Lower bounds on dominating functions in graphs *

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ABSTRACT. We study the signed domination number γ_s, the minus domination number γ^- and the majority domination number γ_{maj}. In this paper, we establish good lower bounds for γ_s, γ^- and γ_{maj}, and give sharp lower bounds for γ_s, γ^- for trees.

1 Introduction

Let $G = (V, E)$ be a simple graph. The *order* of G is the number of vertices. The *size* of G is the number of edges; it is denoted by $\epsilon(G)$. For a vertex $v \in V$, the *degree* of v is $d(v) = |N(v)|$. A vertex v is called odd vertex if $d(v)$ is odd. The *minimum degree* and *maximum degree* of the vertices of G are repectively denoted by $\delta(G)$ and $\Delta(G)$, when no ambiguity can occur, we often simple write ϵ, δ and Δ instead of $\epsilon(G)$, $\delta(G)$ and $\Delta(G)$. The *open neighborhood* of v, denoted by $N(v)$, is defined as the set of vertices adjacent to v, i.e., $N(v) = \{u \in V \mid uv \in E\}$. The *closed neighhorhood* of v is $N[v] = N(v) \cup \{v\}$. If g is a real function defined on V and $S \subseteq V$, we write $g(S) = \sum_{v \in S} g(v)$.

A *signed dominating function* of G is defined in [3] as $g\colon V \to \{\pm 1\}$ satisfying $g(N[v]) \geq 1$ for all $v \in V$. A signed dominating function g

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is *minimal* if there does not exist a signed dominating function $h \neq g$ satisfying $h(v) \leq g(v)$ for every $v \in V$. The *signed domination number* of a graph G is defined as $\gamma_s(G) = \min\{g(V) \mid g$ is a minimal signed dominating function of $G\}$.

A *minus dominating function* of a graph G is defined in [5] as a function $g\colon V \to \{0, \pm 1\}$ such that $g(N[v]) \geq 1$ for all $v \in V$. Similarly, we can define a minimal minus dominating function, the minus domination number $\gamma^-(G)$ of G.

A *majority dominating function* of a graph G is defined in [6] as a function $g\colon V \to \{\pm 1\}$ such that for at least half the vertices $v \in V$, $g(N[v]) \geq 1$. Similarly, a minimal majority dominating function, the majority domination number $\gamma_{\mathrm{maj}}(G)$ of G are defined.

2 Signed domination in graphs

Theorem 1. *For any graph G of order n,*

$$\gamma_s(G) \geq \max(n - \frac{2\epsilon - l}{\delta + 1}, n - \frac{2n\Delta - 2\epsilon - l}{\Delta + 1}),$$

where l is the number of odd vertices.

Proof: Let g be a signed dominating function on G satisfying $g(V) = \gamma_s(G)$, and

$$M = \{x_1, x_2, \ldots, x_m\}$$
$$P = \{x_{m+1}, x_{m+2}, \ldots, x_n\}$$

be the sets of vertices that are assigned the value -1 and 1, respectively. For $x_i \in M$ we have

$$|N(x_i) \cap M| \leq \left\lfloor \frac{d(x_i)}{2} \right\rfloor - 1$$

For $x_i \in P$ we have

$$|N(x_i) \cap M| \leq \left\lfloor \frac{d(x_i)}{2} \right\rfloor$$

So

$$\begin{aligned}\sum_{i=1}^{m} d(x_i) &= \sum_{x_i \in V} |N(x_i) \cap M| \\ &\leq \sum_{i=1}^{m} (\frac{d(x_i)}{2} - 1) + \sum_{i=m+1}^{n} \frac{d(x_i)}{2} - \frac{l}{2} \\ &= \epsilon - m - \frac{l}{2} \qquad (1)\end{aligned}$$

Obviously

$$m\delta \le \sum_{i=1}^{m} d(x_i) \tag{2}$$

$$2\epsilon - (n-m)\Delta \le \sum_{i=1}^{m} d(x_i) \tag{3}$$

Combining (1), (2) and (3) we have

$$m \le \min(\frac{2\epsilon - l}{2(\delta+1)}, \frac{2n\Delta - 2\epsilon - l}{2(\Delta+1)})$$

So

$$\gamma_s(G) = n - 2m \ge \max(n - \frac{2\epsilon - l}{\delta+1}, n - \frac{2n\Delta - 2\epsilon - l}{\Delta+1}).$$

□

From this theorem, we get the following corollary:

Corollary 1. [2] *For every k-regular graph G of order n, then $\gamma_s(G) \ge \frac{n}{(k+1)}$ if k is even and $\gamma_s(G) \ge \frac{2n}{(k+1)}$ if k is odd.*

For tree, there is a sharp lower for γ_s.

Theorem 2. *For any tree T or order n,*

$$\gamma_s(T) \ge \frac{n+2+l}{3},$$

where l is the number of odd vertices, and this bound is sharp.

Proof: Let g be a signed dominating function on T with $g(V) = \gamma_s(T)$, M and P be defined as in the proof of Theorem 1. Let s be the number of vertices of degree 1. For $x \in V(T)$, if $d(x) = 1$, then $N[x] \subseteq P$. So

$$\begin{aligned} 2m &\le \sum_{i=1}^{m} d(x_i) \\ &= \sum_{x_i \in V} |N(x_i) \cap M| \\ &\le \sum_{i=1}^{m} (\frac{d(x_i)}{2} - 1) + \sum_{m+1, d(x_i) \ne 1}^{n} \frac{d(x_i)}{2} - \frac{l-s}{2} \\ &= \frac{2\epsilon - s}{2} - m - \frac{l-s}{2} \\ &= \epsilon - m - \frac{l}{2} \end{aligned}$$

yielding

$$m \le \frac{\epsilon - \frac{l}{2}}{3}$$

thus

$$\begin{aligned}\gamma_s(T) &= n - 2m \\ &\ge \frac{3n - 2\epsilon + l}{3} \\ &= \frac{n + 2 + l}{3}\end{aligned}$$

In fact, this bound is sharp. If T is a path on $3k + 2$ vertices, it is easy to check that $\gamma_s(T) = k + 2 = \frac{n+2+l}{3}$. □

This theorem immediately implies the following corollary.

Corollary 2. [3] *For any tree T of order n, $\gamma_s(T) \ge \frac{n+4}{3}$.*

3 Minus domination in graphs

Theorem 3. *If G is a graph of order n, then $\gamma^-(G) \ge n - \frac{2\epsilon}{\delta+1}$.*

Proof: Let g be a minus dominating function on G satisfying $g(V) = \gamma^-(G)$ and

$$\begin{aligned}M &= \{x_1, x_2, \ldots, x_m\} \\ Q &= \{x_{m+1}, x_{m+2}, \ldots, x_{m+q}\} \\ P &= \{x_{m+q+1}, \ldots, x_n\}\end{aligned}$$

be the sets of vertices that are assigned the value -1, 0 and 1, respectively. Let t_i $(i = 1, 2, \ldots, n)$ denotes the number of vertices of weight 0 in $N(x_i)$, then we have

$$|N(x_i) \cap M| \le \begin{cases} \frac{d(x_i) - t_i}{2} - 1 & \text{if } x_i \in M, \\ \frac{d(x_i) - t_i - 1}{2} & \text{if } x_i \in Q, \\ \frac{d(x_i) - t_i}{2} & \text{otherwise.} \end{cases}$$

So

$$\begin{aligned}\sum_{i=1}^{m} d(x_i) &\le \sum_{i=1}^{m} (\frac{d(x_i) - t_i}{2} - 1) + \sum_{i=m+1}^{m+q} \frac{d(x_i) - t_i - 1}{2} \\ &\quad + \sum_{i=m+q+1}^{n} \frac{d(x_i) - t_i}{2} \\ &= \frac{1}{2} \sum_{i=1}^{n} d(x_i) - \frac{1}{2} \sum_{i=1}^{n} t_i - \frac{1}{2} q - m \qquad (4)\end{aligned}$$

Obviuosly

$$\sum_{i=1}^{n} t_i = d(x_{m+1}) + \cdots + d(x_{m+q}) \geq \delta q \tag{5}$$

$$\delta m \leq \sum_{i=1}^{m} d(x_i) \tag{6}$$

Combining (4), (5) and (6) we obtain

$$2m + q \leq \frac{2\epsilon}{\delta + 1} \tag{7}$$

using (7) we get

$$\gamma^-(G) = n - (q + 2m) \geq n - \frac{2\epsilon}{\delta + 1}.$$

□

Using this theorem we deduce the following:

Corollary 3. *If G is a k-regular graph, then $\gamma^-(G) \geq \frac{n}{k+1}$.*

For arbitrary graphs, lower bounds for minus domination number are known. For tree, there is a sharp lower bound for γ^-.

Theorem 4. *For every tree T of order n,*

$$\gamma^-(T) \geq \frac{n + 2 - s}{3},$$

where s is the number of vertices of degree 1, and this bound is sharp.

Proof: Let g be a minus dominating function on T with $g(V) = \gamma^-(T)$ and M, P and Q be defined as in the proof of Theorem 3. Set $t_i = |N(x_i) \cap Q|$ $(i = 1, 2, \ldots, n)$, $s_1 = |\{v \in Q \mid d(v) = 1\}|$. For $x \in V(T)$, if $d(x) = 1$, then $N[x] \subseteq P \cup Q$. So we have

$$\begin{aligned} 2m &\leq \sum_{i=1}^{m} d(x_i) \\ &\leq \sum_{i=1}^{m} (\frac{d(x_i) - t_i}{2} - 1) + \sum_{m+1, d(x_i) \neq 1}^{m+q} \frac{d(x_i) - t_i - 1}{2} \\ &\quad + \sum_{m+q+1, d(x_i) \neq 1}^{n} \frac{d(x_i) - t_i}{2} \\ &= \frac{1}{2} \sum_{i=1}^{m} d(x_i) - \frac{1}{2} s - m - \frac{1}{2} \sum_{i=1, d(x_i) \neq 1}^{n} t_i - \frac{1}{2}(q - s_1) \end{aligned} \tag{8}$$

Obviously

$$\begin{aligned}\sum_{i=1,d(x_i)\neq 1}^{n} t_i &\geq \sum_{x_i\in Q} d(x_i) - (s-s_1)\\ &\geq s_1 + 2(q-s_1) - (s-s_1)\\ &= 2q - s \end{aligned} \tag{9}$$

using (8) and (9) we get

$$(2m+q) \leq \frac{2\epsilon + s}{3}$$

So

$$\gamma^-(G) \geq n - (2m+q) \geq \frac{n+2-s}{3}.$$

In fact, this bound is sharp, it is easy to check that $\gamma^-(K_{1,k}) = 1 = \frac{n+2-s}{3}$. □

4 Majority domination in graphs

Theorem 5. *If G is a graph of order n, then*

$$\gamma_{\mathrm{maj}}(G) \geq \frac{n(2\delta - \Delta) - 4\epsilon}{2(\delta+1)}$$

Proof: Let g be a majority dominating function of weight $g(V) = \gamma_{\mathrm{maj}}(G)$, and

$$P = \{v \in V \mid f(v) = 1\}$$
$$M = \{v \in V \mid f(v) = -1\}$$

Furthermore

$$P_1 = \{v \in P \mid f[v] \geq 1\}$$
$$P_2 = \{v \in P \mid f[v] < 1\}$$
$$M_1 = \{v \in M \mid f[v] \geq 1\}$$
$$M_2 = \{v \in M \mid f[v] < 1\}$$

we write $M = \{x_1, x_2, \ldots, x_m\}$, $m_i = |M_i|$ and $p_i = |P_i|$ $(i = 1, 2)$, then we have

$$\begin{aligned}\sum_{i=1}^{m} d(x_i) &= \sum_{x_i\in V} |M \cap N(x_i)| \\ &\leq \sum_{x_i\in P_1} \frac{d(x_i)}{2} + \sum_{x_i\in M_1} (\frac{d(x_i)}{2} - 1) + \sum_{x_i\in P_2} d(x_i) + \sum_{x_i\in M_2} d(x_i) \\ &= \epsilon - m + \sum_{x_i\in P_2} \frac{d(x_i)}{2} + \sum_{x_i\in M_2} (\frac{d(x_i)}{2} + 1) \qquad (10)\end{aligned}$$

Since g is a majority dominating function, then

$$p_2 + m_2 \leq \frac{n}{2} \qquad (11)$$

Combining (10) and (11) we have

$$m \leq \frac{\epsilon}{\delta+1} + \frac{n(\Delta+2)}{4(\delta+1)}$$

So

$$\gamma_{\mathrm{maj}}(G) = n - 2m \geq \frac{n(2\delta-\Delta)-4\epsilon}{2(\delta+1)}.$$

□

Corollary 4. [7] *If G is a k-regular graph, k is even, of order n, then* $\gamma_{maj}(G) \geq \frac{-k}{2(k+1)}n$.

Similarly we have the following results.

Theorem 6. *If G is a graph and for every $x \in V(G)$, $d(x)$ is odd, then*

$$\gamma_{maj}(G) \geq \frac{n(2\delta-\Delta+1)-4\epsilon}{2(\delta+1)}.$$

Corollary 5. [7] *If G is a k-regular graph, k is odd, of order n, then*

$$\gamma_{maj} \geq \frac{(1-k)}{2(k+1)}n.$$

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