Existence of HSOLSSOMs with types h^n and $1^n u^1$*

R.J.R. Abel
School of Mathematics
University of New South Wales
Kensington, NSW 2033, Australia

F.E. Bennett
Department of Mathematics
Mount Saint Vincent University
Halifax, Nova Scotia B3M 2J6, Canada

H. Zhang
Computer Science Department
The University of Iowa
Iowa City, IA 52242, U. S. A.

L. Zhu
Department of Mathematics
Suzhou University
Suzhou 215006, China

ABSTRACT. The existence of holey self-orthogonal Latin squares with symmetric orthogonal mates (HSOLSSOMs) of types h^n and $1^n u^1$ is investigated. For type h^n, new pairs of (h, n) are constructed so that the possible exceptions of (h, n) for the existence of such HSOLSSOMs are reduced to 11 in number. Two necessary conditions for the existence of HSOLSSOMs of type $1^n u^1$ are (1) $n \geq 3u + 1$ and (2) n must be even and u odd. Such an HSOLSSOM gives rise to an incomplete SOLSSOM. For $3 \leq u \leq 15$, the necessary conditions are shown to be sufficient with seven possible exceptions. It is also proved that such an HSOLSSOM exists whenever even $n \geq 5u + 9$ and odd $u \geq 9$.

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1 Introduction

We first give definitions of HSOLSSOM and ISOLSSOM. Let S be a set and $\mathcal{H} = \{S_1, S_2, ..., S_n\}$ be a set of disjoint subsets of S. A *holey Latin square* having *hole set* $\mathcal{H}$ is an $|S| \times |S|$ array L, indexed by S, satisfying the following properties:

(1) every cell of L either contains an element of S or is empty,

(2) every element of S occurs at most once in any row or column of L,

(3) the subarrays indexed by $S_i \times S_i$ are empty for $1 \le i \le n$ (these subarrays are referred to as *holes*),

(4) element $s \in S$ occurs in row or column t if and only if $(s,t) \in (S \times S)\setminus \bigcup_{1\le i \le n}(S_i \times S_i)$.

The *order* of L is $|S|$. Two holey Latin squares on symbol set S and hole set $\mathcal{H}$, say L_1 and L_2, are said to be *orthogonal* if their superposition yields every ordered pair in $(S \times S)\setminus \bigcup_{1\le i \le n}(S_i \times S_i)$. We shall use the notation IMOLS$(s; s_1, ..., s_n)$ to denote a pair of orthogonal holey Latin squares on symbol set S and hole set $\mathcal{H} = \{S_1, S_2, ..., S_n\}$, where $s = |S|$ and $s_i = |S_i|$ for $1 \le i \le n$. If $\mathcal{H} = \emptyset$, we obtain a MOLS(s). If $\mathcal{H} = \{S_1\}$, we simply write IMOLS(s, s_1) for the orthogonal pair of holey Latin squares.

If $\mathcal{H} = \{S_1, S_2, ..., S_n\}$ is a partition of S, then a holey Latin square is called a *partitioned incomplete Latin square* , denoted by PILS. The *type* of the PILS is defined to be the multiset $\{|S_i| : 1 \le i \le n\}$. We shall use an "exponential" notation to describe types: so type $t_1^{u_1}...t_k^{u_k}$ denotes u_i occurrences of $t_i, 1 \le i \le k$, in the multiset. Two orthogonal PILSs of type T will be denoted by HMOLS(T).

A holey Latin square is called *self-orthogonal* if it is orthogonal to its transpose. For self-orthogonal holey Latin squares we use the notations SOLS(s), ISOLS(s, s_1) and HSOLS(T) for the cases of $\mathcal{H} = \emptyset, \mathcal{H} = \{S_1\}$ and a holey partition $\{S_1, S_2, ..., S_n\}$, respectively.

If any two PILS's in a set of t PILS's of type T are orthogonal, then we denote the set by t HMOLS(T). Similarly, we may define t MOLS(s) and t IMOLS(s, s_1).

A *holey* SOLSSOM having partition $\mathcal{P}$ is 3 HMOLS (having partition $\mathcal{P}$), say A, B, C, where $B = A^T$ and $C = C^T$. Here a SOLSSOM stands for a *self-orthogonal Latin square* (SOLS) with a *symmetric orthogonal mate* (SOM). A holey SOLSSOM of type T will be denoted by HSOLSSOM(T). From 3 IMOLS(s, s_1) we can similarly define incomplete SOLSSOM, denoted by ISOLSSOM(s, s_1). It is clear that an HSOLSSOM$(1^{v-n}n^1)$ implies the existence of an ISOLSSOM(v, n). In an HSOLSSOM(T), it is known (see [12]) that if one hole has an odd size, then every hole must have an odd size and the number of holes must also be odd.

HSOLSSOMs have been useful in the construction of resolvable orthogonal arrays invariant under the Klein 4-group [9], Steiner pentagon systems [10], [1], three-fold BIBDs with block size seven [17] and Authentication perpendicular arrays [8]. Not only the uniform type h^n but also the nonuniform type $h^n u^1$ are useful. The existence of an HSOLSSOM(h^n) has been investigated by several researchers (see [11], [16], [12], [5], [4], [3]). The known results can be summarized as follows.

Theorem 1.1 (1) *A SOLSSOM(v) exists if and only if $n \geq 4$, except $v = 6$ and except possibly $v = 10, 14, 66, 70$.*
(2) *An HSOLSSOM(h^n) exists only if $n \geq 5$, where $h \geq 2$ and n must when h is odd. These necessary conditions are also sufficient except possibly for the 28 pairs of (h, n) shown in Table 1.*

h	n
2	14, 18, 22, 24, 28, 32
3	19, 23, 27
6	6, 7, 12, 18, 19, 22, 23, 24, 27
26, 30, 34, 38, 42, 46, 54, 58, 62, 66	22

Table 1

In Section 3, we shall remove 17 pairs of (h, n) from Table 1 and show the following.

Theorem 1.2 *An HSOLSSOM(h^n) exists only if $n \geq 5$, where $h \geq 2$ and n must be odd when h is odd. These necessary conditions are also sufficient except possibly for $h = 3$ and $n = 19, 23, 27$, and $h = 6$ and $n = 7, 12, 18, 19, 22, 23, 24, 27$.*

The main purpose of this paper is to start the investigation of the existence of HSOLSSOMs with nonuniform types. The simplest of such types is the type $1^n u^1$. An HSOLSSOM ($1^n u^1$) is equivalent to an ISOLSSOM($n + u, u$). The necessary condition for the existence of an HSOLSSOM($1^n u^1$) is $n \geq 3u + 1$, where u is odd and n is even. We shall show that such an HSOLSSOM exists whenever even $n \geq 5u + 9$ and odd $u \geq 9$. For small u, $3 \leq u \leq 15$, we shall show that the necessary condition is also sufficient with very few possible exceptions.

2 Preliminaries

Our direct construction is based on difference methods. The following is Lemma 2.1 in [3].

Lemma 2.1 *Let $G = Z_g$ with g even, let H be a subgroup of G, and let X be any set disjoint from G. Suppose there exists a set of 5-tuples $\mathcal{B} \subseteq (G \cup X)^5$ which satisfies the following properties:*

1. for each $i, 1 \leq i \leq 5$, and each $x \in X$, there is a unique $B \in \mathcal{B}$ with $b_i = x$ (b_i denotes the i-th co-ordinate of B);

2. no $B \in \mathcal{B}$ has two co-ordinates in X;

3. for each $i, j (1 \leq i < j \leq 5)$ and each $d \in G \backslash H$, there is a unique $B \in \mathcal{B}$ with $b_i, b_j \in G$ and $b_i - b_j = d$;

4. for $b_5 \in G$, $(b_1, b_2, b_3, b_4, b_5) \in \mathcal{B}$ if and only if $(b_2, b_1, b_4, b_3, b_5) \in \mathcal{B}$;

5. the differences $b_1 - b_2$ and $b_3 - b_4$ are both odd if (b_1, b_2, b_3, b_4, x) and (b_2, b_1, b_4, b_3, y) are both in $\mathcal{B}$ for any $x, y \in X, x \neq y$.

Then there exists an HSOLSSOM($h^{g/h}|X|^1$), where $h = |H|$.

We state some known recursive constructions. Denote by ILS(s, s_1) a holey Latin square of order s when it contains only one hole of size s_1. An element in the hole of an ILS is said to be *evenly distributed* if it does not appear on the main diagonal and if when it appears in one cell, then it must appear also in its symmetric cell. If each element in the hole is evenly distributed, then we say that the ILS is *balanced* . Given 3 IMOLS, if one of the three ILS is balanced and if also each element in the hole determines $s - s_1$ distinct entries above the main diagonal in the other two squares, then we say that the 3 IMOLS are *compatible.*

The following known constructions are Lemmas 2.2.1 and 2.2.3 in [5].

Lemma 2.2 *Suppose q is an odd prime power, $q \geq 7$. Suppose there exist 3 MOLS(m) and compatible 3 IMOLS($m + e_t, e_t$) where m is even, $t = 1, 2, ..., (q-5)/2, k = \sum_{1 \leq t \leq (q-5)/2}(2e_t)$. Then there exists an HSOLSSOM of type $m^{(q-1)}(m + k)^1$.*

Lemma 2.3 *Suppose $q \geq 5, q$ is an odd prime power or $q \equiv \pm 1 (mod\ 6)$. Suppose there exist compatible 3 IMOLS($m + e_t, e_t$) where m is even, $t = 1, 2, ..., (q-1)/2, k = \sum_{1 \leq t \leq (q-1)/2}(2e_t)$. Then there exists an HSOLSSOM of type $m^q k^1$.*

Note that from Lemma 3.2 in [5] compatible 3 IMOLS(v, n) exist for $(v, n) =$ (10, 2) and (k, 1), where k is any odd integer ≥ 5. The following are the variations of the above two Lemmas.

Lemma 2.4 *Suppose q is an odd prime power, $q \geq 7$. Suppose m is even and there exist 3 MOLS(m). Suppose for $t = 1, 2, ..., (q-5)/2, e_t = 0$ or e_t odd > 0, $k = \sum_{1 \leq t \leq (q-5)/2}(2e_t)$, and there is an ISOLSSOM($m + e_t, e_t$) if e_t odd > 0. Then there exists an ISOLSSOM($qm+k+1, u$) for $u = q, m+1$, or $m + k + 1$.*

Lemma 2.5 *Suppose $q \geq 5, q$ is an odd prime power or $q \equiv \pm 1 (mod\ 6)$. Suppose there exist compatible 3 IMOLS$(m + e_t, e_t)$ where m is even, $t = 1, 2, ..., (q-1)/2, k = \sum_{1 \leq t \leq (q-1)/2} (2e_t)$. Then there exists an ISOLSSOM $(qm + k + 1, u)$ for $u = m + 1$ or $k + 1$. If $e_t = 0$ or e_t is odd > 0 and the compatible 3 IMOLS are actually an ISOLSSOM, then there exists an ISOLSSOM$(qm + k + 1, q)$.*

In the proof of Lemma 2.2.1 of [3], the key point is that the initial SOLSSOM(q) contains some pairs of symmetric common transversals intersecting at the top left cell. If we have an HSOLSSOM$(1^n 1^1)$, which is cyclically generated over Z_n, we can utilize it in the place of the SOLSSOM(q) of Lemma 2.2.1 since similar pairs of transversals can also be found. Taking $m = 4$, we state the variation of Lemma 2.2 as follows.

Lemma 2.6 *Suppose there is an HSOLSSOM$(1^n 1^1)$ which is cyclically generated over Z_n. Then there exist an HSOLSSOM$(1^{4n}(4 + t)^1)$ and an HSOLSSOM$(1^{3n+3+t}(n + 1)^1)$ for odd $t \leq n - 3$.*

We also need several other recursive constructions. The first one is simple but useful.

Construction 2.7 *(Filling in Holes)*

(1) Suppose there exists an HSOLSSOM of type $\{s_i : 1 \leq i \leq n\}$. Let $a \geq 0$ be an integer. For each $i, 1 \leq i \leq n-1$, if there exists an HSOLSSOM of type $\{s_{ij} : 1 \leq j \leq k_i\} \cup \{a\}$, where $s_i = \sum_{1 \leq j \leq k_i} s_{ij}$, then there is an HSOLSSOM of type $\{s_{ij} : 1 \leq j \leq k_i, 1 \leq i \leq n-1\} \cup \{a + s_n\}$.

(2) Suppose there exists an HSOLSSOM of type $\{s_i : 1 \leq i \leq n\}$. Suppose there exists also an HSOLSSOM of type $\{t_j : 1 \leq j \leq k\}$, where $s_n = \sum_{1 \leq j \leq k} t_j$. Then there is an HSOLSSOM of type $\{s_i : 1 \leq i \leq n-1\} \cup \{t_j : 1 \leq j \leq k\}$.

The next recursive construction for HSOLSSOMs uses group divisible designs. A *group divisible design* (or GDD), is a triple $(X, \mathcal{G}, \mathcal{B})$ which satisfies the following properties:

(1) $\mathcal{G}$ is a partition of a set X (of *points*) into subsets called *groups*,

(2) $\mathcal{B}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point,

(3) every pair of points from distinct groups occurs in a unique block.

The *group type* of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. A GDD $(X, \mathcal{G}, \mathcal{B})$ will be referred to as a K-GDD if $|B| \in K$ for every block B in $\mathcal{B}$. A TD(k, n) is a GDD of group type n^k and block size k. An RTD(k, n) is a TD(k, n) where the blocks can be partitioned into parallel classes. It is well known that the existence of an RTD(k, n) is equivalent to the existence of a TD$(k+1,n)$, or equivalently, k - 1 MOLS(n). We wish to remark that a special GDD with all groups of size one is essentially a pairwise balanced design (PBD), denoted by $(X, \mathcal{B})$. Let $B(K)$ denote the PBD closure of K. We use [6] as our standard design theory reference.

We can apply Wilson's fundamental construction for GDDs [14] to obtain a similar construction for HSOLSSOM. The following is Construction 2.3.3 in [5].

Construction 2.8 *(Weighting) Suppose $(X, \mathcal{G}, \mathcal{B})$ is a GDD and let $w : X \to Z^+ \cup \{0\}$. Suppose there exists an HSOLSSOM of type $\{w(x) : x \in B\}$ for every $B \in \mathcal{B}$. Then there exists an HSOLSSOM of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.*

Lemma 2.9 *Suppose $v \in B(K)$ and K is a set of some odd integers each ≥ 5. Then there exists an HSOLSSOM of type $(1^{v-k}k^1)$ for every $k \in K$.*

Proof: Consider the PBD as a GDD of type $(1^{v-k}k^1)$ and give each point weight one. Apply the Weighting construction. □

We need a variation of the above weighting construction. First, we introduce the concept of holey GDD or GDD with holes. A *holey group divisible design* (or HGDD), is a quadruple $(X, \mathcal{G}, \mathcal{H}, \mathcal{B})$ which satisfies the following properties:

(1) $\mathcal{G}$ is a partition of a set X (of *points*) into subsets called *groups*,

(2) $\mathcal{H} = \{H_1, ..., H_n\}, H_1, ..., H_n$ are disjoint subsets of X called *holes*,

(3) $\mathcal{B}$ is a set of subsets of X (called *blocks*) such that a block intersects a group or a hole in at most one point,

(4) every pair of points from distinct groups occurs in a unique block if they are not in the same hole.

The *type of the hole $H \in \mathcal{H}$* is the multiset $\{|G \cap H| : G \in \mathcal{G}\}$. If we apply the Weighting Construction to an HGDD, we get an HSOLSSOM with some additional holes which come from the holes in the HGDD. If the holes can be filled in, we may get an HSOLSSOM. We state the variation of Construction 2.8 as follows.

Construction 2.10 *Suppose $(X, \mathcal{G}, \mathcal{H}, \mathcal{B})$ is an HGDD and let $w : X \to Z^+ \cup \{0\}$. Suppose there exists an HSOLSSOM of type $\{w(x) : x \in B\}$ for every $B \in \mathcal{B}$. Suppose there is an HSOLSSOM of type $\{\sum_{x \in G \cap H} w(x) : G \in \mathcal{G}\}$ for every $H \in \mathcal{H}$. Then there exists an HSOLSSOM of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.*

The following product construction is essentially Lemma 3.4 in [11].

Construction 2.11 *Suppose there exists an HSOLSSOM of type h^n. Let $m \geq 4$ and $m \neq 6, 10$. Then there exists an HSOLSSOM of type $(mh)^n$.*

To apply the above constructions the following known results are useful, which are taken from [7].

Theorem 2.12 *For any prime power p, there exists a TD(k,p), where $3 \leq k \leq p+1$.*

Theorem 2.13 *(1) There is a TD(5, m) if $m \geq 4$ and $m \notin \{6, 10\}$. (2) There is a TD(6, m) if $m \geq 5$ and $m \notin \{6, 10, 14, 18, 22\}$. (3) There is a TD(7, m) if $m \geq 7$ and $m \notin$ {10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 39, 46, 54, 60, 62}.*

3 HSOLSSOMs with type (h^n)

We shall prove Theorem 1.2 by constructing 17 pairs of (h, n) in Theorem 1.1(2).

We mainly use Lemma 2.1 and some computer search to find the constructions. To ease the notation we shall always take $G = Z_{hn}$ having a subgroup H of order h. Let $u = 2t$ and $X = \{x_1, ..., x_t, y_1, ..., y_t\}$ such that x_i and y_i form a pair satisfying the condition 5 in Lemma 2.1 for $1 \leq i \leq t$. According to the conditions 4 and 5 in Lemma 2.1, to construct an HSOLSSOM($h^n u^1$) we may record half the 5-tuples instead of listing all of them. In order to save space we write the half of 5-tuples of $\mathcal{B}$ vertically, denoted by $\mathcal{A}$.

Lemma 3.1 *There exists an HSOLSSOM(2^{14}).*

Proof: As mentioned above we take $G = Z_{26}, H = \{0, 13\}, X = \{x, y\}$, and let $\mathcal{A}$ be the following set of fourteen 5-tuples:

1	2	3	5	8	9	11	12	14	17	18	19	25	0
10	20	4	7	24	21	16	15	y	23	22	x	6	11
19	10	2	24	18	5	9	23	17	22	8	20	11	6
7	x	6	1	12	16	25	4	15	14	3	21	y	15
0	0	0	0	0	0	0	0	0	0	0	0	0	x

□

Lemma 3.2 *There exists an HSOLSSOM(2^{18}).*

Proof: Let $G = Z_{34}, H = \{0, 17\}, X = \{x, y\}$, and let $\mathcal{A}$ be the following set of eighteen 5-tuples:

0	x	y	13	20	25	32	8	21	12	5	2	22	33	1	3	30	26
1	28	11	15	23	29	4	16	31	24	19	18	27	6	10	14	9	7
29	1	13	14	11	4	24	26	20	9	10	5	32	19	28	12	15	33
16	16	27	x	y	3	6	22	25	2	8	29	23	30	31	18	7	21
x	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

□

Lemma 3.3 *There exists an HSOLSSOM*(2^{22}).

Proof: Let $G = Z_{34}, H = \{0, 14\}$ and $X = \{x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5\}$ We first construct an HSOLSSOM$(2^{17}10^1)$ with $\mathcal{A}$ as follows.

0	0	0	0	0	x1	x2	x3	x4	x5	y1	y2	y3	y4	y5
3	5	7	9	11	33	32	2	20	18	11	25	4	16	28
25	7	19	18	16	6	4	28	16	33	21	15	25	27	10
18	28	20	33	19	11	29	30	12	5	31	23	13	9	24
x1	x2	x3	x4	x5	0	0	0	0	0	0	0	0	0	0

31	27	19	3	9	14	29	1	26	30	23
10	8	21	7	15	22	5	13	6	12	24
1	22	18	8	19	20	32	2	14	7	3
x1	x2	x3	x4	x5	y1	y2	y3	y4	y5	26
0	0	0	0	0	0	0	0	0	0	0

By filling in the size ten hole with an HSOLSSOM(2^5), we get the desired HSOLSSOM(2^{22}). □

Lemma 3.4 *There exists an HSOLSSOM*(h^{22}) *for* $h \in \{26, 30, 34, 38, 42, 46, 54, 58, 62, 66\}$.

Proof: Start with an HSOLSSOM(2^{22}) and apply Construction 2.11 with $m = h/2$. We get the desired HSOLSSOM(h^{22}). □

Lemma 3.5 *There exists an HSOLSSOM*(2^{24}).

Proof: Let $G = Z_{38}$. We first construct an HSOLSSOM$(2^{19}10^1)$ with $\mathcal{A}$ as follows.

0	0	0	0	0	x1	x2	x3	x4	x5	y1	y2	y3	y4	y5
3	5	7	9	11	33	37	18	23	14	35	22	20	24	28
17	25	12	6	33	20	1	10	11	7	5	16	29	25	32
6	10	15	11	24	37	26	12	15	13	33	8	17	9	18
x1	x2	x3	x4	x5	0	0	0	0	0	0	0	0	0	0

30	15	21	34	8	26	29	17	27	9	10	4	12
7	32	1	2	16	36	3	31	5	13	11	6	25
14	6	4	30	34	35	36	27	28	2	24	22	23
x1	x2	x3	x4	x5	y1	y2	y3	y4	y5	31	21	3
0	0	0	0	0	0	0	0	0	0	0	0	0

By filling an HSOLSSOM(2^5) in the size ten hole, we get the desired HSOLSSOM(2^{24}). □

Lemma 3.6 *There exists an HSOLSSOM*(2^{28}).

Proof: Let $G = Z_{46}$. We first construct an HSOLSSOM($2^{23}10^1$) with $\mathcal{A}$ as follows.

0	0	0	0	0	x1	x2	x3	x4	x5	y1	y2	y3	y4	y5	15
1	3	39	15	9	18	29	36	21	5	2	28	16	30	27	26
5	33	19	44	8	25	31	24	37	42	16	4	41	43	33	19
12	28	40	31	19	40	12	26	13	2	8	22	11	29	45	39
x1	x2	x3	x4	x5	0	0	0	0	0	0	0	0	0	0	0

6	40	7	1	38	33	8	10	32	45	12	35	17	3	34	39
19	11	9	37	44	41	20	42	14	25	31	13	22	24	4	43
30	21	44	35	18	36	17	38	1	34	32	28	3	10	9	5
x1	x2	x3	x4	x5	y1	y2	y3	y4	y5	15	27	7	20	6	14
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

By filling an HSOLSSOM(2^5) in the size ten hole, we get the desired HSOLSSOM(2^{28}). □

Lemma 3.7 *There exists an HSOLSSOM*(2^{32}).

Proof: Let $G = Z_{50}$. We first construct an HSOLSSOM($2^{25}14^1$) with $\mathcal{A}$ as follows.

0	0	0	0	0	0	0	35	28	43	x1	x2	x3	x4	x5
1	7	9	11	13	15	23	6	36	13	3	17	23	42	20
3	8	20	5	19	43	12	3	12	22	1	32	6	28	49
14	49	13	20	40	46	45	2	16	20	31	8	11	41	18
x1	x2	x3	x4	x5	x6	x7	0	0	0	0	0	0	0	0

x6	x7	y1	y2	y3	y4	y5	y6	y7	21	41	46	37	12	15
49	33	45	1	39	44	30	2	9	24	8	48	27	18	11
7	43	37	44	21	24	35	9	33	27	42	17	39	36	48
34	15	5	38	29	10	19	47	23	x1	x2	x3	x4	x5	x6
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

34	40	47	16	10	31	14	7
22	4	29	32	38	5	19	26
45	26	14	13	4	46	40	30
x7	y1	y2	y3	y4	y5	y6	y7
0	0	0	0	0	0	0	0

By filling an HSOLSSOM(2^7) in the size 14 hole, we get the desired result. □

To complete the proof of Theorem 1.2 we need only to show the existence of an HSOLSSOM (6^6), which will be done in the next lemma.

Lemma 3.8 *There exists an HSOLSSOM*(6^6).

Proof: Let $G = Z_{30}$ and $H = \{0, 5, 10, 15, 20, 25\}$. We construct an HSOLSSOM($6^5 6^1$) with $\mathcal{A}$ as follows.

0	0	0	x1	x2	x3	y1	y2	y3	8	22	19	21	1	6	17	18	2
1	13	9	29	16	7	23	9	11	12	24	3	27	4	13	28	26	14
23	14	21	12	18	13	9	2	7	6	11	21	4	23	24	26	29	28
4	21	8	3	22	19	17	16	8	x1	x2	x3	y1	y2	y3	14	27	1
x1	x2	x3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

□

4 HSOLSSOMs with type $(1^n u^1)$

We shall first discuss the cases $u = 3, 5$. With the known results we then prove the existence for $n \geq 5u + 9$ and $u \geq 9$. We shall also give the almost complete result for the cases $u = 7, 9, 11, 13$ and 15.

Lemma 4.1 *Suppose there exists a TD$(k, m), k \geq 7$. Then an HSOLSSOM $(1^n u^1)$ exists for odd $u \leq 2m + 1$ and even $n \in [10m + 4, 2(k-1)m]$.*

Proof: Start with a TD(k, m). Keep the first five groups unchanged and delete some points from each of the remaining groups such that the sixth group has exactly $(u-1)/2$ points, the seventh group has at least two points and for other groups each contains zero or at least two points. Give weight two to each point of the GDD and apply the weighting construction. We get an HSOLSSOM since, by Theorem 1.2, all the required HSOLSSOM(2^t) exist for $t \geq 5$. Add one new point and fill in the holes with HSOLSSOM(1^e) for odd $e \geq 5$ leaving the sixth hole empty. We get the desired HSOLSSOM. □

Lemma 4.2 *For odd $u, 3 \leq u \leq 15$, there exists an HSOLSSOM$(1^n u^1)$ for all even $n \geq 74$.*

Proof: From Theorem 2.13, a TD(7, m) exists for all $m \geq 63$. Applying Lemma 4.1 with $k = 7$ and $m \geq 63$ gives an interval $[10m + 4, 12m]$. This solves the cases $n \geq 634$. For small n we take prime powers $m = 7, 8, 11, 19$ and apply Lemma 4.1 to get the intervals $[10m + 4, 2m^2]$, which are overlapped to cover the interval [74, 634]. □

Lemma 4.3 *Suppose odd $q \geq 5$ is a prime power and $m \geq 4$ is even, $m \neq 6, 10, 14, 66, 70$. Then there exists an ISOLSSOM$(qm+u, u)$, or equivalently an HSOLSSOM$(1^{qm} u^1)$, for all odd $u \leq q$.*

Proof: Apply Lemma 2.5 with $e_t = 0$ or 1, and Theorem 1.1(1). □

Lemma 4.4 *If there is a TD$(6, m)$, then there exists an ISOLSSOM$(10m + u, u)$, or equivalently, an HSOLSSOM$(1^{10m} u^1)$, for all odd $3 \leq u \leq 2m + 1$.*

Proof: Delete some points from the last group of the TD(6, m). Give weight two to each point and apply Weighting construction. Add one new point and fill in the first five holes. □

To construct an HSOLSSOM($1^n u^1$) by the direct construction, we shall take $G = Z_n$ having a subgroup $H = \{0\}$ of order 1. Let $u = 2t+1$ and $X = \{x_1, .., x_t, y_1, ..., y_t, z\}$ such that x_i and y_i form a pair satisfying the condition 5 in Lemma 2.1 for $1 \le i \le t$. We will record about half the 5-tuples, denoted by $\mathcal{A}$, as we did in the previous sections.

Lemma 4.5 *There exists an HSOLSSOM*($1^{12}3^1$).

Proof: Let $G = Z_{12}$. We list $\mathcal{A}$ as follows.

0	0	1	11	4	2	8	10	9
6	11	x	y	z	6	5	3	7
3	10	5	1	9	10	3	11	8
9	5	7	2	6	x	y	z	4
z	x	0	0	0	0	0	0	0

□

We are now ready to handle the cases $u = 3$ and 5.

Lemma 4.6 *There exists an HSOLSSOM*(1^n3^1) *for all even* $n \ge 10$ *except possibly for* $n \in \{10, 48, 54\}$.

Proof: From Lemma 4.2 we need only to consider even $n \le 72$. For $n \equiv 0$ (mod 4), we first deal with $n = 20$ by taking $q = 5$ and $m = 4$ in Lemma 4.3. By taking $m = 4$, or 8, or 12, the other n are similar except $n = 12, 16, 24, 32, 64$, where the case $n = 12$ has been solved in Lemma 4.5. From p.199 of [7] we have a {7, 9}-GDD of type (8^82^1). Give weight one to each point and apply Weighting construction. We get an HSOLSSOM of the same type. Add one new point and fill in size eight holes, this solves the case $n = 64$.

For the other four cases we give the direct constructions as follows using $G = Z_n$ and $X = \{x, y, z\}$.

$n = 16$:

0	0	1	4	12	15	2	8	10	13	14
8	15	x	y	z	9	6	3	7	11	5
9	7	5	7	10	14	12	13	2	15	4
1	12	3	11	9	x	y	z	8	6	1
z	x	0	0	0	0	0	0	0	0	0

$n = 24$:

0	0	19	14	16	12	20	4	6	18	3	21	8	1	15
12	7	x	y	z	13	22	7	10	23	9	5	17	11	2
3	16	17	10	18	21	2	14	11	1	20	15	7	16	19

15	21	8	9	22	x	y	z	5	12	6	13	4	23	3
z	x	0	0	0	0	0	0	0	0	0	0	0	0	0

$n = 32$:

0	0	24	4	23	12	7	9	22	29	13	19	8	18	14
16	5	x	y	z	11	5	6	26	3	20	27	17	28	25
24	25	12	25	13	22	10	15	21	16	7	23	6	26	27
8	10	17	19	24	x	y	z	28	4	31	9	2	3	5
z	x	0	0	0	0	0	0	0	0	0	0	0	0	0

30	21	1	16
10	2	15	31
14	18	29	11
1	20	30	8
0	0	0	0

We next discuss the case $n \equiv 2\ (mod\ 4)$. Taking $m = 5, 7$ in Lemma 4.4 solves the cases $n = 50, 70$. For $n = 58$, 62 and 66, we apply Lemma 2.6 with an initial HSOLSSOM of type 1^{15} over Z_{14} from [13]. Taking $t = 1, 5, 9$, we get an HSOLSSOM of type $1^n 15^1$ for $n = 46, 50, 54$. Filling in the size 15 hole with an HSOLSSOM$(1^{12}3^1)$, we get the desired HSOLSSOM of type $1^n 3^1$ for $n = 58, 62, 66$. From p.196 of [7] we have a {5, 7}-GDD of type $(6^7 2^1)$. Give weight one to each point and apply Weighting construction, we get an HSOLSSOM of the same type. Add one new point and fill in six eight holes, this solves the case $n = 42$. The remaining $n = 14, 18, 22, 26, 30, 34, 38$ are directly constructed as follows.

$n = 14$:

0	0	2	10	1	8	12	9	13	3
7	5	x	y	z	4	11	7	5	6
6	3	12	1	8	9	6	4	11	2
13	2	3	13	5	x	y	z	7	10
z	x	0	0	0	0	0	0	0	0

$n = 18$:

0	0	16	9	3	17	2	8	4	5	6	7
9	17	x	y	z	1	14	12	15	13	11	10
10	14	4	12	11	3	9	10	5	2	17	6
1	11	15	14	7	x	y	z	13	8	16	1
z	x	0	0	0	0	0	0	0	0	0	0

$n = 22$:

0	0	6	16	1	2	18	11	3	10	14	7	21	9
11	7	x	y	z	20	13	5	4	12	17	15	8	19
5	6	3	19	16	13	8	2	10	4	15	11	20	1
16	3	7	18	9	x	y	z	12	21	5	17	6	14
z	x	0	0	0	0	0	0	0	0	0	0	0	0

$n = 26$:

0	0	15	20	7	24	19	8	2	21	9	10	3	4	11	13
13	5	x	y	z	23	17	5	6	1	16	18	12	14	22	25
1	14	4	5	11	7	22	13	25	17	2	20	16	6	23	12
14	11	9	15	3	x	y	z	1	18	24	8	10	21	14	19
z	x	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$n = 30$:

0	0	2	10	17	20	3	21	4	22	5	6	15	13	26	16	29	25
15	7	x	y	z	19	1	18	8	27	11	14	24	23	7	28	12	9
23	19	20	2	7	15	5	19	3	16	18	27	26	10	4	22	11	12
8	10	24	21	23	x	y	z	9	6	25	29	1	28	17	14	8	13
z	x	0	0	8	0	0	0	0	0	0	0	0	0	0	0	0	0

$n = 34$:

0	0	6	15	30	18	29	27	22	5	32	14	16	3	8
17	9	7	17	33	x	y	z	26	10	4	21	24	13	19
2	32	17	25	10	12	24	23	21	11	29	3	8	22	13
19	17	x	y	z	32	2	14	19	27	28	7	31	16	20
z	x	0	0	0	0	0	0	0	0	0	0	0	0	0

23	12	31	28	20
1	25	11	9	2
5	30	9	15	1
26	4	33	18	6
0	0	0	0	0

$n = 38$:

0	0	34	8	11	22	26	1	13	31	29	12	9	30	21	15
19	5	35	10	14	x	y	z	17	37	36	20	18	2	32	27
13	23	3	13	19	16	21	35	12	28	18	4	7	17	5	34
32	26	x	y	z	9	1	11	10	27	22	31	30	8	33	29
z	x	0	0	0	0	0	0	0	0	0	0	0	0	0	0

3	19	28	7	25	6
16	33	5	23	4	24
20	23	6	25	2	32
26	36	14	37	24	15
0	0	0	0	0	0

$n = 46$:

0	0	45	10	20	2	15	26	37	41	23	16	43	28	24	38
23	17	27	40	12	x	y	z	1	6	30	25	29	9	36	42
24	20	12	22	25	36	30	13	8	20	4	34	18	24	35	28
1	23	x	y	z	45	42	19	9	3	6	14	39	38	16	43
z	x	0	0	0	0	0	0	0	0	0	0	0	0	0	0

3	22	17	8	39	21	7	18	11	35
5	44	32	13	33	34	4	19	31	14

31	17	2	7	23	40	21	10	15	26
44	1	41	29	27	32	11	5	33	37
0	0	0	0	0	0	0	0	0	0

□

Lemma 4.7 *There exists an HSOLSSOM*(1^n5^1) *for all even* $n \geq 16$.

Proof: From Lemma 4.2 we need only to consider even $n \leq 72$. For $n \equiv 0$ (*mod* 4) and $n \geq 16$, we write $n = 4k - 4$ with $k \geq 5$ and fill in the holes of HSOLSSOM(4^k) by adding an infinite point.

We next discuss the case $n \equiv 2$ (*mod* 4). The direct construction gives the first two.

$n = 18$:

0	0	0	4	7	13	14	17	2	3	6	8	16	15
9	11	17	10	12	9	11	1	x1	x2	y1	y2	z	5
14	9	16	1	14	5	17	16	15	11	12	13	2	8
5	12	11	x1	x2	y1	y2	z	9	4	10	3	6	7
z	x1	x2	0	0	0	0	0	0	0	0	0	0	0

$n = 22$:

0	0	0	10	13	7	18	17	16	19	1	8	2	21	20	15
11	7	9	9	11	4	14	12	x1	x2	y1	y2	z	5	6	3
7	5	14	12	21	11	9	13	19	8	7	17	3	16	14	5
18	4	19	x1	x1	y1	y2	z	6	10	15	1	18	20	4	2
z	x1	x2	0	0	0	0	0	0	0	0	0	0	0	0	0

In Lemma 2.4 let $m = 4$ and $q = 7, 9, 11$. This solves the cases $n = 26, 34, 42$. In Lemma 2.5 let $m = 4$ and $q = 7, 9, 11, 13, 17$. This solves the cases $n = 30, 38, 46, 50, 54, 58, 70$. For the remaining two cases $n = 62$ and 66, add one new point to a TD(5, 14), we see that $71 \in B(5, 15)$. We then apply Lemma 2.9 to solve the case $n = 66$. From [15] there is a 5-GDD of type 6^{11}. By adding one new point we have $67 \in B(5, 7)$. This solves the last case $n = 62$. □

We are now in a position to show the existence of an HSOLSSOM(1^nu^1) for any even $n \geq 5u + 9$ and odd $u \geq 9$.

Lemma 4.8 *For odd* $u \geq 13$ *there exists an HSOLSSOM*(1^nu^1) *whenever even* $n \geq 5u + 9$.

Proof: Let m and m_0 be both odd and $m \geq m_0$, $m_0 = (u - 1)/2 + \delta$, where $\delta = 0$ or 1. Since $u \geq 13$, we have $m \geq 7$. By Theorem 2.13 there is a TD($7, m$) except $m = 15, 39$. Apply Lemma 4.1, we get an HSOLSSOM(1^nu^1) for $n \in [10m + 4, 12m]$. These intervals are overlapped since m is odd and $m \geq 7$ except for $m = 15, 39$. That is, the intervals [154, 180] and [394, 468] are missing for $u \in \{29, 31\}$ and $u \in \{77, 79\}$ respectively.

For the first interval we start with a TD(8, 13) and delete at most five points from each of the last three groups. Give weight two to each point of the GDD and apply Weighting construction, then add three or five new points and fill in the holes. By Lemmas 4.5 and 4.6, we get an HSOLSSOM($1^n u^1$) for $u = 29, 31$ and $n \in [152, 182]$.

Similarly, we start with a TD(9, 37) for the second interval. Delete at least 15 and at most 29 points from each of the last four groups. The Weighting construction and the filling in holes construction give the interval [360, 472] for $u = 77, 79$.

The overlapping of the intervals show that an HSOLSSOM($1^n u^1$) exists whenever even $n \geq 10m_0 + 4 = 5u - 1 + 10\delta$. This completes the proof. □

Lemma 4.9 *For $u = 9, 11$ there exists an HSOLSSOM($1^n u^1$) whenever even $n \geq 5u + 9$.*

Proof: From Lemma 4.2 we need only to consider even $n \in [5u + 9, 72]$. For $u = 9$ and $64 \leq n \leq 72$, write $n + u = 9 \times 8 + k$, $1 \leq k \leq 9$, and apply Lemma 2.4 or Lemma 2.5. For $54 \leq n \leq 60$ write $n + u = 7 \times 8 + k$, $7 \leq k \leq 13$, and apply Lemma 2.5 using compatible 3 IMOLS(10, 2). For the case $n = 62$ adjoin one infinite point x to the groups of a TD(5, 7) and delete a point different from x so as to form a $\{5, 8\}$-GDD of type $4^7 7^1$. Give each point weight two and then adjoin one infinite point to the resulting HSOLSSOM.

For $u = 11$ and $64 \leq n \leq 72$, Lemma 2.4 takes care of $n = 64$ if we write $75 = 9 \times 8 + 3$. The case $n = 66$ can be done by Construction 2.11 since $77 = 7 \times 11$. Take an HSOLSSOM(10^8) from Theorem 1.2 and adjoin one infinite point. The filling in holes construction solves the case $n = 70$. Finally, Lemma 2.5 takes care of $n = 68, 70$ since $79 = 17 \times 4 + 11$ and $83 = 9 \times 8 + 11$, where compatible 3 IMOLS(10, 2) are again needed. □

Now we are to show the existence of an HSOLSSOM($1^n 7^1$) for almost all even $n \geq 22$.

Lemma 4.10 *There exists an HSOLSSOM($1^n 7^1$) for all even $n \geq 22$ except possibly for $n = 58$.*

Proof: By Lemma 4.2, we consider only $n \leq 72$. Take an HSOLSSOM(6^m) from Theorem 1.2 and add a new point. The filling in holes construction solves the cases when $n = 6(m - 1)$ for $5 \leq m \leq 13$ and $m \neq 7, 12$.

For $n = 28, 36, 40, 44, 52, 56, 68$, apply Lemma 2.5 with $n = qm$ and $u = k + 1 = 7$. For $n = 22, 26, 50$, apply Lemma 2.5 with $q = 7$, and m and k as follows: $(m, k) =$ (4, 0) for $n = 22$, (4, 4) for $n = 26$ and (8, 0) for $n = 50$.

Apply Lemma 2.2 with $q = 9, 17, m = 4$ and $k = 2$, we get an HSOLSSOM of type $4^8 6^1$ and $4^{16} 6^1$. The filling in holes construction solves the cases $n = 32$ and $n = 64$. Since $77 = 7 \times 11$, the product construction takes care of the case $n = 70$.

From p.199 of [7] we have a $\{5,7\}$-GDD of type $8^6 4^1$. So, $53 \in B(5,7,9)$. Applying Lemma 2.9 solves $n = 46$. Take a 7-GDD of type 3^{15} from p.191 of [7]. Delete one block and one group which intersect. We get a $\{5,6,7\}$-GDD of type $3^8 2^6$. Give each point weight two and then fill in the holes by adjoining an infinite point. This solves the case $n = 66$. The above 7-GDD contains two intersecting blocks which intersect exactly 11 groups. Delete one point from each of the 11 groups to get a $\{5,6,7\}$-GDD of type $2^{11} 3^4$. The similar constructions solve the case $n = 62$. The remaining two cases $n = 34, 38$ are done by direct constructions as follows:

$n = 34$:

0	0	0	0	5	28	13	15	3	21	17	29	20	9	32	2
17	11	13	15	6	30	16	19	8	27	24	x1	x2	x3	y1	y2
5	30	14	17	18	29	23	24	11	25	20	15	7	31	21	26
22	23	15	8	x1	x2	x3	y1	y2	y3	z	19	33	13	8	16
z	x1	x2	x3	0	0	0	0	0	0	0	0	0	0	0	0

23	1	10	22	4	33	12	25
y3	z	18	31	14	11	26	7
14	27	9	6	1	28	10	2
17	12	3	4	30	5	32	22
0	0	0	0	0	0	0	0

$n = 38$:

0	0	0	0	29	24	23	34	12	33	14	10	37	35	36	17
19	7	15	17	28	22	20	30	7	27	6	x1	x2	x3	y1	y2
23	36	13	22	9	32	35	37	1	2	5	6	16	18	28	12
4	9	30	31	x1	x2	x3	y1	y2	y3	z	7	11	31	22	19
z	x1	x2	x3	0	0	0	0	0	0	0	0	0	0	0	0

5	1	16	9	2	3	8	18	26	31
y3	z	25	19	13	15	21	32	4	11
29	33	4	8	27	23	24	15	30	36
25	17	34	20	3	26	14	13	10	21
0	0	0	0	0	0	0	0	0	0

□

Lemma 4.11 *For $u = 9, 11, 13, 15$, there exists an HSOLSSOM$(1^n u^1)$ whenever even $n \geq 3u+1$ except possibly for the pairs $(n, u) \doteq (54, 13), (58, 13), (58, 15)$.*

Proof: Most of the cases can be done routinely by Lemmas 2.4, 2.5 and others. We then omit the details and only mention some specially treated cases.

For $(n, u) = (42, 9)$ and (46, 9), delete 5 points from a block in a TD(6, 5) to form a $\{5, 6\}$-GDD of type $4^5 5^1$. Weighting and filling in holes constructions solve the first case. Since $55 \in B(5, 7, 9)$ from Lemma 9.4 of [2], applying Lemma 2.9 gives the second case.

For $(n, u) = (62, 11)$, consider an RTD(6, 7) as a $\{6, 7\}$-GDD of type 6^7 and delete one block of size 6 to form a $\{5, 6\}$-GDD of type $5^6 6^1$; then use a weighting and filling in holes construction. The case $(n, u) = (58, 11)$ can be done directly:

$n = 58$:

0	0	0	0	0	0	17	3	47	35	31	7	9	12	6	41
29	13	15	17	19	21	18	5	50	39	36	13	16	20	15	51
13	53	54	34	50	27	34	12	3	36	46	32	11	16	27	53
42	36	43	37	27	26	x1	x2	x3	x4	x5	y1	y2	y3	y4	y5
z	x1	x2	x3	x4	x5	0	0	0	0	0	0	0	0	0	0

48	57	40	34	4	27	38	33	28	52	2	37	56	11	29	8
1	x1	x2	x3	x4	x5	y1	y2	y3	y4	y5	z	10	25	45	26
6	35	19	14	43	9	21	17	13	38	47	25	55	8	20	30
z	23	39	1	48	40	29	51	50	42	22	31	45	57	5	37
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

24	32	23	19	30	53	22	14
44	54	46	43	55	21	49	42
18	4	26	52	56	24	41	7
54	2	44	33	28	10	15	49
0	0	0	0	0	0	0	0

For $(n, u) = (62, 13)$ and (66, 13), delete 3 or 5 points from a block of an RTD(6, 7) to form a $\{5, 6, 7\}$-GDD of type $6^6 3^1$ or a $\{5, 6\}$-GDD of type $6^5 7^1$. Give each point weight two and then fill in the holes by adjoining an infinite point. These solve the two cases.

For $n = 15$ and $u = 50, 52, 54, 62, 66$, applying Lemma 2.6 with an initial HSOLSSOM(1^{15}) gives the first three as we did in the proof of Lemma 4.5. Truncate a TD(6, 7) to form a $\{5, 6\}$-GDD of type $7^5 m^1$ where $m = 3$ or 5. Weighting and filling in holes constructions solve the last two cases. □

We now summarize the results of this section in the following theorem.

Theorem 4.12 *For any odd $u \geq 9$ there exists an HSOLSSOM($1^n u^1$) whenever even $n \geq 5u + 9$. For any odd $u, 3 \leq u \leq 15$, such an HSOLSSOM exists if and only if $n \geq 3u + 1$ with the possible exceptions shown in Table 2.*

n	u
10, 48, 54	3
58	7
54	11
58	13
58	15

Table 2

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