Numbers of Vertices and Edges of Magic Graphs

Marián Trenkler
University of P.J. Šafárik
Jesenná 5
041 54 Košice
Slovakia
email: trenkler@duro.upjs.sk

Dedicated to Professor E. Jucovič on the occasion of his 70th birthday

ABSTRACT. We prove: A connected magic graph with n vertices and q edges exists if and only if $n = 2$ and $q = 1$ or $n \geq 5$ and $\frac{5n}{4} < q \leq \frac{n(n-1)}{2}$.

In the paper only finite, undirected graphs without loops and multiple edges are considered. By a *magic valuation* of a graph **G** is meant such an assignment $\mathcal{F}$ of the edges of **G** by pairwaise different positive numbers that the sum of assignments of edges meeting of the same vertex is constant. A graph is called *magic* if it allows a magic valuation. This notion has been introducted by J.Sedláček in [5]. Two different characterizations of magic graphs were published in [2] and [3] and one characterization for regular magic graphs is in [1].

A spanning subgraph **F** of the graph **G** is called a (1-2)-*factor* of **G** if each of its components is an isolated edge or a circuit. We say that a (1-2)-factor *separates the edges* e and f if at least one of them belongs to **F** and neither the edge part nor the circuit part contains both of them. In [2] the following theorem is proved.

Theorem 1. *A graph* **G** *is magic if and only if every edge belongs to a (1-2)-factor, and every pair of edges* e, f *is separated by a (1-2)-factor.*

It follows, then, that a graph is magic with real labels if and only if it is magic with integer labels.

If e and f is an arbitrary couple of edges of a bipartite magic graph **G** then it has a 1-factor which contains e and does not contain f. Evidently, every bipartite magic graph is an elementary graph (see [4] p.122), which has a 1-factor for its arbitrary edge.

The aim of this paper is to prove the following theorem:

Theorem 2. *A connected magic graph with n vertices and q edges exists if and only if $n = 2$ and $q = 1$ or $n \geq 5$ and $\frac{5n}{4} < q \leq \frac{n(n-1)}{2}$.*

Proof: Since, except of the complete graph $\mathbf{K}_2$ of order 2, no graph with less that 5 vertices is magic we confine ourselves to graphs of order $n \geq 5$.

1. Neccesary conditions.

Every vertex of a magic graph has degree ≥ 2 and no edge has both its end-vertices of degree 2. Therefore the number of vertices of degree 2 is at most the number of vertices of degree $n > 2$, i.e. $q \geq \frac{5n}{4}$.

If $\mathbf{G}$ is a bipartite magic graph with the bipartition V_1, V_2, then $q > \frac{5n}{4}$. In the opposite case all vertices of degree 2 form the set V_1 and all vertices of degree ≥ 3 the set V_2. This is not possible because a bipartite magic graph must be balanced, i.e. $|V_1| = |V_2|$.

Let $\mathbf{G}$ be a non-bipartite magic graph. The non-equality holds from the fact that every odd circuit of length m has at most $\frac{m-1}{2}$ vertices of degree 2.

2. The construction of a magic graph with n vertices and a minimal possible number of edges starts from a circuit $\mathbf{C}$ with vertices $v_1, v_2, \dots, v_n$. We add p chords, $\frac{n}{4} < p \leq \frac{n+4}{4}$, to circuit $\mathbf{C}$.

For $n = 5$ or $n = 6$ or $n = 7$ we add two edges v_1v_4, v_2v_5 or v_1v_4, v_3v_6 or v_1v_3, v_5v_7, respectively. (Note. There exists no non-bipartite graph with 6 vertices and 8 edges but it exists with three chords v_1v_4, v_2v_6, v_3v_5.) For $n = 8$ we add three edges v_1v_4, v_5v_8, v_3v_6 or v_1v_4, v_6v_8, v_3v_7.

Let $n \geq 9$, we consider four cases

if $n \equiv 0 (mod\ 4)$, then we add $v_1v_{\frac{n}{2}}$, $v_{\frac{n+2}{2}}v_n$, $v_{\frac{n-2}{2}}v_{\frac{n+2}{2}}$ and v_iv_{n-i+1} or $v_1v_{\frac{n}{2}}$, $v_{\frac{n+4}{2}}v_n$ and v_iv_{n-i+2} for $i = 3, 5, \dots, \frac{n-3}{2}$

if $n \equiv 1 (mod\ 4)$, then we add $v_1v_{\frac{n+3}{2}}$ and v_iv_{n-i} for $i = 3, 5, \dots, \frac{n-5}{2}$

if $n \equiv 2 (mod\ 4)$, then we add $v_1v_{\frac{n}{2}}$ and v_iv_{n+3-i} or $v_1v_{\frac{n}{2}}$ and v_iv_{n+2-i} for $i = 3, 5, \dots, \frac{n}{2}$

if $n \equiv 3 (mod\ 4)$, then we add $v_1v_{\frac{n-1}{2}}$, $v_{\frac{n+1}{2}}v_n$ and v_iv_{n-i+1} for $i = 3, 5, \dots, \frac{n-5}{2}$

(In Figure 1 $n = 10, 12$ (bipartite graphs) and $n = 9, 11$.)

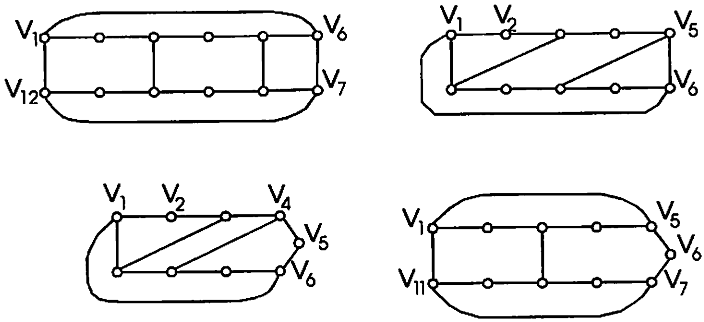

Figure 1

For cases $8 \leq n \equiv 0(mod\ 2)$ two graphs are described, the first one is bipartite and the second one non-bipartite.

3. Now we show that these graphs are magic.

From our construction follows that every graph **G**, except from a non-bipartite graph of order $\equiv 0(mod\ 2)$, has a factor **F** which consists of one circuit **C** and isolated edges. We consider only such factors **F**, that at most one edge of **C** is a chord of the circuit $(v_1v_2 \ldots v_{n-1}v_n)$. Evidently, we can choose the factor **F** in such a way that:

(i) If e is a chord, then there exists the factor **F** such, that e lies on its circuit **C**.

(ii) If e and f are two arbitrary edges of **G**, which have no common vertex of degree 2, then there exists the factor **F** such that exactly one of edges e and f lies on its circuit.

(iii) If e and f are two arbitrary edges of **G** with a common vertex of degree 2, then there exists the factor **F** such, that its circuit **C** has only one vertex of e and no vertex of f. (Note. The edge e does not belong to this factor and f is its isolated edge.)

From the existence of such a factor it follows that every edge belongs to some (1-2)-factor and edges e and f are separated by a **F** factor.

If $n \equiv 0(mod\ 2)$ and **G** is non-bipartite, some couples of edges are separated by a factor **F** which consists of two circuits (having no common edge) of odd length and possibly some isolated edges.

In the described ways all couples of edges can be separated.

4. By adding one new edge e to a non-bipartite magic graph **G** we obtain a magic graph. Let $\mathcal{F}$ be a magic valuation of **G**. By its modification we obtain a new magic valuation of **G**.

We consider two cases:

If a new edge e forms a new even circuit $\mathbf{C}$, we chage the valuation $\mathcal{F}$ of $\mathbf{C}$ that e has value ϵ as it is shown on Figure 2, where $\epsilon < \min\{\mathcal{F}(e) : e \in E(\mathbf{C})\}$.

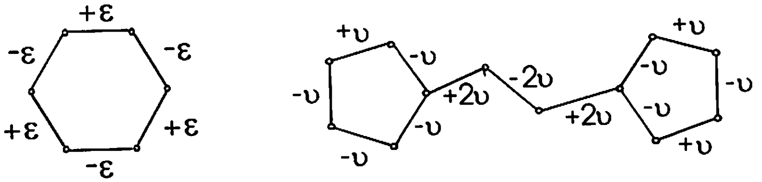

Figure 2

If e does not lie on any even circuit then e forms a new odd circuit $\mathbf{C}_1$. In $\mathbf{G}$ exists (from our construction) another odd circuit $\mathbf{C}_2$ joined with $\mathbf{C}_1$ by a path $\mathbf{P}$. These two circuits and the path form a subgraph $\mathbf{D}$. We change the valuation $\mathcal{F}$ as it is shown on Figure 2 in such a way, that the value of e is v, where $v < \frac{1}{2}\min\{\mathcal{F}(e) : e \in E(\mathbf{D})\}$.

Remarks. 1. Every connected component of a magic graph $\mathbf{G}$ is a magic graph. At most one of components of $\mathbf{G}$ can be $\mathbf{K}_2$. So the bound in theorem 2 can be replaced by $q < \frac{5(n-2)}{4} + 1 = \frac{5n-6}{4}$.

2. If we identify two vertices of a complete graph $\mathbf{K}_{n-2}$ with end-vertices of a path $\mathbf{P}_3$ of order 4 then we obtain a graph of order n with $\frac{(n-2)(n-3)}{2} + 3$ edges which has no magic valuation.

3. A Zikovian product of a total disconnected graph $\mathbf{D}_{\frac{n}{2}}$ and a complete graph $\mathbf{K}_{\frac{n}{2}}$ (every vertex of $\mathbf{D}_{\frac{n}{2}}$ is joined with every vertex of $\mathbf{K}_{\frac{n}{2}}$) has all vertices of degre $\geq \frac{n}{2}$ and it is a Hamiltonian graph but it is not a magic graph.

References

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