ON THE DECOMPOSITION OF GRAPHS INTO CLIQUES

Gregory F. Bachelis, Troy Barcume, and Xiang-Ying Su
Department of Mathematics
Wayne State University
Detroit, MI 48202 USA
greg @ math.wayne.edu

ABSTRACT

We show by an elementary argument that, given any greedy clique decomposition of a graph G with n vertices, the sum of the orders of the cliques is less than $\frac{5}{8}n^2$. This gives support to a conjecture of Peter Winkler.

Introduction. Let G be a graph with n vertices. By a *Clique Decomposition of G* we mean a partition of the edges of G into cliques. The size of the decomposition is the number of cliques, and the order of a clique is the number of vertices it contains. Erdős, Goodman and Posa [3] showed that any graph has a clique decomposition of size at most $\frac{1}{4}n^2$, and that a minimum such decomposition has size $\lfloor \frac{1}{4}n^2 \rfloor$ only when G is the complete bipartite graph $K_{\lfloor \frac{1}{2}n \rfloor, \lceil \frac{1}{2}n \rceil}$. Later, Chung [2] and Gyori and Kostochka [4] showed that any G has a clique decomposition such that the sum of the orders of the cliques is at most $\frac{1}{2}n^2$. A clique decomposition is called *greedy* if it is obtained by successively removing maximal cliques from G. More recently Winkler [1] made two conjectures:

(I) Any greedy decomposition of G has size at most $\frac{1}{4}n^2$.

(II) For any greedy decomposition of G, the sum of the orders of the cliques is at most $\frac{1}{2}n^2$.

McGuinness [5] proved (I) and showed that the bound $\lfloor \frac{1}{4}n^2 \rfloor$ is achieved only for the appropriate complete bipartite graph. As mentioned by Winkler, the latter is not the case for (II), and we give an example below. Both McGuinness [6,7] and the third author [8] have made progress on (II). In [6] a proof is given for K_4-free graphs, and in [7] a proof is given for the case when a maximum (not just maximal) clique is removed at each stage. In [8] a greedy breadth-first-search polynomial time algorithm for clique decomposition is given in which the sum of the orders of the cliques is at most $\frac{1}{2}n^2$, and for which the bound $\lfloor \frac{1}{2}n^2 \rfloor$ is achieved only for the appropriate complete bipartite graph [9]. In this note we prove the following:

THEOREM. For any clique decomposition of G of size at most $\frac{1}{4}n^2$ (in particular for any greedy clique decomposition), the sum of the orders of the cliques is at most $\frac{5}{8}n^2 - \frac{1}{4}n$.

Our proof is elementary and uses only the numerical properties of the objects involved. After giving the proof we will show that the conclusion of the theorem is sharp, and we will indicate how our method can be extended to other decompositions.

Proof of the Theorem. Consider the following equations, written for convenience in matrix form:

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 3 & 4 & \cdots & n \\ \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \cdots & \binom{n}{2} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} N \\ S \\ E \end{pmatrix}$$

If $2 \leq j \leq n$, then $2j \leq 3 + \binom{j}{2}$, and it is easy to see that $2S \leq 3N + E$ if $x_j \geq 0$. Now suppose that G has a clique decomposition of size no more than $\frac{1}{4}n^2$, and let x_j be the number of cliques of order j, $2 \leq j \leq n$. Then N is the size of the decomposition, so $N \leq \frac{1}{4}n^2$. E is the number of edges of G, so $E \leq \binom{n}{2}$. Finally, S is the sum of the orders of the cliques in the decomposition, so the above inequality gives $2S \leq \frac{3}{4}n^2 + \frac{1}{2}n(n-1)$ and $S \leq \frac{5}{8}n^2 - \frac{1}{4}n$, as was to be proved.

Remarks. (1) The conclusion of the theorem is sharp. Let n be even, and consider the following (non-greedy!) clique decomposition of K_n. Choose a set of $\binom{n/2}{2} = \frac{1}{8}n^2 - \frac{1}{4}n$ edge-disjoint triangles.[1] This leaves $\frac{1}{2}n^2 - \frac{1}{2}n$ −

[1] This can be done by, say, numbering the vertices from 1 to n and then choosing all triangles with vertices {a, b, a+b mod n}, where a and b are distinct odd numbers.

$3(\frac{1}{8}n^2 - \frac{1}{4}n) = \frac{1}{8}n^2 + \frac{1}{4}n$ edges, so there are $\frac{1}{8}n^2 - \frac{1}{4}n + \frac{1}{8}n^2 + \frac{1}{4}n = \frac{1}{4}n^2$ cliques, and the sum of their orders is $\frac{3}{8}n^2 - \frac{3}{4}n + \frac{1}{4}n^2 + \frac{1}{2}n = \frac{5}{8}n^2 - \frac{1}{4}n$.

(2) It is clear that our method of proof generalizes to other types of decompositions of the edges of G in which each subgraph of order j has, say e_j edges. Then $\binom{j}{2}$ in the matrix above is replaced by e_j, and we get some different inequality involving N, S and E. We hope to explore this at a future time.

An Example. We conclude with an example of an extremal situation, if (II) is true, in which the graph G is not bipartite. A different example is given in [6]. Let n be an even integer, and let G be the graph K_n with a perfect matching $\frac{1}{2}n\,K_2$ removed. Then G has $\frac{1}{2}n^2 - n$ edges. One greedy clique decomposition of G (although not by breadth-first search) consists of two vertex-disjoint copies of $K_{n/2}$ plus all the remaining edges joining their vertex sets, which are $\frac{1}{4}n^2 - \frac{1}{2}n$ in number. Thus this decomposition has size $\frac{1}{4}n^2 - \frac{1}{2}n + 2$, and the sum of the orders is $2 \cdot \frac{1}{2}n + 2(\frac{1}{4}n^2 - \frac{1}{2}n) = \frac{1}{2}n^2$.

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