mathrm{dist}_{F_2}(r_2, x_{012}) \leq 2$ for any vertex $x \notin U_3$ for similar reasons as above. For a vertex b_3 we have $\mathrm{dist}_{F_2}(r_2, b_3) \leq 2$ (because of the path $r_2 w_0 b_3$ in F_2) and for w_3 we have $\mathrm{dist}_{F_2}(r_2, w_3) \leq 2$ (due to the path $r_2 b_5 w_3$). Since there are no other vertices in U_3, we have again $\mathrm{dist}_{F_2}(r_2, x) \leq 2$ for every vertex x. Hence in either case we have a vertex r_{12} with $\mathrm{ecc}_{F_2} r_{12} = 2$. This yields $\mathrm{diam} F_2 \leq 4$, which is a contradiction. Therefore $U_3 \cup U_4 \cup U_5$ contains a red vertex, as claimed.

Next, we show that for each non-white vertex $\bar{w}$ we have $\mathrm{dist}_{F_2}(w_0, \bar{w}) \leq 2$. It is obvious that $\mathrm{dist}_{F_2}(w_0, \bar{w}_{2345}) = 1$ for any vertex $\bar{w}_{2345}$ which is not white and has in F_1 distance 2,3,4 or 5 from w_0. For a vertex x_1 that is neither white nor blue we have $\mathrm{dist}_{F_2}(w_0, x_1) = 2$, because in F_2 there always is a path $w_0 b_5 x_1$. The remaining possible non-white vertex to consider, b_1, has $\mathrm{dist}_{F_2}(w_0, b_1) = 2$, because (as we saw above) there exists a vertex r_{345} which yields a path of the form $w_0 r_{345} b_1$ in F_2. Thus, indeed, $\mathrm{dist}_{F_2}(w_0, \bar{w}) \leq 2$ for any non-white vertex $\bar{w}$.

Let us now check the distances $\mathrm{dist}_{F_2}(w_0, w_i)$. We have $\mathrm{dist}_{F_2}(w_0, w_{23}) = 2$, because of the F_2-path $w_0 b_5 w_{23}$, and $\mathrm{dist}_{F_2}(w_0, w_5) = 2$, because there always exists a non-white vertex $\bar{w}_{23} \in U_2 \cup U_3$, and therefore a path $w_0 \bar{w}_{23} w_5$ in F_2. Thus, $\mathrm{dist}_{F_2}(w_0, x) \leq 2$ for any vertex x different from w_4. Hence if neither of x and y is a white vertex from U_4, then $\mathrm{dist}_{F_2}(x, y) \leq \mathrm{dist}_{F_2}(w_0, x) + \mathrm{dist}_{F_2}(w_0, y) \leq 4$.

Collecting all the above information, in order to arrive at a contradiction (and thus complete the proof), it only remains to be shown that $\mathrm{dist}_{F_2}(w_4, x) \leq 4$ for any vertex x. If there is a non-white vertex $\bar{w}_2 \in U_2$, then in F_2 there is a path $w_0 \bar{w}_2 w_4$ and $\mathrm{dist}_{F_2}(w_0, w_4) = 2$. It then follows that $\mathrm{dist}_{F_2}(w_4, x) \leq \mathrm{dist}_{F_2}(w_4, w_0) + \mathrm{dist}_{F_2}(w_0, x) \leq 4$ for any $x \neq w_4$ (see the preceding paragraph). But due to the supposed existence of $\bar{w}_2$, the distance in F_2 between any two white vertices from U_4 is equal to 2, and so $\mathrm{dist}_{F_2}(w_4, x) \leq 4$ for any vertex x, and we are done. Hence we may assume that U_2 contains white vertices only. Recalling the existence of a vertex r_{345} again, we have $\mathrm{dist}_{F_2}(w_4, \bar{w}_{345}) \leq 4$, because in F_2 there always is a path of the form $w_4 x_1 c_{345} w_0 \bar{w}_{345}$ where c is either red or blue. Because all vertices in U_2 are white and $\mathrm{dist}_{F_2}(w_4, \bar{w}_1) = 1$, we conclude that $\mathrm{dist}_{F_2}(w_4, \bar{w}) \leq 4$ for any non-white vertex $\bar{w}$. Similarly, we always have $\mathrm{dist}_{F_2}(w_4, w_{02}) \leq 4$; this follows by considering the F_2-paths $w_4 \bar{b}_1 b_5 w_{02}$, $w_4 b_1 r_{45} w_{02}$, or $w_4 b_1 r_3 b_5 w_{02}$, depending on where the red vertex of $U_3 \cup U_4 \cup U_5$ belongs and whether or not U_1 contains a blue vertex. Finally, for the distance between w_4 and any other white vertex w'_{45} we have $\mathrm{dist}_{F_2}(w_4, w'_{45}) = 2$, because of the path $w_4 \bar{w}_1 w'_{45}$. This completes the analysis (note that now there are no white vertices in U_2), and hence $\mathrm{dist}_{F_2}(w_4, x) \leq 4$ for each vertex x.

Summing up, we have shown that $\mathrm{dist}_{F_2}(x, y) \leq 4$ for any pair of vertices, and so $\mathrm{diam} F_2 \leq 4$, a contradiction. □

We shall keep to the terminology and notation introduced before the statement (and in the proof) of Lemma 1 in our subsequent considerations.

Lemma 2. *Let $K_{m_1,m_2,\ldots,m_r}$, $r \geq 3$, have a 5-halving into factors F_1 and F_2. Let z be a white vertex of eccentricity 5 in F_1 and let U_i be the set of all vertices at distance i from z in F_1. Then, all non-white vertices of $U_1 \cup U_4$ are of the same colour.*

Proof: We first show that the set U_1 contains vertices of one colour only. Suppose this is not the case and let the (white) vertex $z = w_0$ be adjacent in F_1 to a blue and a red vertex, say, $b_1, r_1 \in U_1$. Our goal is to arrive at the same contradiction as in Lemma 1.

Note that any two vertices c_{345}, c'_{345} of $U_3 \cup U_4 \cup U_5$ (no matter what the colours c and c' are) have in F_2 distance at most 2, because there are vertices of 3 different colours in $U_0 \cup U_1$. Further, in $U_4 \cup U_5$ there are vertices of at least 2 colours. By Lemma 1 one of these colours is w; let a be the other. Consider now the distance in F_2 of vertices $c_{012}, c'_{012} \in U_0 \cup U_1 \cup U_2$. If one of the colours a, w is different from both c and c', then clearly $\mathrm{dist}_{F_2}(c_{012}, c'_{012}) \leq 2$. In the opposite case we may suppose that $c = w$ and $c' = a$. Let $x \in \{b, r\} \setminus \{a\}$. Then the vertices c_{345} and c'_{345} have a common neighbour x_{01} in F_2, which yields a path $c_{012}c'_{45}x_{01}c_{45}c'_{012}$ in F_2. All this shows that $\mathrm{dist}_{F_2}(u, v) \leq 4$ if $u, v \in U_0 \cup U_1 \cup U_2$ or if $u, v \in U_3 \cup U_4 \cup U_5$.

It remains to consider the F_2-distance of c_{012} and c'_{345} for any two colours c and c'. Assume first that $c = c'$. Of course, c differs from one of a, w, as well as from one of b, r. Hence, for suitable $x \in \{a, w\}$ and $y \in \{b, r\}$ we have in F_2 a path $c_{012}x_{45}y_{01}c_{345}$, that is, $\mathrm{dist}_{F_2}(c_{012}, c_{345}) \leq 3$. Now, let $c \neq c'$ and, to be specific, let $c_{012} = c_i \in U_i$ for some i, $0 \leq i \leq 2$ and let $c'_{345} = c'_j \in U_j$ for some j, $3 \leq j \leq 5$. If $j - i > 1$, then c_i and c'_j are adjacent in F_2. If $i = 2$ and $j = 3$, then (similarly) in F_2 there is a path $c_2x_{45}y_{01}c'_3$ for suitable $x \in \{a, w\}$ and $y \in \{b, r, w\}$; so $\mathrm{dist}_{F_2}(c_{012}, c'_{345}) \leq 3$. But this all means that $\mathrm{diam} F_2 \leq 4$, which is impossible. It follows that all vertices in U_1 must be of the same colour, say, blue.

Next, assume that U_4 contains vertices of at least two different colours except w, say, r and some other colour c. If c is not blue then it can be seen quickly that $\mathrm{diam} F_2 \leq 4$; we may therefore assume that U_4 only contains vertices coloured blue, red, and possibly white. Now, proceeding as above, we see that $\mathrm{dist}_{F_2}(c_{012}, c'_{012}) \leq 2$, and also $\mathrm{dist}_{F_2}(c_{345}, c'_{345}) \leq 2$ if $\{c, c'\} \neq \{b, w\}$. In the case when (say) $c = b$ and $c' = w$ we have in F_2 a path $c_{345}w_0r_4b_1c'_{345}$. At last, it is easy to see that $\mathrm{dist}_{F_2}(c_{012}, c'_{345}) \leq 3$, and so $\mathrm{diam} F_2 \leq 4$, a contradiction again.

To finish the proof it remains to show that U_4 does not contain red vertices. Suppose the contrary; then (as we have seen in the preceding paragraph) U_4 is coloured *only* red and white. Then, $\mathrm{dist}_{F_2}(b_1, c_{012}) \leq 2$, because there is always one of the paths $b_1r_4c_{012}$ or $b_1w_5c_{012}$ in F_2, depending on the colour c. Also, $\mathrm{dist}_{F_2}(b_1, b_3) = 2$, because of the F_2-path $b_1w_5b_3$. Further, $\mathrm{dist}_{F_2}(b_1, c_3) = 1$ if $c \neq b$, $\mathrm{dist}_{F_2}(b_1, c_4) = 1$ because now $c \in \{r, w\}$, and $\mathrm{dist}_{F_2}(b_1, c_5) = 1$ (as $c = w$ by Lemma 1). Hence $\mathrm{ecc}_{F_2} b_1 = 2$, which yields $\mathrm{diam} F_2 \leq 4$, a contradiction. □

The *neighbourhood* of a vertex v in a graph G, denoted $N_G(v)$, is the set of all vertices adjacent to v in G. If A is a set of vertices of G, then $N_G(A)$ is the union of neighbourhoods of all vertices of A. We say that a subset B of vertices of $K_{m_1,m_2,\ldots,m_r}$ is monochromatic if all vertices of B are of the same colour.

Lemma 3. *Let $K_{m_1,m_2,\ldots,m_r}$, $r \geq 3$, have a 5-halving into factors F_1 and F_2. Let A_i be the set of all vertices of eccentricity 5 in F_i, $i = 1, 2$. Then $A_1 \cap A_2 = \emptyset$, the set $A_1 \cup A_2$ is monochromatic, and the set $N_{F_1}(A_1) \cup N_{F_2}(A_2)$ is monochromatic as well.*

Proof: Referring to the preceding two lemmas and keeping to the notation used in their proofs, we may assume that $U_0 \cup U_5$ is monochromatic (white), and the unique vertex $w_0 \in U_0$ is adjacent in F_1 to a blue vertex b_1. Let u, v be any two vertices for which $\mathrm{dist}_{F_2}(u, v) = 5$. It follows from Lemma 1 that u, v have the same colour; we show first that this colour must be white.

Assume that $u = c$ and $v = c'$ are of colour $c \notin \{b, w\}$. By Lemma 2 we have $c, c' \in U_2 \cup U_3$ and so they are both adjacent to w_0 in F_2, contradiction. Next, suppose that $u = b$ and $v = b'$ are both blue. They cannot both be in U_1, because then they would have a common neighbour w_5 in F_2. There is no blue vertex in U_2 (because *all* of U_1 is necessarily blue), and any two blue vertices of $U_3 \cup U_4$ have a common neighbour w_0 in F_2. It follows that, w.l.o.g., $b = b_1$ and $b' = b_{34}$. Due to the existence of a (say) red vertex r_{23}, the path $b_1 w_5 r_{23} w_0 b_{34}$ yields $\mathrm{dist}_{F_2}(b_1, b_{34}) \leq 4$, a contradiction again.

Hence, *all* vertices in A_2 (i.e., of eccentricity 5 in F_2) are white. The same argument with the roles of F_1 and F_2 interchanged shows that A_1 contains white vertices only; in particular, $A_1 \cup A_2$ is monochromatic (white).

Now we prove that $A_1 \cap A_2 = \emptyset$. Suppose this is not the case; we may w.l.o.g. assume that $w_0 \in A_2$. We check the distances $\mathrm{dist}_{F_2}(w_0, w')$ for each white vertex w'. For $w' \in U_2$ we have in F_2 a path $w_0 b_4 w_2'$. Using the existence of a vertex r_{23}, for $w' \in U_3$ we have in F_2 a path $w_0 r_{23} w_5 b_1 w_3'$, and for $w' \in U_5$ a path $w_0 r_{23} w_5$. Thus, $\mathrm{dist}_{F_2}(w_0, w') \leq 4$ for each w'. But by Lemma 1 (applied to F_2), each vertex of distance 5 in F_2 from w_0 must be white. It follows that $\mathrm{ecc}_{F_2} w_0 < 5$ and so $A_1 \cap A_2 = \emptyset$.

To complete the proof, notice that each white vertex w is in F_2 adjacent to at least one blue vertex (either b_1 or b_4). Invoking Lemma 2, this means that each vertex in A_2 has only blue neighbours. Repeating the same consideration with F_1 in place of F_2 shows that the set $N_{F_1}(A_1) \cup N_{F_2}(A_2)$ is indeed monochromatic (blue). □

We use the following immediate consequence of Lemma 3 later in the case of complete multipartite graphs with at most one odd part.

Corollary 1. *Let $K_{m_1,m_2,\dots,m_r}$ have a 5-halving and let $r \geq 3$ and $m_1 \geq m_2 \geq \cdots \geq m_r$. Then $m_1 \geq 4$ and $m_2 \geq 2$.*

Proof: Each factor F_i contains at least 2 vertices with eccentricity 5, hence $|A_i| \geq 2$. Because $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2$ is monochromatic, we have $m_1 \geq 4$. Obviously $|N_{F_1}(A_1)| \geq 2$; since this set is monochromatic as well (and of different colour as $A_1 \cup A_2$) we see that $m_2 \geq 2$. □

Next, we show that any isomorphism ϕ: $F_1 \to F_2$ fixes both the neighbourhood of vertices of eccentricity 5 as well as the corresponding colour class. As usual, we keep all notation introduced above .

Lemma 4. *Let $K_{m_1,m_2,\dots,m_r}$, $r \geq 3$, have a 5-halving into factors F_1 and F_2 and let ϕ: $F_1 \to F_2$ be an isomorphism. Then $N_{F_1}(A_1) = N_{F_2}(A_2)$, or equivalently $\phi(N_{F_1}(A_1)) = N_{F_1}(A_1)$. Moreover, if X is the colour class containing $N_{F_1}(A_1)$, then $\phi(X) = X$.*

Proof: By Lemma 3, the sets $A_1 \cup A_2$ and $N_{F_1}(A_1) \cup N_{F_2}(A_2)$ are monochromatic (white and blue, respectively). As before, we fix a white vertex w_0 and let U_i be the set of all vertices at distance i from w_0 in F_1, $0 \leq i \leq 5$. We shall focus on $\text{dist}_{F_2}(w, w')$ between white vertices w, w'. Taking into account the existence of blue vertices b_1 and b_4, we see that $\text{dist}_{F_2}(w, w') = 2$ except when (w.l.o.g.) $w = w_2$ and $w' = w_{345}$. Since $U_2 \cup U_3$ contains also vertices of colours different from $\{b, w\}$, say, red, $\text{dist}_{F_2}(w_2, w_4) \leq 4$ because of the F_2-path $w_2 b_4 w_0 r_{23} w_5$. The same principle shows that $\text{dist}_{F_2}(w_2, w_4) \leq 4$ if there exists a non-white vertex in U_2. Thus, if $\text{dist}_{F_2}(w, w') = 5$ then (w.l.o.g.) $w = w_2$ and either $w' = w_3$, or $w' = w_4$ and all vertices in U_2 are white. In any case, each such w' (which obviously belongs to A_2) is adjacent in F_2 to *all* vertices in U_1. Since $U_1 = N_{F_1}(w_0)$, we obtain $N_{F_1}(w_0) \subset N_{F_2}(A_2)$. This yields $N_{F_1}(A_1) \subset N_{F_2}(A_2)$, as the vertex w_0 can be chosen arbitrarily among white vertices of eccentricity 5 in F_1. Interchanging the roles of the factors yields $N_{F_2}(A_2) \subset N_{F_1}(A_1)$, and hence $N_{F_1}(A_1) = N_{F_2}(A_2)$.

Let X be the set of all blue vertices in our complete multipartite graph; we show that $\phi(X) = X$ for any graph isomorphism ϕ: $F_1 \to F_2$. Suppose this is not the case. Then there exists a non-blue vertex $c \notin X$ such that $\phi(c) \in X$. Since $\phi(c)$ is blue, it is *not* adjacent in F_2 to *any* vertex of $N_{F_2}(A_2)$ (because these are all blue). Now, as we already saw, $N_{F_1}(A_1) = N_{F_2}(A_2) = N_{\phi(F_1)}(\phi(A_1))$, and therefore c is not adjacent in F_1 to any vertex of $N_{F_1}(A_1)$. It follows that c must be adjacent to *all* vertices of $N_{F_1}(A_1) = N_{F_2}(A_2)$ in F_2. Then, for any vertex $w \in A_2$ we have $\text{dist}_{F_2}(w, c) = 2$ which yields $\text{dist}_{F_2}(w, w') = 4$ for *any* pair of vertices of A_2, which is impossible. This contradiction shows that $\phi(X) = X$, as claimed. □

The main result now follows easily.

Theorem 3. *Let $r \equiv 0 \pmod 4$ and let $K_{m_1,m_2,\ldots,m_r}$ have a 5-halving. Then at least 3 of the numbers $m_1, m_2, \ldots, m_r$, are even.*

Proof: We begin with analysing the number t of odd parts (i.e., the number of odd m_i's) in a $K_{m_1,m_2,\ldots,m_r}$ which has a halving (not necessarily a 5-halving). Clearly, the number of edges $e = \sum_{i<j} m_i m_j$ of such a graph must be even. We may assume w.l.o.g. that $m_1, m_2, \ldots, m_t$ are odd while the remaining $r - t$ m_i's are even. Then,

$$e \equiv \sum_{1 \le i < j \le t} m_i m_j \equiv \binom{t}{2} \pmod 2,$$

and the latter is an even number only if $t \equiv 0, 1 \pmod 4$. Therefore, to prove our theorem we only need to show that *one* of the numbers $m_1, m_2, \ldots, m_r$ is even.

Assume that $K_{m_1,m_2,\ldots,m_r}$ has a 5-halving for some r divisible by 4. Employing the previous notation again, we know that $N_{F_1}(A_1) \subset X$ for a colour class X; without loss of generality we may assume that $|X| = m_r$. If m_r is even, we are done. Otherwise, Lemma 4 implies $\phi(X) = X$, and so $\phi(G \setminus X) = G \setminus X$. Consequently, the graph $G \setminus X = K_{m_1,m_2,\ldots,m_{r-1}}$ has a halving as well. Since $r - 1 \equiv 3 \pmod 4$, the counting argument at the beginning of the proof shows that at least one of the m_i's $(1 \le i \le r-1)$ must be even. The proof is complete. □

The nonexistence result for 5-halvings of complete 4-partite graphs with all parts odd follows immediately from the preceding theorem, setting $r = 4$.

Theorem 4. *The graph K_{m_1,m_2,m_3,m_4} has no 5-halving for any odd numbers m_1, m_2, m_3, m_4.* □

4 Complete 4-partite graphs with at most one odd part

In this section we completely determine the spectrum of the complete 4-partite graphs with at most one odd part which have a d-halving. First we state a theorem of Gangopadhyay [5] dealing with complete 4-partite graphs decomposable into two factors with the same diameter.

Theorem 5. [5] *Let n and d be given numbers. Then, a complete 4-partite graph of order n decomposable into two factors of diameter d exists if and only if one of the following conditions applies:*

(a) *$d = 2$ and $n \ge 7$,*

(b) *$d = 3$ and $n \ge 4$,*

(c) *$d = 4$ and $n \ge 6$,*

(d) $d = 5$ and $n \geq 8$.

In what follows, we shall use constructions based on extensions of isomorphic factors of smallest graphs which do have a suitable type of halving. For instance, assume that we have a graph K_{m_1,m_2,m_3,m_4} with parts $V_1, \ldots, V_4$ which has a halving into isomorphic factors F_1 and F_2 of diameter d, and an isomorphism ϕ: $F_1 \to F_2$. If there exists a vertex $v \in V_1$ such that $\phi(v) = v$, we can extend the part V_1 by adding a new vertex v' adjacent (in each factor) precisely to the neighbours of the vertex v. The factors F_1' and F_2' of the new graph K_{m_1+1,m_2,m_3,m_4} are also isomorphic and have the same diameter d. Similarly, if there are vertices $v_1, v_2 \in V_2$ such that $\phi(v_1) = v_2$ and $\phi(v_2) = v_1$, we can extend the part V_2 by adding new vertices v_1' and v_2' so that v_1' (v_2') has in both factors the same neighbours as v_1 (v_2, respectively). These new factors F_1' and F_2' form again a d-halving of the graph K_{m_1,m_2+2,m_3,m_4}; the corresponding isomorphism ϕ' which extends ϕ is given by $\phi'(v_1') = v_2'$, $\phi'(v_2') = v_1'$, and $\phi' = \phi$ elsewhere. An exact proof of the following theorem is easy and can be found in [3].

Theorem 6. *Let the graph K_{m_1,m_2,m_3,m_4} with parts $V_i = \{v_{i1}, v_{i2}, \ldots, v_{im_i}\}$, $i = 1, 2, 3, 4$ have a d-halving into factors F_1 and F_2. Let $2 \leq q \leq 4$ and let ϕ: $F_1 \to F_2$ be an isomorphism such that $\phi(v_{11}) = (v_{11})$, $\phi(v_{i1}) = v_{i2}$, and $\phi(v_{i2}) = v_{i1}$ for $i = 2, \ldots, q$. Let k_i, $1 \leq i \leq q$, be arbitrary non-negative integers. Then the graph K_{M_1,M_2,M_3,M_4} has a d-halving for every admissible quadruple M_1, M_2, M_3, M_4 such that $M_1 = m_1 + k_1$, $M_i = m_i + 2k_i$ for $i = 2, \ldots, q$ and $M_i = m_i$ for $i > q$.*

Now we are ready to prove the main result of this section.

Theorem 7. *A complete 4-partite graph K_{m_1,m_2,m_3,m_4}, $m_1 \leq m_2 \leq m_3 \leq m_4$, with at most one odd part, has a d-halving for a finite diameter d if and only if one of the following conditions holds:*

(a) $d = 2$ and $m_1 \geq 1$, $m_2, m_3, m_4 \geq 2$,

(b) $d = 3$ and $m_1 \geq 1$, $m_2, m_3, m_4 \geq 2$,

(c) $d = 4$ and $m_1 \geq 1$, $m_2, m_3, m_4 \geq 2$,

(d) $d = 5$ and $m_1 \geq 1$, $m_2, m_3 \geq 2$, $m_4 \geq 4$.

Proof: The necessity in the cases (a)-(c) is obvious. The graph $K_{1,2,2,2}$ is in all these cases the smallest admissible 4-partite graph of this class of order not less than the corresponding smallest value found in [5]. In the case (d), by [5] the smallest order is 8, but Corollary 1 shows that a complete 4-partite graph with a 5-halving has at least one part of order 4 or more. Obviously, $K_{1,2,2,4}$ is the smallest admissible graph with this property.

To prove the sufficiency we show that for each of the above minimal graphs there is an isomorphism $\phi\colon F_1 \to F_2$ satisfying assumptions of Theorem 7.

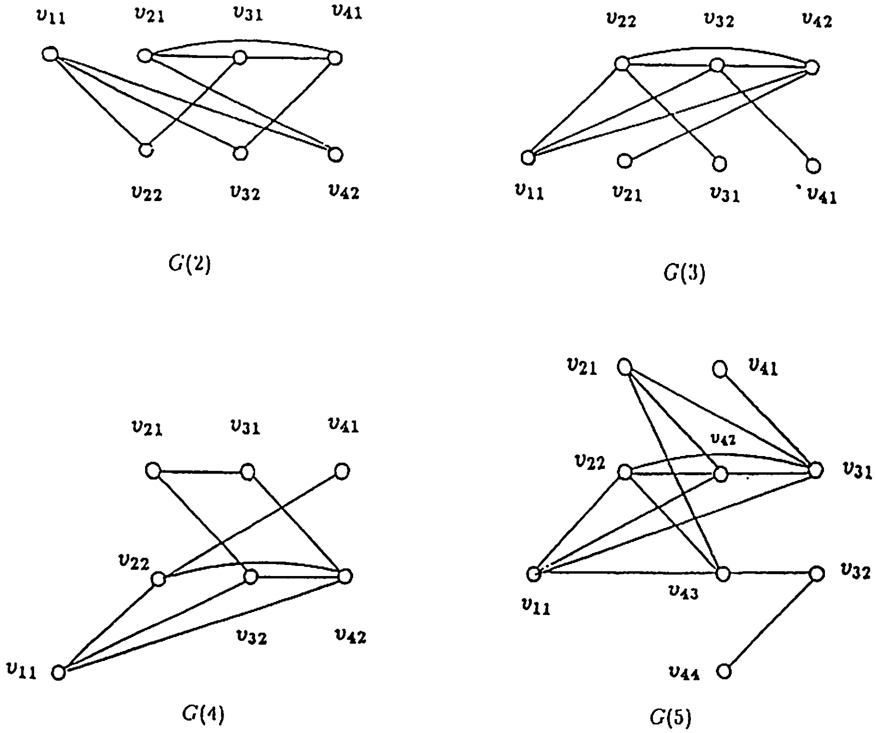

Figure 1

(a) Consider the factor $F_1 \cong G(2)$ of the graph $K_{1,2,2,2}$ of Figure 1 and the isomorphism

$$\phi_2\colon F_1 \to F_2\colon (v_{11})(v_{21}v_{22})(v_{31}v_{32})(v_{41}v_{42}).$$

The distance for any pair of vertices $\text{dist}(v_{ij}, v_{kl}) \leq 2$, and so $\text{diam} F_1 = 2$.

(b) Consider the factor $F_1 \cong G(3)$ of the graph $K_{1,2,2,2}$ in Figure 1 and the isomorphism ϕ_2 from (a). The distance $\text{dist}(v_{21}, v_{41}) = 3$ and for each other pair of vertices we see that $\text{dist}(v_{ij}, v_{kl}) \leq 3$, hence $\text{diam} F_1 = 3$.

(c) Take the factor $F_1 \cong G(4)$ of the graph $K_{1,2,2,2}$ in Figure 1 and again the isomorphism ϕ_2 from (a). The distance $\text{dist}(v_{21}, v_{41}) = 4$ and for all remaining pairs of vertices we have $\text{dist}(v_{ij}, v_{kl}) \leq 3$, thus $\text{diam} F_1 = 4$.

(d) Consider the factor $F_1 \cong G(5)$ of the graph $K_{1,2,2,4}$ in Figure 1 and the isomorphism

$$\phi_5\colon F_1 \to F_2\colon (v_{11})(v_{21}v_{22})(v_{31}v_{32})(v_{41}v_{42})(v_{43}v_{44}).$$

Here $\text{dist}(v_{41}, v_{42}) = 5$ and for any other pair of vertices we have $\text{dist}(v_{ij}, v_{kl}) \leq 4$, hence $\text{diam}F_1 = 5$.

Because both ϕ_2 and ϕ_5 satisfy conditions of Theorem 7, we can always extend a d-halving of the above minimal graphs to a d-halving of any complete 4-partite graph of order n with at most one odd part, for each $n \geq 7$ (in the cases (a)-(c)) or $n \geq 9$ (in the case (d)). □

5 All parts odd: positive results

As we have seen in Section 2, Theorem 1, if $r \neq s$ are odd numbers, then $K_{r,r,r,s}$ has no d-halving for any d. Also, for any r, the graph $K_{1,1,r,r}$ has no 2-halving (cf. Theorem 2). Our third negative result (Theorem 4) says that no complete 4-partite graph with all parts odd has a 5-halving. In contrast to this, we now prove that for all odd r and s the graphs $K_{r,r,s,s}$ have a d-halving for each $d = 3, 4$ (with the obvious exception when $d = 4$ and $r = s = 1$); for $d = 2$ we prove the existence of a 2-halving in the case $r = s \geq 3$.

It should be pointed out that in order to prove the announced results, the method used in the proof of Theorem 7 (i.e., extending a known "smallest halving") will not work because of lack of small examples of complete 4-partite graphs with all parts odd which would have a suitable halving (that is, a halving satisfying the assumptions of Theorem 6). We shall therefore work with direct constructions.

Theorem 8. *Let r, s be odd integers. A complete 4-partite graph $K_{r,r,s,s}$ has a d-halving for a finite diameter d if*

(a) $d = 4$ *and* $\max\{r, s\} \geq 3$, *or*

(b) $d = 3$ *and* $r, s \geq 1$, *or*

(c) $d = 2$ *and* $r = s \geq 3$.

Proof: We start with the case (a). If $r \neq s$, say $r < s$, take a complete graph $K_{2(r+s)}$ and partition its vertex set into 8 subsets $X_1, \ldots, X_4$, $Y_1, \ldots, Y_4$ where for each $i = 1, 2, 3, 4$, $|X_i| = r$ and $|Y_i| = (s - r)/2 = t$. Let $X_i = \{x_{i1}, x_{i2}, \ldots, x_{ir}\}$ and $Y_i = \{y_{i1}, y_{i2}, \ldots, y_{it}\}$ for $i = 1, 2, 3, 4$. First we construct isomorphic factors F_1 and F_2 as follows: F_1 contains all edges $x_{ij}x_{i+1,k}$, where $i = 1, 2, 3$ and $j, k = 1, 2, \ldots, r$, all edges $y_{ij}y_{i+1,k}$ where $i = 1, 2, 3$ and $j, k = 1, 2, \ldots, t$, and all edges $y_{1j}x_{ik}$ and $y_{4j}x_{ik}$, where $1 \leq i \leq 4$, $1 \leq j \leq t$ and $1 \leq k \leq r$. Furthermore, F_1 contains all edges $x_{2i}x_{2j}$ and $x_{3i}x_{3j}$ where $i \neq j$; $i, j = 1, 2, \ldots, r$, and $y_{2i}y_{2j}$ and $y_{3i}y_{3j}$ where $i \neq j$; $i, j = 1, 2, \ldots, t$ (see Figure 2).

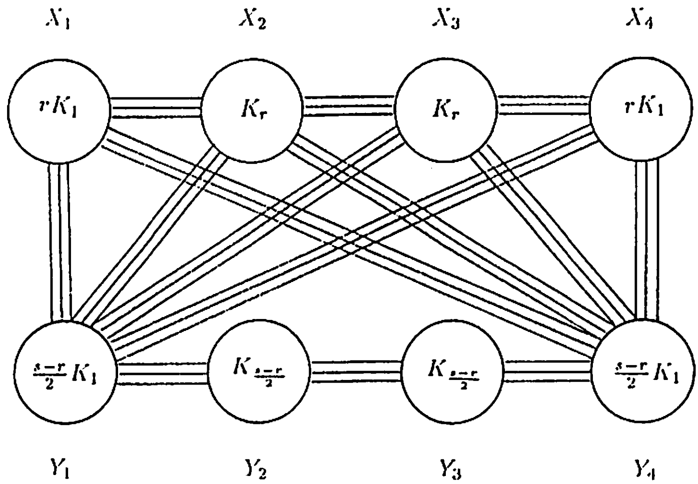

Figure 2

Define now $F_2 = K_{2(r+s)} \setminus F_1$. Let ϕ: $F_1 \to F_2$ be a mapping which, when considered as a permutation of the vertex set of $K_{2(r+s)}$, has cycles $(x_{3i}x_{1i}x_{2i}x_{4i})$ and $(y_{3j}y_{1j}y_{2j}y_{4j})$ for $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, t$. It is easy to verify that ϕ is a graph isomorphism. Now we remove from the complete graph $K_{2(r+s)}$ all edges of its complete subgraphs $\langle x_{21}, x_{22}, \ldots, x_{2r}\rangle \cong \langle x_{41}, x_{42}, \ldots, x_{4r}\rangle \cong K_r$, $\langle y_{21}, y_{22}, \ldots, y_{2t}, y_{31}, y_{32}, \ldots, y_{3t}, x_{31}, x_{32}, \ldots, x_{3r}\rangle \cong K_s$ and $\langle y_{11}, y_{12}, \ldots, y_{1t}, y_{41}, y_{42}, \ldots, y_{4t}, x_{11}, x_{12}, \ldots, x_{1r}\rangle \cong K_s$, in order to obtain $K_{r,r,s,s}$. If we remove the above edges also from the factors F_1, F_2 of $K_{2(r+s)}$, we obtain factors F_1, F_2 of $K_{r,r,s,s}$ with the parts $V_1 = X_2$, $V_2 = X_4$, $V_3 = X_3 \cup Y_2 \cup Y_3$, $V_4 = X_1 \cup Y_1 \cup Y_4$. The isomorphism ϕ': $F_1' \to F_2'$ is then induced by the isomorphism ϕ: $F_1 \to F_2$.

The factor F_1' now contains the edges $x_{ij}x_{i+1,k}$ with $i = 1, 2, 3$; $j, k = 1, 2, \ldots, r$; $y_{1j}y_{2k}$ and $y_{3j}y_{4k}$ with $j, k = 1, 2, \ldots, t$; $y_{1i}x_{jk}$ with $i = 1, 2, \ldots, t$; $j = 2, 3, 4$; $k = 1, 2, \ldots, r$, and $y_{4i}x_{jk}$ again with $i = 1, 2, \ldots, t$; $j = 2, 3, 4$; $k = 1, 2, \ldots, r$, as shown in Figure 3. One can verify that $\text{dist}_{F_1'}(y_{2i}, y_{3j}) = 4$ for any $i, j = 1, 2, \ldots, t$. Taking into account Theorems 4 and 5 it follows that $\text{diam} F_1' = 4$.

For $r = s$ we need a different construction. We take the graph K_{4r} with vertices $x_{i1}, x_{i2}, x_{i3}, x_{i4}$, $i \leq r$ and construct first its selfcomplementary factors F_1 and F_2 as follows. The factor F_1 contains r paths of length 3 induced by the vertices $x_{i1}, x_{i2}, x_{i3}, x_{i4}$ with the edges $x_{i1}x_{i2}, x_{i2}x_{i3}, x_{i3}x_{i4}$ for each $i = 1, 2, \ldots, r$, and for each $i = 2, \ldots, r$ moreover all edges $x_{i2}x_{jk}$ and $x_{i3}x_{jk}$ where $j = 1, 2, \ldots, i-1$ and $k = 1, 2, 3, 4$. An example for $r = 3$ is shown in Figure 4.

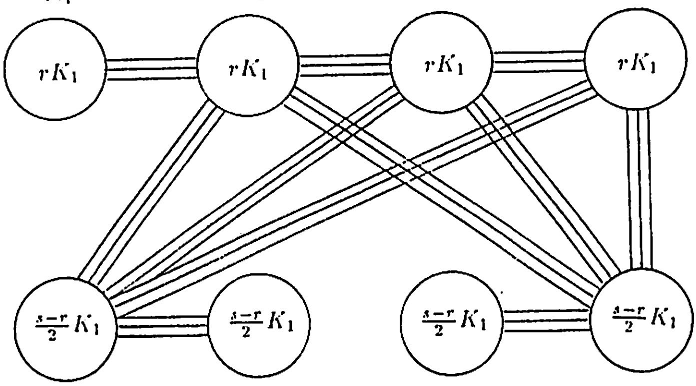

Figure 3

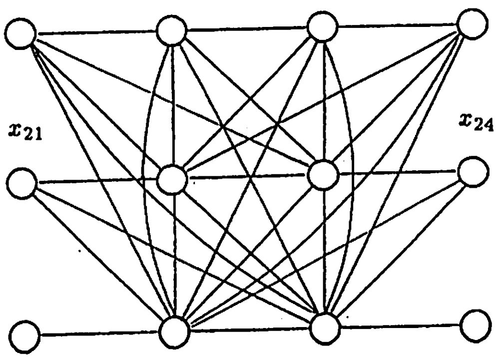

Figure 4

F_2 is then the complement of F_1; the complementing permutation ϕ has cycles $(x_{i3}x_{i1}x_{i2}x_{i4})$ for $i = 1, 2, \ldots, r$. The factors are known to be self-complementary of diameter 3. To obtain the graph $K_{r,r,r,r}$ and its selfcomplementary factors F_1' and F_2', we remove all edges of the induced complete subgraphs $\langle x_{21}, x_{31}, \ldots, x_{r1}, x_{r4} \rangle$, $\langle x_{14}, x_{24}, \ldots, x_{r-1,4}, x_{11} \rangle$, $\langle x_{12}, x_{22}, \ldots, x_{r-1,2}, x_{13} \rangle$, and $\langle x_{23}, x_{33}, \ldots, x_{r3}, x_{r2} \rangle$. The factor F_1' for $r = 3$ is shown in Figure 5.

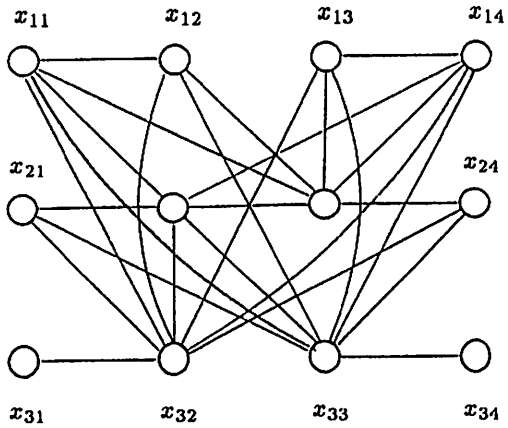

Figure 5

The isomorphism $\phi'\colon F_1' \to F_2'$ is again induced by the isomorphism $\phi\colon F_1 \to F_2$. The vertices x_{r1} and x_{r4} are now at distance $\text{dist}_{F_1'}(x_{r1}, x_{r4}) = 4$ and hence $\text{diam}F_1' = 4$ as it again follows from Theorems 4 and 5.

In the case (b) we take again the graph $K_{2(r+s)}$ with the vertex set split into the same subsets $X_1, \ldots, X_4$, $Y_1, \ldots, Y_4$ as in part (a) and construct the factors F_1 and F_2 in a slightly different way. The factor F_1 will now contain the edges $x_{ij}x_{i+1k}$ where $i = 1, 2, 3$ and $j, k = 1, 2, \ldots, r$, all edges $y_{ij}y_{i+1k}$, where $i = 1, 2, 3$ and $j, k = 1, 2, \ldots, t$, and all edges $x_{2i}y_{jk}$ and $x_{3i}y_{jk}$, where $i = 1, 2, \ldots, r$ and $j, k = 1, 2, \ldots, t$. Furthermore, F_1 will contain all edges $x_{2i}x_{2j}$ and $x_{3i}x_{3j}$ where $i \neq j$; $i = 1, 2, \ldots, r$ and $y_{3i}y_{3j}$ where $i \neq j$; $i, j = 1, 2, \ldots, t$. A diagram is shown in Figure 6.

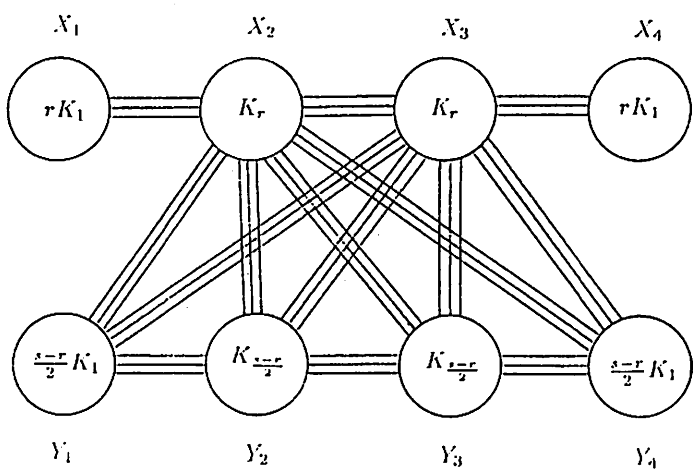

Figure 6

We now remove from F_1 all edges of the induced complete subgraphs $\langle x_{31}, x_{32}, \ldots, x_{3r}\rangle \cong K_r$ and $\langle y_{21}, y_{22}, \ldots, y_{2t}, y_{31}, y_{32}, \ldots, y_{3t}, x_{21}, x_{22}, \ldots, x_{2r}\rangle \cong K_s$. From F_2 we remove the edges of the induced subgraphs $\langle x_{41}, x_{42}, \ldots, x_{4r}\rangle \cong K_r$ and $\langle y_{11}, y_{12}, \ldots, y_{1t}, y_{41}, y_{42}, \ldots, y_{4t}, x_{11}, x_{12}, \ldots, x_{1r}\rangle \cong K_s$. The resulting graphs F_1', F_2' are certainly factors of the graph $K_{r,r,s,s}$ with the parts $V_1 = X_3$, $V_2 = X_4$, $V_3 = X_2 \cup Y_2 \cup Y_3$, $V_4 = X_1 \cup Y_1 \cup Y_4$; a diagram of F_1' is shown in Figure 7. The isomorphism between them is again defined as above, i.e. $\phi'\colon F_1' \to F_2'$, with cycles $(x_{3i}x_{1i}x_{2i}x_{4i})$ and $(y_{3j}y_{1j}y_{2j}y_{4j})$ for $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, t$.

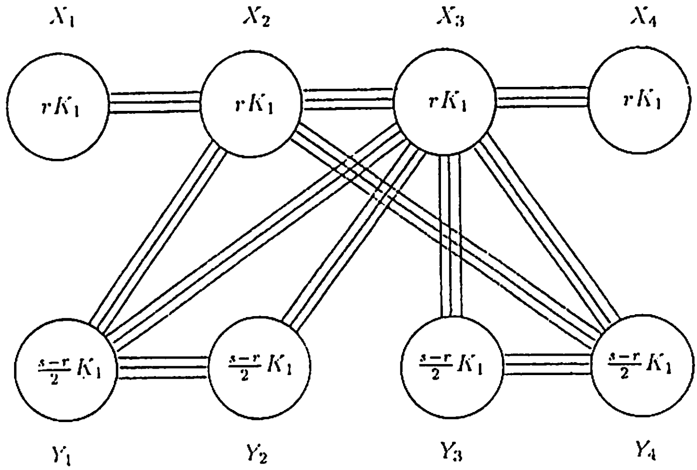

Figure 7

It can be checked that he factors indeed have diameter 3. For instance, $\mathrm{dist}_{F_1}(x_{1i}, x_{4j}) = 3$ for any $i, j = 1, 2, \ldots, r$ and for any other pair of vertices the distance does not exceed 3.

For diameter 2 (case (c)) we decompose the equipartite graph $K_{r,r,r,r}$, $r = 2t + 1$ with parts $V_i = \{v_{i1}, v_{i2}, \ldots, v_{ir}\}$, $i = 1, 2, 3, 4$ as follows. For every $j = 1, 2, \ldots, r$ the factor F_1 contains the path P_4^j with edges $v_{1j}v_{2j}$, $v_{2j}v_{3j}$, $v_{3j}v_{4j}$, and both terminal vertices of P_4^j, v_{1j} and v_{4j}, are connected to all vertices of the paths $P_4^{j+1}, P_4^{j+2}, \ldots, P_4^{j+t}$ which do not belong to the same part as v_{1j} or v_{4j}, respectively. The superscripts are taken mod r. More precisely, each vertex v_{1j} is adjacent to the vertices $v_{2\,j}, v_{2\,j+1}, v_{3\,j+1}, v_{4\,j+1}, v_{2\,j+2}, v_{3\,j+2}, v_{4\,j+2}, \ldots, v_{2\,j+t}, v_{3\,j+t}, v_{4\,j+t}$ and to $v_{4\,j+t+1}, v_{4\,j+t+2}, v_{4\,j-2}, v_{4\,j-1}$. The neighbours of v_{4j} are defined analogously, replacing vertices v_{4k} with v_{1k} and vertices v_{2j} with v_{3j}. The vertex v_{2j} is adjacent to $v_{1\,j}, v_{3\,j}, v_{1\,j+t+1}, v_{1\,j+t+2}, \ldots, v_{1\,j-1}, v_{4\,j+t+1}, v_{4\,j+t+2}, \ldots, v_{4\,j-1}$. Again, the neighbourhood of v_{3j} is described analogously, replacing vertices v_{1j} with v_{4j} and v_{3k} with v_{2k}. An example for $r = 3$ is presented in Figure 8. The factor F_2 consisting of the remaining edges of $K_{r,r,r,r}$ is isomorphic to F_1 and the halving isomorphism is a permutation with cycles $(v_{1j}, v_{2j}, v_{4j}, v_{3j})$ for each $j = 1, 2, \ldots, r$.

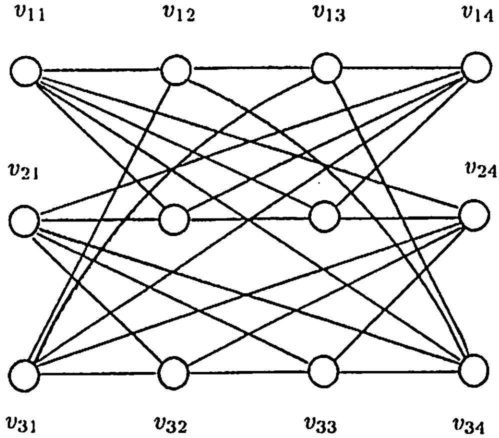

Figure 8

Since there is an automorphism ψ_1 of F_1 with cycles $(v_{1j}v_{4j})$ and $(v_{2j}v_{3j})$ for each $j = 1, 2, \ldots, r$, and another, ψ_2, defined as $\psi_2(v_{ij}) = v_{ij+1}$ for $i = 1, \ldots, 4$; $j = 1, 2, \ldots, r$, it suffices to check only the distances from v_{11} and v_{21} to all others to prove that diam$F_1 = 2$. We start with v_{11}. The vertices not adjacent in F_1 to v_{11} are $v_{31}, v_{41}, v_{12}, v_{13}, \ldots, v_{1\,t+1}$ and $v_{i\,t+1}, v_{i\,t+2}, v_{ir}$ for $i = 1, 2, 3$. But v_{31} is adjacent to v_{21} and v_{41} to v_{22} which are both neighbours of v_{11}. Each vertex v_{1j}, $j = 2, 3, \ldots, t+1$ is also adjacent to a neighbour of v_{11}, v_{2j}, and all vertices $v_{i\,t+2}, v_{i\,t+3}, \ldots, v_{ir}$ are adjacent to another neighbour of v_{11}, namely $v_{4\,t+1}$. Hence $\text{ecc}_{F_1} v_{11} = 2$ and from what we mentioned above it follows that $\text{ecc}_{F_1} v_{1j} = \text{ecc}_{F_1} v_{4j} = 2$ for all $j = 1, 2, \ldots, r$. Therefore we have now only to check the distance from v_{21} to all vertices v_{2j} and v_{3j}, $j = 1, 2, \ldots, r$. All vertices v_{2j} and v_{3j}, $j = 1, 2, \ldots, t+1$ are adjacent to v_{11} which is a neighbour of v_{21} and v_{31} is adjacent to v_{21} itself. Furthermore, for each $j = t+2, t+3, \ldots, r$, every vertex v_{2j} is adjacent to the vertex v_{1j} and, similarly, each such v_{3j} is adjacent to v_{4j}. Because the vertices v_{1j} and v_{4j} are for $j = t+2, t+3, \ldots, r$ adjacent to v_{21}, we have $\text{ecc}_{F_1} v_{21} = 2$. It immediatelly follows that $\text{ecc}_{F_1} v_{2j} = \text{ecc}_{F_1} v_{3j} = 2$ and hence diam$F_1 = 2$. □

6 Summary and remarks

To begin with, we recall once more that if a complete fourpartite graph has a halving then either at most one of its parts is odd, or else all four parts are odd. A complete solution to the problem of the existence of a d-halving of K_{m_1,m_2,m_3,m_4} in the case when at most one m_i is odd was presented in

Theorem 7 (see Section 4). The situation seems to be more complicated if all parts are odd. In Section 3, Theorem 4, we proved that K_{m_1,m_2,m_3,m_4} has no 5-halving if all m_i are odd; combined with Theorem 7 this gives a complete solution of the 5-halving problem for complete fourpartite graphs (regardless of the parity of parts). For $2 \leq d \leq 4$ we answered the question of the existence of a d-halving of K_{m_1,m_2,m_3,m_4} for the following quadruples (m_1, m_2, m_3, m_4) of odd numbers:

$(n, n, n, m), n, m$ odd, $n \neq m$	no d-halving for $2 \leq d \leq 4$ (Theorem 1)
$(1, 1, r, r), r$ odd	no 2-halving (Theorem 2)
$(r, r, s, s), r, s$ odd, $\max\{r, s\} \geq 3$	4-halving exists (Theorem 8 (a))
$(r, r, s, s), r, s$ odd	3-halving exists (Theorem 8 (b))
(r, r, r, r), odd $r \geq 3$	2-halving exists (Theorem 8 (c)).

It also follows that we have a complete solution to the d-halving problem of complete fourpartite graphs for $3 \leq d \leq 4$ in the case when there are at most two different part sizes.

The problem of determining if K_{m_1,m_2,m_3,m_4} admits a d-halving remains therefore open for the following instances:

$(q, q, r, s), q, r, s$ odd and distinct	d-halving open for $2 \leq d \leq 4$
(q, r, s, t), all odd and distinct	d-halving open for $2 \leq d \leq 4$
$(r, r, s, s), r, s$ odd and distinct	2-halving open.

Although the complete 4-partite graphs with two or three odd parts have an odd number of edges and therefore cannot have a halving, one can study *almost complete 4-partite graphs* (i.e. complete 4-partite graphs with one missing edge), in analogy with almost complete graphs, introduced by Das [2]. Decompositions of this category of graphs have not been studied yet.

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