Step Domination in Graphs

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ABSTRACT. A set $S = \{v_1, v_2, \ldots, v_n\}$ of vertices in a graph G with associated sequence $k_1, k_2, \ldots, k_n$ of nonnegative integers is called a *step domination set* if every vertex of G is at distance k_i from v_i for exactly one i $(1 \leq i \leq n)$. The minimum cardinality of a step domination set is called the *step domination number* of G. This parameter is determined for several classes of graphs and is investigated for trees.

1 Definitions and Examples

For a connected graph G, the *distance* $d(u, v)$ between two vertices u and v is the length of a shortest $u - v$ path in G. If $d(u, v) = 1$ for two vertices u and v of G, then u and v are *adjacent*. The distance from a vertex v to a vertex furthest from v is the *eccentricity* of v and is denoted $e(v)$. The maximum eccentricity is the *diameter* diam G of the graph G. The minimum eccentricity is the *radius* rad G of the graph G.

The set of vertices at distance k from a vertex v in a graph G is called the *k-neighborhood of* v and is denoted by $N_k(v)$. That is,

$$N_k(v) = \{u \in V(G) \mid d(v, u) = k\}.$$

When $k = 1$, then we refer to the 1-neighborhood of v simply as the neighborhood or *open neighborhood of* v, which we also denote by $N(v)$.

In a graph G, a vertex v is said to *dominate* itself and each of its neighbors. A set S of vertices of G is a *domination* (or *dominating*) *set* if every vertex of G is dominated by some vertex of S. Nonadjacent vertices u and v are

*Research supported in part by Office of Naval Research Grant N00014-91-J-1060.

said to be *independent*. A set S of vertices of G is an *independent set* if every pair of vertices in S is independent.

A set $S = \{v_1, v_2, \ldots, v_n\}$ of vertices in a graph G is defined in [1] as a *step domination set* for G if there exist nonnegative integers $k_1, k_2, \ldots, k_n$ such that the sets $\{N_{k_i}(v_i)\}$ form a partition of $V(G)$. The partition $\{N_{k_i}(v_i)\}$ of $V(G)$ is called the *step domination partition associated with* S. The sequence $k_1, k_2, \ldots, k_n$ is called a *distance domination sequence associated with* S, while k_i is called the *step* of v_i and written step $v_i = k_i$. Each vertex u in $N_{k_i}(v_i)$ is said to be *step dominated by* v_i, and v_i *step dominates* u. As a consequence of the definition of a step domination set, each set $N_{k_i}(v_i)$ is nonempty. Thus, $0 \le k_i \le e(v_i)$ for each integer k_i in a distance domination sequence associated with S. Since a vertex in a step domination set S cannot step dominate both itself and other vertices, the cardinality of a step domination set for G is at least 2 unless $G = K_1$. Certainly, the cardinality of a step domination set of a graph G cannot exceed the order of G.

Consider the graph $G = P_6$ of Figure 1 and observe that $N_2(v) = \{x\}$, $N_2(w) = \{u, y\}$, $N_2(x) = \{v, z\}$, and $N_2(y) = \{w\}$. Thus, $S = \{v, w, x, y\}$ is a step domination set for G with associated distance domination sequence 2, 2, 2, 2. The distance domination set associated with S is $\{2\}$. The step domination partition associated with S is $\{\{x\}, \{w\}, \{u, y\}, \{v, z\}\}$. We indicate this in Figure 1 by representing the vertices of S by solid circles and labeling these vertices with their associated distances.

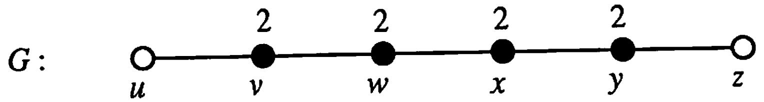

Figure 1

Other step domination sets exist for the graph G of Figure 1. This is illustrated in Figure 2, where in each case the step domination set is $\{u, w, x, z\}$. These examples show that there can be more than one distance domination sequence and step domination partition for a given step domination set.

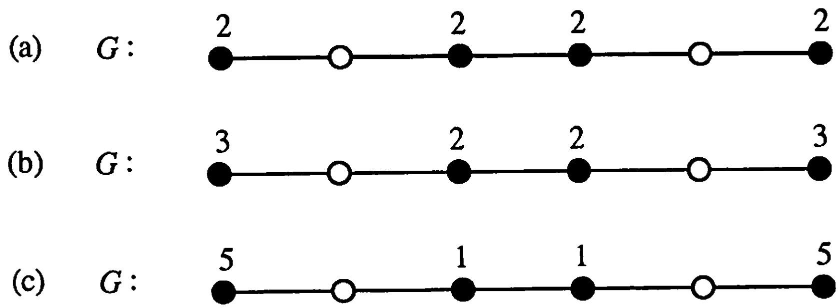

Figure 2

It is not necessary that some value in the distance domination sequence repeat as the graph in Figure 3 shows.

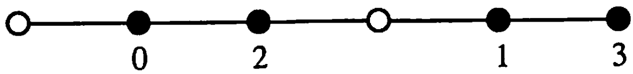

Figure 3

For each positive integer n, there exists a graph G having n distinct distance domination sets and, consequently, n distinct distance domination sequences. For example, for $G = P_{n+1}$ each set $\{0, i\}$, where $1 \leq i \leq n$, is a distance domination set for G. To see this, let $v_0, v_1, \ldots, v_n$ denote this path. For an integer i with $i \leq n/2$, the set $V(G) - \{v_{2i}\}$ is a step domination set for G with distance domination sequence $\{k_j\}$, where $k_j = i$ if $j = 0$ or $j = i$, and $k_j = 0$ otherwise. If $i > n/2$, then $V(G)$ is a step domination set for G using the same distance domination sequence described above. In each case, the associated distance domination set is $\{0, i\}$.

2 The Step Domination Number of a Graph

If G is a graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$, then the set $\{N_0(v_i)\colon i = 1, 2, \ldots, p\}$ is a step domination partition of $V(G)$ corresponding to the step domination set $S = V(G)$. Thus, every graph has some step domination set. This leads us to the *step domination number* $\gamma_s(G)$ of a graph G, defined in [1] as the minimum cardinality of a step domination set for G. Consequently, $\gamma_s(G)$ is well-defined for every graph G and $\gamma_s(G) \leq |V(G)|$. Also, from our earlier observation, $\gamma_s(G) \geq 2$ unless $G = K_1$. In this latter case, $\gamma_s(G) = 1$, of course.

Formulas for the step domination number for paths, cycles, and complete bipartite graphs were determined in [1].

Theorem A. *For every positive integer n,*

$$\gamma_s(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 0, 1, 3 \pmod 4 \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Theorem B. *For every integer $n \geq 3$,*

$$\gamma_s(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 0, 1, 3 \pmod 4 \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Theorem C. *Every complete bipartite graph has step domination number 2.*

A formula can also be found for the class of complete graphs.

Theorem 1. *For every positive integer n, $\gamma_s(K_n) = n$.*

Proof: Let $S = \{v_1, v_2, \ldots, v_t\}$ be a step domination set for K_n where $t = \gamma_s(K_n)$. If some term k_i in the distance domination sequence associated with S is nonzero, then $k_i = 1$ since diam $K_n = 1$. Hence, v_i step dominates $V(G) - \{v_i\}$ and there is no vertex to step dominate v_i. Thus, $k_i = 0$ for all $v_i \in S$, and v_i step dominates only itself. Since all vertices in G must be step dominated, $S = V(K_n)$. □

More can be said about graphs with step domination number 2. Let G be a connected graph of order $p \geq 3$ with $\gamma_s(G) = 2$ and $S = \{v_1, v_2\}$. Then $v_2 \in N_{k_1}(v_1)$ and $v_1 \in N_{k_2}(v_2)$, that is, each of v_1 and v_2 must step dominate the other. Thus, $k_1 = k_2$ for any distance domination sequence. If $k_1 \neq 1$, then there is a vertex of G distinct from v_1 and v_2 on a shortest $v_1 - v_2$ path, but such a vertex is step dominated by neither v_1 nor v_2. Thus, $k_1 = k_2 = 1$. This implies that diam $G \leq 3$. In fact, such graphs G can be classified completely [1].

Theorem D. *For a graph G, the step domination number $\gamma_s(G) = 2$ if and only if $G = 2K_1$ or G has a spanning double star with (adjacent) centers u and v such that $\{N_1(u), N_1(v)\}$ is a partition of $V(G)$.*

The value of $\gamma_s(G)$ can be determined precisely for a select class of regular graphs.

Theorem 2. *If G is an r-regular graph with diameter 2 and girth at least 5, then $\gamma_s(G) = 2 + (r-1)^2$.*

Proof: Let $v_0 \in V(G)$ and let $v_1, v_2, \ldots, v_r$ be the r neighbors of v_0. Since $g(G) \geq 5$, it follows that $v_iv_j \notin E(G)$ for integers i and j with $1 \leq i \neq j \leq r$. Without loss of generality, one can assume that u is adjacent to v_1. If $uv_j \in E(G)$ for some integer j with $2 \leq j \leq r$, then u, v_j, v_0, v_1, u forms a 4-cycle. Thus, $uv_j \notin E(G)$ for all integers j with $2 \leq j \leq r$. Let $S = V(G) - N(v_0) - N(v_1)$. Then the set $\{N_1(v_i) : i = 0, 1\} \cup \{N_0(v) : v \in S\}$ is a partition of $V(G)$. Since each vertex v_i, $1 \leq i \leq r$, has $r-1$ unique neighbors, $|S| \geq (r-1)^2$. If there is a vertex w not adjacent to any vertex v_i, $0 \leq i \leq r$, then $d(v_0, w) \geq 3$, contrary to hypothesis. Thus, $|S| = (r-1)^2$ and $\gamma_s(G) \leq 2 + (r-1)^2$.

For $r = 2$, the only 2-regular graph of diameter 2 with girth at least 5 is $G = C_5$. Theorem B gives $\gamma_s(C_5) = 3$. Assume then that $r \geq 3$ and G is a graph having the given properties. Let $S = \{x_1, x_2, \ldots, x_t\}$ be a step domination set for G, where $t = \gamma_s(G)$, and let $k_1, k_2, \ldots, k_t$ be a distance domination sequence associated with S. Since diam $G = 2$, we have $k_i = 0$, 1, or 2 for $i = 1, 2, \ldots, t$. We consider two cases.

Case 1. *Assume that $k_i = 2$ for some integer i with $1 \leq i \leq t$.* Suppose that the vertices whose distance from x_i is at most 2 are those indicated in

Figure 4. Then all vertices $w_1, w_2, \ldots, w_{r^2-r}$ are step dominated by x_i. If a vertex u_j $(1 \leq j \leq r)$ step dominates x_i, then some of the vertices w_k $(1 \leq k \leq r^2 - r)$ are step dominated twice. Thus, a vertex w_j $(1 \leq j \leq r^2 - r)$ step dominates x_i. Since each vertex w_n is only adjacent to $r-1$ vertices in $\{w_m : 1 \leq m \leq r^2 - r\}$, both w_j and x_i step dominate some vertices in $\{w_m : 1 \leq m \leq r^2 - r\}$. Thus, $k_i \neq 2$ for each integer i with $1 \leq i \leq t$.

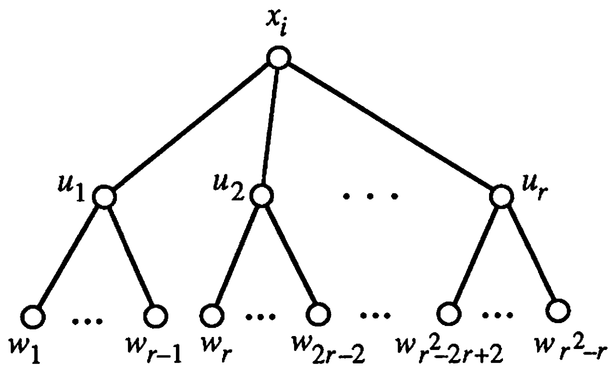

Figure 4

Case 2. *Assume that at least three of the numbers $k_1, k_2, \ldots, k_t$ equal 1, say $k_1 = k_2 = k_3 = 1$.* Suppose that the distance between some two of x_1, x_2, and x_3 is 2, say $d(x_1, x_2) = 2$ and that x_1, x, x_2 is an $x_1 - x_2$ path in G. Then x is step dominated by both x_1 and x_2, which is impossible. Hence, since diam $G = 2$, it follows that $\langle\{x_1, x_2, x_3\}\rangle = K_3$; but this says that each of x_1, x_2, and x_3 step dominates the other two, which cannot occur.

Therefore, at most two of the numbers $k_1, k_2, \ldots, k_t$ are 1 in any distance dominating sequence for G. Thus, $\gamma_s(G) \geq 2 + (r-1)^2$. □

The preceding theorem verifies that the step domination number of the Petersen graph is 6. The requirement that G has diameter 2 in Theorem 2 implies that the girth of G is exactly 5. Thus, if G meets the conditions of Theorem 2, then G is an $(r, 5)$-cage of order $r^2 + 1$. From [2, p. 38] we see that this occurs if and only if $r = 2, 3, 7$, and possibly 57.

Before proceeding further, we present the following lemma.

Lemma 3. *For a graph G and nontrivial connected graphs $G_1, G_2, \ldots, G_n$, no term of a distance domination sequence for $H = G + \cup_{i=1}^{n} G_i$ equals 1.*

Proof: Assume, to the contrary, that there is a step domination set $S = \{v_1, v_2, \ldots, v_t\}$, and corresponding distance domination sequence $k_1, k_2, \ldots, k_t$ where $k_\ell = 1$ for some ℓ with $1 \leq \ell \leq t$. We consider two cases.

Case 1. *Assume that $v_\ell \in V(G)$.* Then all vertices of $V(H) - V(G)$ and all vertices adjacent in G to v_ℓ are step dominated by v_ℓ. If there is a vertex v_j in G_m for some m $(1 \leq m \leq n)$ that step dominates v_ℓ, then $k_j = 1$. Now, since G_m is connected and nontrivial, there is a vertex v_r in G_m for which $v_jv_r \in E(G_m)$. Thus, v_r is step dominated twice, producing a contradiction. So a vertex v_j that step dominates v_ℓ must belong to G.

If $k_j = 1$, then the vertices in $V(G_i)$ for $1 \leq i \leq n$ are step dominated twice, so $k_j = 2$ and $v_\ell v_j \notin E(G)$. Suppose that there is a vertex v_r (necessarily in G) that step dominates v_j. Again, $k_r = 2$ and $v_rv_j \notin E(G)$. If $v_\ell v_r \notin E(G)$, then v_ℓ is step dominated by both v_j and v_r; so $v_\ell v_r \in E(G)$. This implies that both v_j and v_ℓ step dominate v_r, producing a contradiction.

Case 2. *Assume that $v_\ell \notin V(G)$, say $v_\ell \in V(G_m)$, where $1 \leq m \leq n$.* Then all vertices of G and some vertices of the graph G_m are step dominated by v_ℓ. Since the vertices of G are step dominated by v_ℓ, no vertex adjacent to v_ℓ in G_m can step dominate v_ℓ. Also, since there is a vertex in G_m already step dominated by v_ℓ, no vertex of G can step dominate v_ℓ. Thus, a vertex v_j with $d(v_j, v_\ell) = 2$ must step dominate v_ℓ. So $k_j = 2$.

If $v_j \in V(G_s)$, where $s \neq m$, then a vertex in G_m is step dominated twice. Thus, $v_j \in V(G_m)$. Suppose that the vertex v_r step dominates v_j. By the reasoning above, $k_r = 2$ and $v_r \in V(G_m)$. Also, $v_rv_j \notin E(G)$ but $v_rv_\ell \in E(G_m)$. Therefore, v_r is step dominated twice, producing a contradiction.

Thus, $k_i \neq 1$ for all $1 \leq i \leq t$. □

Of course, $0 < \gamma_s(G)/|V(G)| \leq 1$ for every graph G. We show in fact that every rational number in the interval $(0, 1]$ can be realized as the value $\gamma_s(G)/|V(G)|$ for a suitably chosen graph G. To see this we present the following result.

Theorem 4. *Let $G = K_m + (G_1 \cup G_2)$, where m is a positive integer and G_1 and G_2 are connected graphs with radius 1. Then $\gamma_s(G) = m + 2$.*

Proof: Let v_i be a vertex of eccentricity 1 in G_i, for $i = 1, 2$, and let $v_3, v_4, \ldots, v_{m+2}$ denote the vertices of K_m. Then $S = \{v_1, v_2, \ldots, v_{m+2}\}$ is a step domination set with distance domination sequence $2, 2, 0, 0, \ldots, 0$. Thus, $\gamma_s(G) \leq m + 2$.

Now let $S = \{v_1, v_2, \ldots, v_t\}$ be a step domination set for G with distance domination sequence $k_1, k_2, \ldots, k_t$ where $t = \gamma_s(G)$. By Lemma 3, $k_i \neq 1$ for all i $(1 \leq i \leq t)$. Thus, each vertex of $V(K_m)$ is required to be in S. Since $G - K_m$ is disconnected, at least two vertices are needed to step dominate $G - K_m$. Hence, $\gamma_s(G) \geq m + 2$. □

An argument similar to the proof of Theorem 4 also shows that $\gamma_s(\overline{K_m} + (G_1 \cup G_2)) = m + 2$ if both G_1 and G_2 are connected graphs with radius 1. With the aid of the graph G defined in the statement of Theorem 4, the desired result can now be established.

Theorem 5. *For every rational number r with $0 < r \leq 1$, there exists a connected graph G such that $\gamma_s(G)/|V(G)| = r$. Furthermore, there exists such a graph G with diameter 2.*

Proof: Let r be a rational number with $0 < r \leq 1$. Then there exist integers a and b with $3 \leq a \leq b$ such that $r = a/b$. Let $G = K_{a-2} + (K_{b-a} \cup K_2)$. Then, G has diameter 2 and by Theorem 4, $\gamma_s(G) = a - 2 + 2 = a$. Also, $|V(G)| = a - 2 + b - a + 2 = b$. Therefore, $\gamma_s(G)/|V(G)| = a/b$. □

The minimum number of vertices in a graph G needed to dominate $V(G)$ is called the domination number of G. It is interesting to note that no apparent relationship exists between the domination number of a graph and the step domination number of a graph. For the graph K_n, the domination number $\gamma(K_n) = 1$; however, the step domination number $\gamma_s(K_n) = n$. Also, the graph G shown in Figure 5 has $\gamma_s(G) = 4$ and $\gamma(G) = n + 1$.

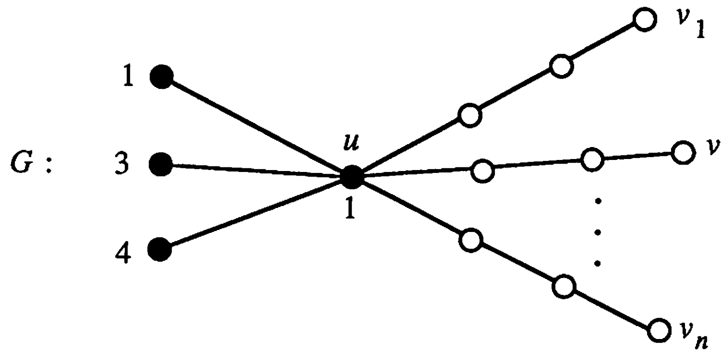

Figure 5

Since $\gamma(G) \leq \beta(G)$, one might question as to whether some relationship between $\gamma_s(G)$ and $\beta(G)$ exists. We see that this is not the case.

Theorem 6. *For every two integers $a \geq 2$ and $b \geq 1$, there exists a graph, G with $\gamma_s(G) = a$ and $\beta(G) = b$.*

Proof: We consider four cases.

Case 1. *Assume that $a = 2$ and $b \geq 2$.* Consider a double star G as shown in Figure 6, which has independence number b. By Theorem D, $\gamma_s(G) = 2$.

Case 2. *Assume that $a \geq 2$ and $b = 1$.* Consider $G = K_a$. Certainly $\beta(G) = 1$. By Theorem 1, $\gamma_s(G) = a$.

Case 3. *Assume that $a \geq 3$ and $b = 2$.* Consider $G = K_{a-2} + \overline{K}_2$, which has independence number 2. By Lemma 3, $\gamma_s(G) = a$.

Case 4. *Assume that $a \geq 3$ and $b \geq 3$.* Consider $G = K_{a-2} + (K_{1,1} \cup K_{1,b-1})$. Now $\beta(G) = b$. By Theorem 4, $\gamma_s(G) = a - 2 + 2 = a$. □

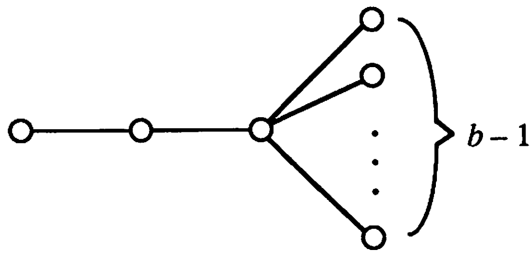

Figure 6

For each graph whose step domination number has been evaluated, only small nonnegative integers have been used in a step domination sequence. However, graphs G exist having a step domination set S with $\gamma_s(G) = |S|$ and a distance domination sequence containing arbitrarily large positive integers.

Theorem 7. *For every integer $n \geq 3$, there exists a graph G_n for which every distance domination set having cardinality $\gamma_s(G_n)$ contains a value that is at least n.*

Proof: For $n \geq 3$, let G_n denote the graph shown in Figure 7. The set $S = \{v_0, v_1, \ldots, v_n\}$ with distance domination sequence

$$k_i = \begin{cases} 1 & i = 0, 1 \\ i+1 & \text{otherwise} \end{cases}$$

is a step domination set for G_n. Thus, $\gamma_s(G_n) \leq n + 1$.

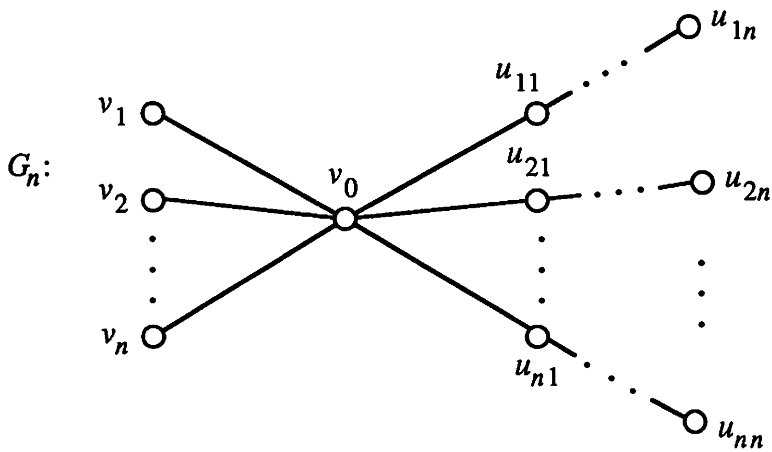

Figure 7

Suppose that $S = \{w_1, w_2, \dots, w_t\}$ is a step domination set for G_n with cardinality $\gamma_s(G)$ labeled so that w_i step dominates at least as many vertices of G_n as w_{i+1} for $1 \le i \le t-1$. Then w_1 step dominates at most $2n$ vertices and w_i step dominates at most n vertices for all $2 \le i \le t$. Thus,

$$2n + n(t-1) \ge |V(G_n)| = n + 1 + n^2.$$

This gives $t \ge n+1$. Thus $\gamma_s(G_n) \ge n+1$.

Let S be a step domination set for G_n having cardinality $\gamma_s(G_n)$. Also, let $x_i \in S$ denote the vertex step dominating u_{in} and let k_i denote the value associated with x_i in a distance domination sequence for S. Assume, to the contrary, that $k_i < n$ for all i with $1 \le i \le n$. Thus, $x_i = u_{ij}$ for some j with $1 \le j \le n$. If some $x_i = u_{ij}$ for $j \ge n/2$, then x_i step dominates only two vertices with $k_i < n/2$. By the same reasoning as above, this may only occur once since $|S| = \gamma_s(G)$. If $k_r = k_s$ for some integers r and s with $n/2 - 1 \le r \ne s < n$, then some vertices are step dominated by both x_r and x_s. Thus, there are $n-1$ distinct numbers represented by the set $\{k_i : 1 \le i \le n\}$ for which each k_i satisfies $n/2 < k_i < n$, producing a contradiction. Therefore, $k_i \ge n$ for some $1 \le i \le n$. □

3 Step Domination of Trees

We now turn our attention to the step domination number of trees. A tree T is a *caterpillar* if the removal of the end-vertices of T produces a path. A natural lower bound exists for $\gamma_s(T)$ where T is a tree. Let v be a vertex in a step domination set for a tree T. At most two vertices on any diametrical path can be step dominated by v. Hence, we have the following result.

Theorem 8. *If T is a tree of diameter d, then $\gamma_s(T) \ge \lceil \frac{d+1}{2} \rceil$.*

Using this result we can determine $\gamma_s(T)$ for trees with small diameter.

Theorem 9. *If T is a tree of diameter d, where $1 \le d \le 5$, then*

$$\gamma_s(T) = \begin{cases} 2 & \text{if } d = 1 \text{ or } d = 2 \\ d-1 & \text{if } 3 \le d \le 5. \end{cases}$$

Proof: If $d = 1$ or $d = 2$, then $T = K_{1,n}$ and $\gamma_s(T) = 2$. If $d = 3$, then T contains a spanning double star and $\gamma_s(T) = 2$. We consider two cases.

Case 1. *Assume that $d = 4$.* By Theorem 8, $\gamma_s(T) \ge 3$. The fact that $\gamma_s(T) = 3$ results from the labeling presented in Figure 8.

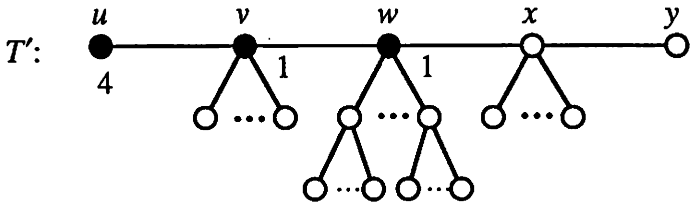

Figure 8

Case 2. *Assume that* $d = 5$. Thus, $\gamma_s(T) \leq 4$ and T is a subtree of the tree T' shown in Figure 9 containing the path P: $u_1, u_2, \ldots, u_6$.

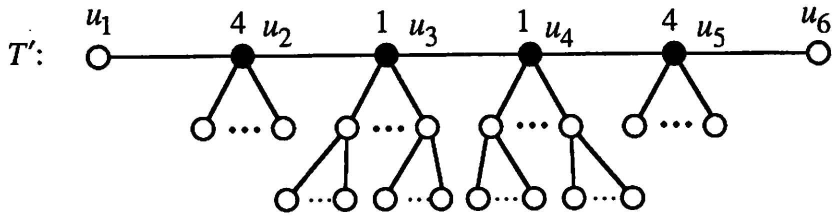

Figure 9

By Theorem 8, $\gamma_s(T) \geq 3$. Assume, to the contrary, that $\gamma_s(T) = 3$. Let $S = \{v_1, v_2, v_3\}$ be a step domination set for T and k_1, k_2, k_3 a distance domination sequence for S such that $u_1 \in N_{k_1}(v_1)$ and $u_2 \in N_{k_2}(v_2)$. Since $\gamma_s(T) = 3$, each vertex of S must step dominate exactly two vertices of P. Since T is a tree, two adjacent vertices on P cannot be step dominated by the same vertex of S. Now, $u_1 \in N_{k_1}(v_1)$ implies that either $u_3 \in N_{k_1}(v_1)$ or $u_5 \in N_{k_1}(v_1)$.

If $u_5 \in N_{k_1}(v_1)$, then $u_2 \in N_{k_2}(v_2)$ and no vertex step dominates u_3 and u_6. Therefore, $u_3 \in N_{k_1}(v_1)$. Thus, either $u_4 \in N_{k_2}(v_2)$ or $u_6 \in N_{k_2}(v_2)$. If $u_4 \in N_{k_2}(v_2)$, then no vertex step dominates u_5 and u_6. If $u_6 \in N_{k_2}(v_2)$, then no vertex step dominates u_4 and u_5, producing a contradiction. Thus, $\gamma_s(T) = 4$. □

By Theorem 9, then, the value of $\gamma_s(T)$ for a tree T depends only on its diameter d, where $1 \leq d \leq 5$. However, $\gamma_s(T)$ does not have a fixed value for trees T of diameter 6.

Theorem 10. *There exist trees T_1 and T_2 of diameter 6 such that $\gamma_s(T_1) = 4$ and $\gamma_s(T_2) = 5$.*

Proof: By Theorem 8, $\gamma_s(T) \geq 4$ if diam $T = 6$. Figure 10 shows step domination sets of cardinality 4 for any caterpillar T of diameter 6. Thus, $\gamma_s(T) = 4$ for all caterpillars T. Let T_1 be a caterpillar of diameter 6.

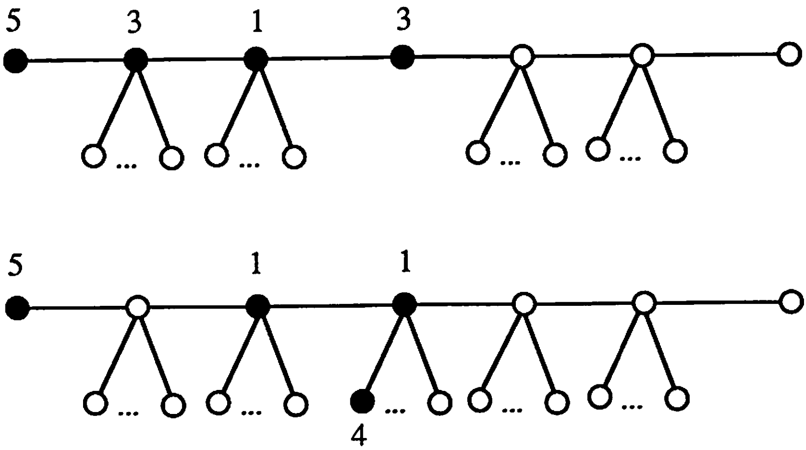

Figure 10

Let T_2 be the tree shown in Figure 11. Then the step domination set given shows that $\gamma_s(T_2) \leq 5$. Assume, to the contrary, that $\gamma_s(T_2) = 4$. Let $S = \{w_1, w_2, w_3, w_4\}$ be a step domination set for T_2 and let k_1, k_2, k_3, k_4 be the distance domination sequence associated with S.

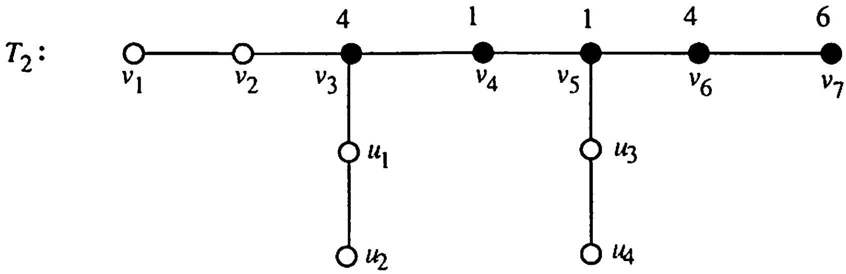

Figure 11

Since the order of T_2 is 11, some vertex of S must step dominate at least three vertices. Assume, without loss of generality, that w_1 step dominates at least three vertices. We consider four cases.

Case 1. *Assume that* $w_1 = v_4$ *and* $k_1 = 2$. Then w_1 step dominates v_2, u_1, u_3 and v_6. Assume that w_2 step dominates w_1. If $w_2 = v_i$ for $1 \leq i \leq 6$, then either u_1 or u_3 is step dominated twice. Similarly, if $w_2 = u_i$ for $1 \leq i \leq 4$, then v_2 or v_6 is step dominated twice, producing a contradiction.

Case 2. *Assume that* $w_1 = v_4$ *and* $k_1 = 3$. Then w_1 step dominates v_1, u_2, u_4 and v_7. Assume that w_2 step dominates w_1. If $w_2 = v_3$, then $k_2 = 1$ and v_2, u_1 and v_4 are step dominated. In order to step dominate w_2 we must have $w_3 = v_1$. Thus, w_4 must step dominate both v_5 and v_6, which is impossible. By symmetry, $w_2 \neq v_5$.

If $w_2 = v_2$, then $k_2 = 2$ and both u_1 and v_4 are step dominated. To step dominate w_2, we must have $w_3 = v_1$. Again, w_4 must step dominate both v_5 and v_6, producing a contradiction. By symmetry, w_2 is none of u_1, u_3 or v_6.

If $w_2 = v_1$, then $k_2 = 3$ and, again, u_1 and v_4 are step dominated. Now, without loss of generality, w_3 must step dominate v_2 and one of v_3, v_5 and v_6. Since $w_3 \neq v_4$ this is impossible. By symmetry, w_2 is also not u_2, u_4 or u_7.

Case 3. *Assume that* $w_1 = v_3$ *and* $k_1 = 2$. Then w_1 step dominates v_1, u_2 and v_5. Assume that w_2 step dominates w_1. Thus, by symmetry, w_2 is one of v_1, v_5, v_6 and v_7, and the vertices v_2, v_4, v_6 and v_7 must be step dominated by w_3 and w_4. Without loss of generality, assume that w_3 step dominates v_7. Then w_3 does not step dominate any of v_2, v_4 and v_6 and no vertex w_4 exists to step dominate all three of these vertices.

Case 4. *Assume that* $w_1 = v_3$ *and* $k_1 = 1$. Then w_1 step dominates v_2, u_1 and v_4. Assume that w_2 step dominates w_1. If w_2 is v_1, v_2, v_6 or v_7, then u_2, u_3, v_5, v_6 and v_7 must be step dominated by w_3 and w_4, which is impossible. If w_2 is v_4, then $k_2 = 1$ and v_1, u_2, u_3, u_4, v_6 and v_7 must be step dominated by w_3 and w_4, which is impossible. If w_2 is v_5, then $k_2 = 2$ and v_1, u_2, u_3, u_4, v_5 and v_6 must be step dominated by w_3 and w_4, which is impossible.

Therefore, $\gamma_s(T) = 4$. □

Also, for trees of diameter $d = 4n - 1$, the value of $\gamma_s(T)$ is not fixed.

Theorem 11. *There exist trees* T_1 *and* T_2 *of diameter* $d \equiv 3 \pmod 4$ *such that* $\gamma_s(T_1) \neq \gamma_s(T_2)$.

Proof: By Theorem 8, any tree T with diameter $d = 4n-1$ has $\gamma_s(T) \geq 2n$. The tree T_1 shown in Figure 12 has a step domination set of cardinality $2n$. So, $\gamma_s(T_1) = 2n$.

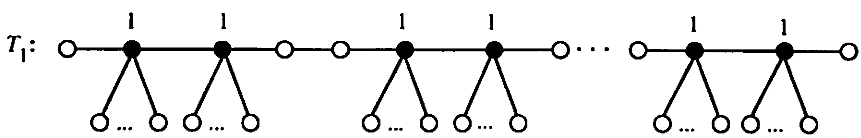

Figure 12

Consider the tree T_2 shown in Figure 13. Assume, to the contrary, that T_2 has $\gamma_s(T_2) = 2n$.

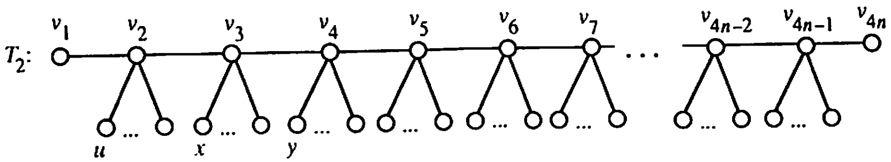

Figure 13

Let $S = \{w_1, w_2, \ldots, w_{2n}\}$ and let $k_1, k_2, \ldots, k_{2n}$ be a distance domination sequence for S. Thus, each vertex w_i of S step dominates exactly two vertices of the set $\{v_1, v_2, \ldots, v_{4n}\}$. Since v_1 and v_{4n} can only step dominate one vertex on the spine of T, it follows that neither v_1 nor v_{4n} are in S. Let w_i denote the vertex that step dominates u.

If $w_i = v_2$, then $k_i = 1$ and v_3 is step dominated by w_i. The only vertex that can step dominate y is $w_j = v_3$ with $k_j = 2$. However, then, v_1 is step dominated twice. Thus, $w_i \neq v_2$ and $k_i \geq 2$. Therefore, v_1 is step dominated by w_i. Let w_j be the vertex that step dominates v_2. By the same reasoning, $k_j \geq 2$ and x and some vertex on the spine of T are step dominated by w_j. We repeat this process until v_r and v_{r+1} are the only vertices in $\{v_i \mid 1 \leq i \leq r+1\}$ that remain to be step dominated. Thus, v_r and v_{r+1} need to be step dominated without step dominating v_{r-1}, v_{r+2}, and the end-vertices of T a second time. If the vertex w_k that step dominates v_r is to the left of v_r in Figure 14, then a is step dominated twice. If w_k is to the right of v_r, then c is step dominated twice, producing a contradiction.

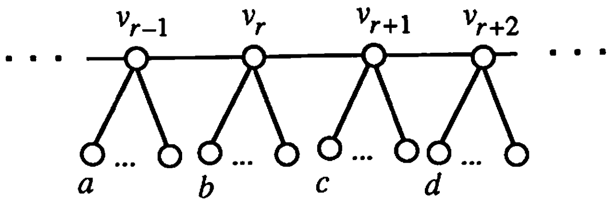

Figure 14

Hence $\gamma_s(T_2) > 2n$. □

For caterpillars T of diameter at least 6 we can find an upper bound for $\gamma_s(T)$.

Theorem 12. *If T is a caterpillar of diameter $d \geq 6$, then*

$$\gamma_s(T) \leq \begin{cases} d-1 & \text{if } d = 6 \text{ or } d = 8 \\ d-2 & \text{if } d = 7 \text{ or } d = 9 \\ d-3 & \text{if } 10 \leq d \leq 16 \\ \lceil \frac{7d+1}{8} \rceil & \text{if } d \geq 17. \end{cases}$$

Proof: Let $P\colon v_1, v_2, \ldots, v_{d+1}$ be a diametrical path for T. We consider two cases.

Case 1. *Assume that $d \geq 7$ is odd.* Let $Q = \{6, 12, 18\} \cup \{30 + 8j, 36 + 8j\colon 0 \leq j \leq \lceil \frac{d-33}{16} \rceil\}$ and define $R = \{v_{\frac{d+5+k}{2}}, v_{\frac{d-k-1}{2}} : k \in Q\}$. Define $S = \{v_2, v_3, \ldots, v_d\} - R$ and the distance domination sequence for S by

$$k_i = \begin{cases} 1 & \text{if } i = \frac{d+1}{2} \text{ or } i = \frac{d+3}{2} \\ 4 & \text{if } i = \frac{d+5}{2} \text{ or } i = \frac{d-1}{2} \\ 4+k & \text{if } i = \frac{d+5+k}{2} \text{ or } i = \frac{d-1-k}{2} \\ & \text{for } 2 \leq k \leq d-5 \text{ where } k \text{ is even and } k \notin Q. \end{cases}$$

Thus, every vertex of T is step dominated by S except possibly v_2 and v_d. If this is the case, then add v_1 and v_{d+1} to the set S each with step 1. Then S is a step domination set for T. The cardinality of R is at most $3 + 2\left(\lceil \frac{d-33}{16} \rceil + 1\right)$, so

$$|S| = d + 1 - 5 - \left\lceil \frac{d-33}{8} \right\rceil = \left\lceil \frac{7d+1}{8} \right\rceil.$$

Thus,

$$|S| \leq \begin{cases} d-2 & \text{if } d = 7 \text{ or } d = 9 \\ d-4 & \text{if } 11 \leq d \leq 15 \\ \lceil \frac{7d+1}{8} \rceil & \text{if } d \geq 17. \end{cases}$$

Case 2. *Assume that $d \geq 6$ is even.* Let $R = \{v_{\frac{d+4+k}{2}}, v_{\frac{d-k-2}{2}} : k \in Q\}$. Define $S = \{v_1, v_2, \ldots, v_{d-1}\} - R$ and the distance domination sequence for S by

$$k_i = \begin{cases} 1 & \text{if } i = \frac{d}{2} \text{ or } i = \frac{d+2}{2} \\ 4 & \text{if } i = \frac{d+4}{2} \text{ or } i = \frac{d-2}{2} \\ 4+k & \text{if } i = \frac{d+k}{2} + 2 \text{ or } i = \frac{d-k}{2} - 1 \\ & \text{for } 2 \leq k \leq d-6 \text{ where } k \text{ is even and } k \notin Q. \end{cases}$$

Then S is a step domination set for T with the same modification as above, if needed. Thus,

$$|S| \leq \begin{cases} d-1 & \text{if } d = 6 \text{ or } d = 8 \\ d-3 & \text{if } 10 \leq d \leq 16 \\ \lceil \frac{7d+1}{8} \rceil & \text{if } d \geq 18. \end{cases}$$

□

The step domination numbers of trees were studied in [1] where the following result appeared.

Theorem E. *If T is a caterpillar of diameter $d \geq 2$, then*

$$\gamma_s(T) \leq \begin{cases} d-1 & \text{if } d \text{ is odd} \\ d+1 & \text{if } d \text{ is even.} \end{cases}$$

Using previous theorems we can consider the sharpness of Theorem E. The caterpillars of diameter $d = 3$ or $d = 5$, have $\gamma_s(T) = d - 1$. Thus, Theorem E is sharp for caterpillars of small odd diameter d. However, by Theorem 12 an improvement can be made for caterpillars with large diameter. In [1] an upper bound for the step domination number for caterpillars was given in terms of their orders.

Theorem F. *If T is a caterpillar of order p, then $\gamma_s(T) \leq \lceil \frac{p}{2} \rceil + 1$.*

Next we consider the sharpness of Theorem F. Let $T = P_{4k+2}$. Thus, $\gamma_s(T) = 2k + 2 = \lceil \frac{4k+2}{2} \rceil + 1$. Now, consider the caterpillar T_n as shown in Figure 15. Thus, $\gamma_s(T_n) = 2$ but the upper bound gives $\lceil \frac{n}{2} \rceil + 1$. Therefore, the difference between the values can be arbitrarily far apart and no improvement in the upper bound of Theorem F is possible for caterpillars.

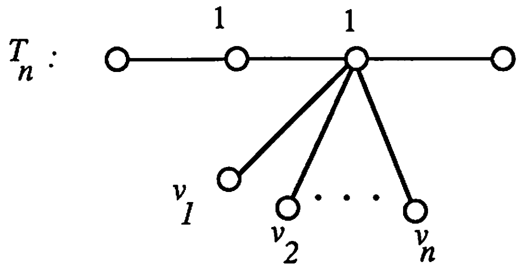

Figure 15

For completeness we state another result in [1] which gives an upper bound for trees in general in terms of their orders.

Theorem G. *If T is tree of order $p \geq 3$, then $\gamma_s(T) \leq p - \lfloor \sqrt{\frac{p}{2}} \rfloor$.*

References

[1] G. Chartrand, M. Jacobson, E. Kubicka, and G. Kubicki, The step domination number of a graph. In progress.

[2] G. Chartrand and L. Lesniak, *Graphs & Digraphs* (second edition). Wadsworth and Brooks/Cole, Monterey, CA (1986).