On L(2,1)-labeling of the Cartesian product of a cycle and a path

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Abstract: Sharp bounds are presented for the λ-number of the Cartesian product of a cycle and a path, and of the Cartesian product of two cycles.

Keywords: $L(2,1)$-labeling, Cartesian product, cycle, path

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1 Introduction

The problem of $L(2,1)$-*labeling* graphs is a variation of the problem of assigning frequencies to radio transmitters subject to certain restrictions imposed by the distances between the transmitters. Formally, an $L(2,1)$-labeling of a graph G is an assignment f of non-negative integers to the vertices of G such that

- $|f(u) - f(v)| \geq 2$ if $d(u,v) = 1$, and
- $|f(u) - f(v)| \geq 1$ if $d(u,v) = 2$.

The difference between the largest label and the smallest label assigned by f is called the span of f, and the minimum span over all $L(2,1)$-labelings of G is called the λ-*number* of G, denoted by $\lambda(G)$. The $L(2,1)$-labeling was first introduced by Griggs and Yeh [8], and has since been an object of extensive research [1, 2, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 16]. Note that the general problem of determining $\lambda(G)$ is NP-hard [7].

Whittlesey et al [15] determined $\lambda(P_{n_1}\Box\cdots\Box P_{n_r})$ where n_i's satisfy certain conditions. Our contribution consists of the following:

- $5 \leq \lambda(C_m\Box P_2) \leq 6$ with equality at the lower end if $m \equiv 0 \pmod 3$,
- $6 \leq \lambda(C_m\Box P_n) \leq 7$ for $n \geq 3$ and with equality at the lower end if $m \equiv 0 \pmod 7$,
- $\lambda(C_{7k}\Box C_{7l}) = 6$; $6 \leq \lambda(C_m\Box C_{4l}) \leq 7$ and $6 \leq \lambda(C_{6k}\Box C_{3l}) \leq 7$.

By a graph is meant a finite, simple, undirected and connected graph. The *Cartesian product* $G\Box H$ of graphs $G = (V,E)$ and $H = (W,F)$ is defined as follows: $V(G\Box H) = V \times W$ and $E(G\Box H) = \{\{(u,x),(v,y)\}$: either $u = v$ and $\{x,y\} \in F$ or $x = y$ and $\{u,v\} \in E\}$. This product (that is commutative and associative in a natural way) is one of the most important graph products, with potential applications. For example, the n-cube Q_n is easily seen to be the Cartesian product of n copies of P_2. It is known that (i) $G\Box H$ is connected if and only if G and H are connected, and (ii) $G\Box H$ is bipartite if and only if G and H are bipartite. Further, $d_{G\Box H}((u,x),(v,y)) = d_G(u,v) + d_H(x,y)$.

A vertex subset S of a graph G is said to be an *independent set* if elements of S are mutually nonadjacent in G. If every x not in S is adjacent to at least one element of S, then S is said to be a *dominating set*. An *independent dominating set* has an obvious definition. The general problem of obtaining an (independent) dominating set of smallest size is NP-hard [3]. Klavžar and Seifter [11] studied domination in the Cartesian products of cycles.

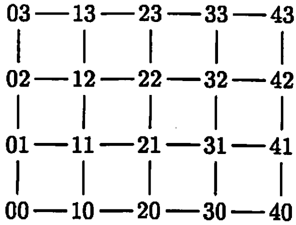


Figure 1: The graph $P_5 \Box P_4$

There is a clear connection between independent sets and λ-numbering of a graph. In particular, if $\lambda(G) = n$, then there is a partition of $V(G)$ into at most $n+1$ independent sets.

Remark: Let $n = 2^k - 1$. Whittlesey et al [15] proved that $\lambda(Q_n) \leq 2n$. This follows also from the fact that Q_n admits of a vertex partition into (equal-size) independent dominating sets $V_0, \cdots, V_n$ where distinct elements of each V_i are at a distance of at least three [9].

For $m \geq 3$ and $n \geq 2$, let C_m denote the *cycle* on m vertices, and let P_n denote the *path* on n vertices, where $V(C_k) = V(P_k) = \{0, \ldots, k-1\}$, $E(P_k) = \{\{i, i+1\} : 0 \leq i \leq k-2\}$ and $E(C_k) = E(P_k) \cup \{\{k-1, 0\}\}$. The graph $P_5 \Box P_4$ appears in Figure 1. For simplicity, a vertex (p, q) has been shown as pq.

The following result consists of a useful lower bound on $\lambda(G)$.

Lemma 1.1 *(Griggs & Yeh [8]) Let G be a graph with maximum degree $\Delta \geq 2$. If G contains three vertices of degree Δ such that one of them is adjacent to the other two, then $\lambda(G) \geq \Delta + 2$.*

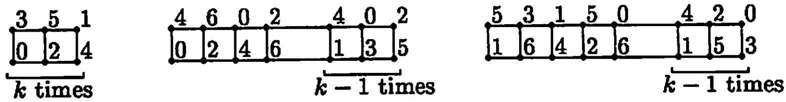
The lower bound of Lemma 1.1 is achievable in certain cases.

Theorem 1.2 *(Whittlesey et al [15])*

1. *If $m \geq 4$, then $\lambda(P_m \Box P_2) = 5$.*
2. *If $m, n \geq 4$ or $m \geq 5$ and $n \geq 3$, then $\lambda(P_m \Box P_n) = 6$.*

2 Main Results

By Lemma 1.1, (i) $\lambda(C_m \Box P_2) \geq 5$, (ii) $\lambda(C_m \Box P_n) \geq 6$, where $n \geq 3$, and (iii) $\lambda(C_m \Box C_n) \geq 6$. Among other things, we show that each of these lower bounds is achievable.



(a) $m = 3k$ (b) $m = 3k + 1$ (c) $m = 3k + 2$

Figure 2: A labeling of $P_m \square P_2$ towards that of $C_m \square P_2$

Theorem 2.1 *Let $m \geq 3$.*

1. *If $m \equiv 0 \pmod 3$, then $\lambda(C_m \square P_2) = 5$.*
2. *If $m \not\equiv 0 \pmod 3$, then $\lambda(C_m \square P_2) \leq 6$.*

Proof. For $m \equiv 0 \pmod 3$, observe that a properly-labeled $P_{3k} \square P_2$ is obtainable by appropriately repeating the basic structure of $P_3 \square P_2$ that appears in Figure 2(a). Further, a "wrap-around" of $P_{3k} \square P_2$ thus obtained leads to $C_{3k} \square P_2$. Labeling remains valid. The span of the labeling being equal to the lower bound, we have $\lambda(C_{3k} \square P_2) = 5$.

Next suppose that $m \equiv 1 \pmod 3$. A proper $L(2,1)$-labeling of $P_{3k+1} \square P_2$ appears in Figure 2(b). A "wrap-around" of this graph graph leads to a properly-labeled $C_{3k+1} \square P_2$. The span of the labeling being equal to 6, we have $\lambda(P_{3k+1} \square P_2) \leq 6$. The remaining case is illustrated in Figure 2(c).∎

Remark: If $m = 4$, then the upper bound on $\lambda(C_m \square P_2)$ appearing in Theorem 2.1(2) corresponds to the exact value. This is because $C_4 \square P_2$ is isomorphic to Q_3, and it is known that $\lambda(Q_3) = 6$ [10, 15].

For the case when $n \geq 3$, we proceed in a manner similar to the above.

Let vertex (i,j) of the graph $P_{7k} \square P_n$ be labeled as follows: $f(i,j) = (5i + 4j) \bmod 7$. The reader may check to see this is a well-defined $L(2,1)$-labeling of $P_{7k} \square P_n$. Among other things, a proof of Theorem 1.2(2) is implicit in it. An illustration appears in Figure 3.

Theorem 2.2 *If $m \equiv 0 \pmod 7$ and $n \geq 3$, then $\lambda(C_m \square P_n) = 6$.*

Proof. The $L(2,1)$-labeling of $P_{7k} \square P_n$ is such that when this graph is "wrapped around" to yield $C_{7k} \square P_n$, the labeling remains valid. (Verification is left to the reader.) The span of the labeling coincides with the lower bound, hence the result.∎

Theorem 2.3 *If $m \equiv 0 \pmod 7$ and $n \equiv 0 \pmod 7$, then $\lambda(C_m \square C_n) = 6$.*

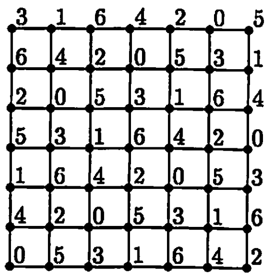


Figure 3: A canonical $L(2,1)$-labeling of $P_7 \square P_7$

Proof. If the graph $C_{7k} \square P_{7l}$ is labeled by the scheme outlined in the proof of Theorem 2.2, and it is "wrapped around" to yield the graph $C_{7k} \square C_{7l}$, the original $L(2,1)$-labeling is not "disturbed."∎

Some remarks are in order.

- The labeling of $C_{7k} \square C_{7l}$ is tight in the sense that if a vertex x is not labeled i, then x is at a distance of at most two from some vertex labeled i.

- $C_{7k} \square C_{7l}$ is a regular graph whose λ-number coincides with the lower bound mentioned in Lemma 1.1. This is unlike Q_n which is an n-regular graph, and for which $\lambda(Q_n) > n + 2$, where $n \geq 3$ [10, 15].

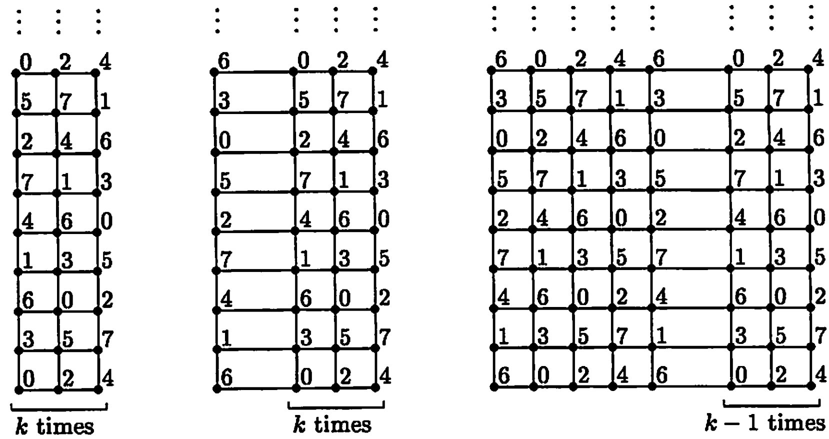
Corollary 2.4 *For $0 \leq i \leq 6$, let V_i be the set of vertices of $C_{7k} \square C_{7l}$ that receive label i in the proof of Theorem 2.3.*

1. *$V_0, \ldots, V_6$ constitute a vertex partition into equal-size independent sets.*

2. *Each of $(V_0 \cup V_1)$, $(V_2 \cup V_3)$ and $(V_4 \cup V_5)$ is an independent dominating set.*

We next consider $L(2,1)$-labeling of $C_m \square P_n$ in general. For this, there are three cases: (i) $m = 3k$, (ii) $m = 3k + 1$, and (iii) $m = 3k + 2$. The labelings of $P_m \square P_n$ appearing in Figure 4 and a "wrap-around" in each case yield the following result.

Theorem 2.5 *If $m, n \geq 3$, then $\lambda(C_m \square P_n) \leq 7$.*



(a) $m = 3k$ (b) $m = 3k+1$ (c) $m = 3k+2$

Figure 4: Labeling of $P_m \square P_n$ towards that of $C_m \square P_n$

3 Additional Results

In this section, we establish the upper bound of 7 for each of $C_m \square C_{4l}$ and $C_{6k} \square C_{3l}$. Lower bound of 6 is obvious.

Theorem 3.1 *If $m \geq 4$ and $n \equiv 0 \pmod 4$, then $\lambda(C_m \square C_n) \leq 7$.*

Proof. Examine the $L(2,1)$-labeling of $P_m \square P_{4l}$ depicted in Figure 5(a) through Figure 5(d) that, respectively, capture the following cases: (a) $m = 4k$, (b) $m = 4k+1$, (c) $m = 4k+2$, and (d) $m = 4k+3$. For each, the labeling is such that when the graph is "wrapped around" to yield $C_m \square C_{4l}$, the labeling is not "disturbed."∎

Remark: If $m = n = 4$, then the upper bound on $\lambda(C_m \square C_n)$ appearing in Theorem 3.1 corresponds to the exact value. This is because $C_4 \square C_4$ is isomorphic to Q_4, and it is known that $\lambda(Q_4) = 7$ [10, 15].

Corollary 3.2 *For $0 \leq i \leq 7$, let V_i be the set of vertices of $C_m \square C_{4l}$ that receive label i in the proof of Theorem 3.1. $(V_0 \cup V_1)$, $(V_2 \cup V_3)$, $(V_4 \cup V_5)$, and $(V_6 \cup V_7)$ constitute a vertex partition into equal-size independent dominating sets.*

Theorem 3.3 *If $m \equiv 0 \pmod 6$ and $n \equiv 0 \pmod 3$, then $\lambda(C_m \square C_n) \leq 7$.*

Proof. The labeling of $P_{6k} \square P_{3l}$ depicted in Figure 6 leads to a valid $L(2,1)$-labeling of $C_{6k} \square C_{3l}$.∎

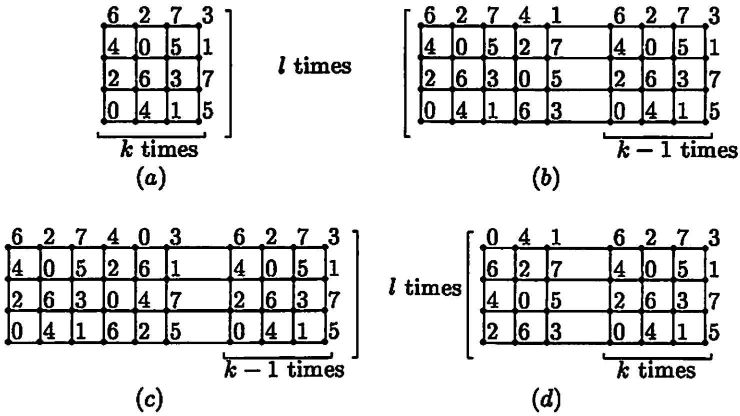


Figure 5: Labeling pattern of $P_m \square P_{4l}$ towards that of $C_m \square C_{4l}$

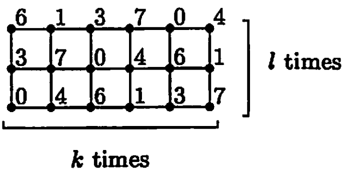


Figure 6: Labeling pattern of $P_{6k} \square P_{3l}$ towards that of $C_{6k} \square C_{3l}$

Corollary 3.4 *For $0 \leq i \leq 7$, let V_i be the set of vertices of $C_{6k} \square C_{3l}$ that receive label i in the proof of Theorem 3.3.* $(V_0 \cup V_1)$, $(V_3 \cup V_4)$ *and* $(V_6 \cup V_7)$ *constitute a vertex partition into equal-size independent dominating sets.*

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