A new infinite series of double Youden rectangles

B.J. Vowden
Institute of Mathematics and Statistics
Cornwallis Building
University of Kent at Canterbury
Canterbury, Kent CT2 7NF, England

Abstract

Bailey (1989) defined a $k \times v$ double Youden rectangle (DYR), with $k < v$, as a type of balanced Graeco–Latin design where each Roman letter occurs exactly once in each of the k rows of the rectangle, and each Greek letter occurs exactly once in each of the v columns. A DYR of a particular size $k \times v$ can exist only if there exists a symmetric 2–design for v treatments in blocks of size k, but existence of a symmetric 2–design does not guarantee the existence of a corresponding DYR, nor does it provide a construction for such a DYR. Vowden (1994) provided constructions of DYRs of sizes $k \times (2k+1)$ where $k > 3$ is a prime power with $k \equiv 3$ (modulo 4). We now provide a general construction for DYRs of sizes $k \times (2k+1)$ where $k > 5$ is a prime power with $k \equiv 1$ (modulo 4). We present DYRs of sizes 9×19 and 13×27.

1 Introduction

Bailey [1] defined a double Youden rectangle (DYR) of size $k \times v$, with $k < v$, as a row and column arrangement of the kv distinct ordered pairs x, y formed when x is drawn from a set X of v elements, and y from a set Y of k elements, and which is organised so that

- each element of X appears exactly once in each row,
- each element of Y appears exactly once in each column,
- each element of X appears at most once in each column, and the sets of elements of X in the columns are the blocks of a symmetric balanced incomplete block design (SBIBD, also known as a symmetric 2–design), so that $v - 1$ divides $k(k - 1)$,
- each element of Y appears n or $n + 1$ times in each row, where n is the integral part of v/k, and either $v - nk = 1$ or, if n occurrences of each element from Y are removed from each row, the remaining sets of $v - nk$ elements of Y in the rows are the blocks of an SBIBD.

The last of these conditions requires $\mu = (v - nk)(v - nk - 1)/(k - 1)$ to be a non–negative integer.

Preece [5,6] reviewed knowledge of DYRs. Existence of an SBIBD for v treatments in blocks of size k with μ integral is necessary but not sufficient for the existence of a $k \times v$ DYR, and, even for smallish values of k, there remain many pairs of values (k, v) for which the existence of a $k \times v$ DYR is an open question. For $k < v - 1$ the only known infinite series of DYRs are those given by Vowden [8], who constructed DYRs of sizes $k \times (2k + 1)$

where $k > 3$ is a prime power with $k \equiv 3$ (modulo 4). We now provide a general construction for DYRs of sizes $k \times (2k+1)$ where $k > 5$ is a prime power with $k \equiv 1$ (modulo 4); for these sizes $n = 2$ and so $v - nk = 1$. Some 5×11 DYRs were presented by Preece [5]: these were generated by a method entirely different from that described here and none fits as the start of our series. The newly constructed DYRs are 'perfect' in the sense of Preece, Vowden and Phillips [7], i.e. within each of two disjoint sets of k columns, the symbols from Y are disposed in a Latin square.

2 An example of a 9 × 19 double Youden rectangle

An example of a 9×19 double Youden rectangle based on the sets $X = \{*, A, B, C, D, E, F, G, H, I, a, b, c, d, e, f, g, h, i\}$ and $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is shown in Table 1.

Table 1: A 9 × 19 double Youden rectangle

A1	*1	g8	d6	F9	C4	i2	H5	e3	B7	a1	G8	D6	c9	h4	I2	b5	E3	f7
B2	e4	*2	h9	g3	D7	A5	C8	I6	f1	E4	b2	H9	G3	a7	i5	d8	c6	F1
C3	i7	f5	*3	B6	h1	E8	d2	A9	G4	I7	F5	c3	g6	H1	b8	D2	e9	a4
D4	B8	h6	E1	*4	a2	g9	I3	F7	c5	e8	H6	i1	d4	A2	G9	f3	b7	C5
E5	F2	C9	i4	h7	*5	b3	a6	G1	D8	g2	f9	I4	H7	e5	B3	A6	d1	c8
F6	g5	D3	A7	c1	i8	*6	E9	b4	H2	G5	h3	d7	C1	I8	f6	a9	B4	e2
G7	C6	I1	f8	E2	b9	H4	*7	d5	a3	i6	e1	F8	h2	B9	c4	g7	D5	A3
H8	d9	A4	G2	I5	F3	c7	b1	*8	e6	D9	g4	f2	a5	i3	C7	B1	h8	E6
I9	H3	e7	B5	a8	G6	D1	f4	c2	*9	d3	E7	h5	A8	b6	g1	F4	C2	i9

As explained by Preece [5], with each $k \times v$ double Youden rectangle we may associate a $v \times v$ square incidence array, and for our present example this array is displayed in Table 2.

Table 2: The square incidence representation of a 9×19 double Youden rectangle

*		11	22	33	44	55	66	77	88	99									
A	11		84	67			25		39					98	42		56		73
B	22	48		95	36					17					79	53	81	64	
C	33	76	59			14		28						61		87		92	45
D	44		63			27	91			58	89		16				32	75	
E	55			41	72		38	69			24	97						13	86
F	66	52			19	83			47			35	78				94		21
G	77			82		96			51	34	65	18		23		49			
H	88	93					74	15		62		46	29	57	31				
I	99		71		85			43	26		37		54		68	12			
a					98	42		56		73	11			85	27		69		34
b						79	53	81	64			22			96	38	15	47	
c					61		87		92	45			33	19		74		26	58
d		89		16				32	75		93		67	44			28	51	
e		24	97						13	86	48	71			55			39	62
f			35	78				94		21		59	82			66	43		17
g		65	18		23		49				52	84		36		91	77		
h			46	29	57	31						63	95	72	14			88	
i		37		54		68	12				76		41		83	25			99

We obtain Table 2 from Table 1 as follows. Assume the rows of our double Youden rectangle in Table 1 are indexed by Y, and the columns by X. Table 2 has both its rows and columns indexed by X. If Table 1 has the pair (x, y) in cell (y', x'), then cell (x, x') of Table 2 contains the pair (y', y). Essentially we interchange in the display the partitioning induced by the rows and that induced by the elements of X. This equivalent representation of a double Youden rectangle will be more convenient for our purpose: subsequent constructions are patterned on this square incidence form.

3 A construction for $k\times(2k+1)$ double Youden rectangles

Suppose the positive integer k is odd and a prime power. Within the finite field $\mathbf{F}$ having k elements let x denote a primitive element, so that the powers x^r, for $r = 0, 1, 2, \ldots, k-2$, constitute the $k-1$ non-zero elements of $\mathbf{F}$, and $x^{k-1} = 1$. We work with $k \times k$ matrices: when doing so we index rows and columns by the elements of $\mathbf{F}$ and index arithmetic is performed within $\mathbf{F}$. Suppose $M = [m_{ij}]$ is the $k \times k$ matrix for which

$$m_{ij} = \begin{cases} 1 & \text{if } i-j \text{ is an even power of } x, \\ -1 & \text{if } i-j \text{ is an odd power of } x, \\ 0 & \text{if } i=j. \end{cases}$$

If the matrices I and J denote, respectively, the identity matrix and the matrix of all ones, then $MJ = JM = 0$ and $MM' = M'M = kI - J$. To verify the latter relationship, note that the (i,j)th entry of MM' is $\sum_h m_{ih}m_{jh} = \sum_{h\neq i,j} m_{ih}m_{jh}$, and this sum equals $k-1$ when $i = j$, but when $i \neq j$

$$\begin{aligned} \sum_{h\neq i,j} m_{ih}m_{jh} &= \sum_{h\neq i,j} m_{i-h,0}m_{j-h,0} \\ &= \sum_{g\neq 0,1} m_{g0} = \sum_{g\neq 0} m_{g0} - 1 = -1\ , \end{aligned}$$

where $g = (i-h)/(j-h)$. The matrix $A = \frac{1}{2}\{(J-I)+M\}$ is the zero–one matrix whose (i,j)th entry is one if $i-j$ is an even power of x, but is zero otherwise. Likewise $B = \frac{1}{2}\{(J-I)-M\}$ is the zero–one matrix whose (i,j)th entry is one if $i-j$ is an odd power of x, but is zero otherwise. Consequently $AJ = JA = \frac{1}{2}(k-1)J$, $BJ = JB = \frac{1}{2}(k-1)J$, $AB = BA$, $I + A + B = J$ and $AA' + BB' = \frac{1}{2}(k+1)I + \frac{1}{2}(k-3)J$. These relations imply that the $(2k+1) \times (2k+1)$ block matrix

$$N = \begin{bmatrix} 0 & \mathit{1}' & \mathit{0}' \\ \mathit{1} & A & B \\ \mathit{0} & B' & I+B' \end{bmatrix}$$

where $\mathit{1}$ and $\mathit{0}$ are column vectors of, respectively, ones and zeros, satisfies $NN' = \frac{1}{2}(k+1)I + \frac{1}{2}(k-1)J$, and correspondingly that N represents the incidence matrix of an SBIBD whose parameters are $2k+1$, k and $\frac{1}{2}(k-1)$.

If $k \equiv 3$ (modulo 4) then $B = A'$ and the matrix A itself is the incidence matrix of an SBIBD having parameters k, $\frac{1}{2}(k-1)$ and $\frac{1}{4}(k-3)$. The incidence matrix N corresponds to an SBIBD denoted as C_3 by Bhat and Shrikhande[2] and appears amongst the incidence matrices for SBIBDs previously employed by Vowden [8] for his construction of some infinite series of DYRs of sizes $k \times (2k+1)$ where $k > 3$ is a prime power with $k \equiv 3$ (modulo 4). Our concern here is with the complementary case $k \equiv 1$ (modulo 4) and now the matrices A and B are symmetric, so that the incidence matrix N may be rewritten

$$N = \begin{bmatrix} 0 & \mathit{1}' & \mathit{0}' \\ \mathit{1} & A & B \\ \mathit{0} & B & I+B \end{bmatrix}.$$

The SBIBD that corresponds to this incidence matrix may also be derived from a construction for Hadamard matrices due to Paley [4] (see also [3], Chapter 14).

Employing the square incidence form described in Section 2, we build DYRs based on the incidence matrix N. Those entries of N which are zero we discard, but entries which are one we replace by pairs of elements drawn from the field $\mathbf{F}$. To express this more precisely we refer to N as a 3×3 block matrix with blocks $(1,1),(1,2),\ldots,(3,3)$. Then we revise the entries which are one in the following way:

1. The k pairs i,i where the index i ranges through $\mathbf{F}$, are entered as a column in the (2,1) block of N, as a row in the (1,2) block, and in the diagonal of the (3,3) block.

2. The (2,2) block in N is the matrix A, whose (i,j)th entry is one when $i-j$ is an even power of x. Select for this block distinct integers a_1 and a_2 from the range $1,\ldots,k-2$. When $i-j=x^{2r}$ install, as replacement entry, the pair of field elements

$$x^{2r+a_1}+j, \quad x^{2r+a_2}+j\,.$$

3. The (3,2) block in N is the matrix B, whose (i,j)th entry is one when $i-j$ is an odd power of x. Again, select for this block distinct integers b_1 and b_2 from the range $1,\ldots,k-2$. When $i-j=x^{2r+1}$ the replacement entry is

$$x^{2r+1+b_1}+j, \quad x^{2r+1+b_2}+j\,.$$

Likewise, within N the (2,3) block and the non–diagonal contribution to the (3,3) block is again B. Entry replacements are made in the same fashion after the selection of two further integer pairs c_1, c_2 and d_1, d_2.

For this array to comply with the constraints required of a DYR, the eight parameters $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ introduced in its construction must satisfy certain consistency conditions. Firstly, to ensure that each column of the array contains all the elements of the field $\mathbf{F}$, separately in both the first and second components of the array entries, the integers a_1 and b_1 should have the same parity, i.e. either both are even or both are odd. Likewise, a_2 and b_2 should have the same parity, whereas c_1 and d_1 should have opposite parity, as should c_2 and d_2. Next, to accommodate the corresponding consideration for rows, we employ the notation introduced by Vowden [8] whereby, for any integer m selected from the range $1, \ldots, k-2$, $\overline{m}$ denotes the integer in the range $0, 1, \ldots, k-2$ (but avoiding $\frac{1}{2}(k-1)$) which satisfies $x^{\overline{m}} = x^m - 1$. Then we must require that $\overline{a}_1$ and $\overline{c}_1$ have the same parity, and also $\overline{a}_2$ and $\overline{c}_2$, but $\overline{b}_1$ and $\overline{d}_1$ must have opposite parity, as must $\overline{b}_2$ and $\overline{d}_2$. Finally consider the association between the first and second components of the pairs of elements, from the field $\mathbf{F}$, installed in our array. Within the (2,2) block 0 as a first component is paired with second components $x^{2r}(x^{a_2} - x^{a_1})$; within the (3,2) block the pairing is with $x^{2r-1}(x^{b_2} - x^{b_1})$, and likewise for the (2,3) block and the off–diagonal entries of the (3,3) block. The within–rows replication of elements of the set Y in our original row and column description of a DYR given in Section 1 is thus achieved if we require that exactly two of $x(x^{a_2} - x^{a_1})$, $x^{b_2} - x^{b_1}$, $x^{c_2} - x^{c_1}$, and $x^{d_2} - x^{d_1}$ be even powers of x.

When $k = 5$ the conditions we thus impose on the eight parameters $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ are incompatible, so that our construction process does not succeed. However Preece [5], used a quite different approach to produce some 5×11 DYRs.

The next case to consider is $k = 9$. The finite field $\mathbf{F} = \mathbf{F}_9$ is of characterisitic 3 and has a primitive element x which satisfies $x^2 = 2x+1$, so that the nine field elements are $0, x^0 = 1$, $x^1 = x$, $x^2 = 2x+1$, $x^3 = 2x+2$, $x^4 = 2$, $x^5 = 2x$, $x^6 = x+2$, $x^7 = x+1$. It is easily verified that the parameter assignment $a_1 = b_1 = c_1 = 1$, $d_1 = 2$, $a_2 = b_2 = c_2 = 2$, $d_2 = 3$ meets our requirements and the 9×19 DYR of our construction is that displayed in Table 1 when the elements of the set Y used there are identified with the elements of $\mathbf{F}_9$ via the correspondence $1 \leftrightarrow 0$, $2 \leftrightarrow 1$, $3 \leftrightarrow 2$, $4 \leftrightarrow x$, $5 \leftrightarrow x+1$, $6 \leftrightarrow x+2$, $7 \leftrightarrow 2x$, $8 \leftrightarrow 2x+1$, $9 \leftrightarrow 2x+2$.

4 An infinite series of double Youden rectangles

Section 3 showed how to construct a $k \times (2k+1)$ DYR when k is a prime power congruent to 1 (modulo 4) provided that a choice could be made for the eight parameters $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ introduced into our construction, where each of these parameters is drawn from the integer range 1 to $k-2$ and collectively they satisfy the following conditions

$$a_1 \neq a_2, b_1 \neq b_2, c_1 \neq c_2, d_1 \neq d_2,$$

$$\left.\begin{array}{l} a_1 \text{ and } b_1 \text{ have the same parity} \\ c_1 \text{ and } d_1 \text{ have opposite parity} \\ \overline{a}_1 \text{ and } \overline{c}_1 \text{ have the same parity} \\ \overline{b}_1 \text{ and } \overline{d}_1 \text{ have opposite parity} \end{array}\right\} \quad \begin{array}{l} \text{and as well the corresponding set} \\ \text{of conditions for } a_2, b_2, c_2, d_2, \end{array}$$

exactly two of $x(x^{a_2} - x^{a_1})$, $x^{b_2} - x^{b_1}$, $x^{c_2} - x^{c_1}$, $x^{d_2} - x^{d_1}$ are even powers of x.

As $\mathbf{F}$ represents the finite field having k elements, within which x is a primitive element, elements in $\mathbf{F}$ of the form $x^m - 1$, for $m = 1, 2, \ldots, k-2$ exhaust the whole of $\mathbf{F}$ with the exception of the two elements 0 and $-1 = x^{\frac{1}{2}(k-1)}$. Thus the exponent $\overline{m}$ that we introduced via the relation $x^m - 1 = x^{\overline{m}}$ assumes all integer values from 1 to $k-2$ inclusive, apart from the even integer $\frac{1}{2}(k-1)$. The identity

$$(x^m - 1)(x^{(k-1)-m} - 1) = x^{(k-1)-m+\frac{1}{2}(k-1)}(x^m - 1)^2$$

shows that $\overline{m}$ and $\overline{(k-1)-m}$ have the same parity if m is even, but that if m is odd they have opposite parity. Because $k \equiv 1$ (modulo 4) we may write $k = 4\lambda + 1$ for some positive integer λ. We see now that when m is even the values $x^m - 1$ provide $\lambda - 1$ even powers of x and λ odd powers, and when m is odd the values $x^m - 1$ provide both λ even powers and λ odd powers. This information is summarised in Table 3 in terms of the parity combinations of m and $\overline{m}$.

Table 3: Counts of integers satisfying the different parity combinations

m	$\overline{m}$	Number of possibilities
Even	Even	$\lambda - 1$
Even	Odd	λ
Odd	Even	λ
Odd	Odd	λ

Table 4: A 13×27 *double Youden rectangle*

(In this representation x, y and z denote 10, 11 and 12 respectively.)

A0	*0	Fy	k9	C7	H5	m3	e1	jz	bx	G8	L6	d4	I2	a0	gy	K9	f7	l5	M3	E1	Jz	Bx	c8	i6	D4	h2
B1	J3	*1	Gz	lx	D8	I6	a4	f2	k0	cy	H9	N7	e5	i3	b1	hz	Lx	g8	m6	A4	F2	K0	Cy	d9	j7	E5
C2	f6	K4	*2	H0	my	E9	J7	b5	g3	l1	dz	Ix	A8	F6	j4	c2	i0	My	h9	a7	B5	G3	L1	Dz	ex	k8
D3	B9	g7	L5	*3	I1	az	Fx	K8	c6	h4	m2	e0	Jy	l9	G7	k5	d3	j1	Az	ix	b8	C6	H4	M2	E0	fy
E4	Kz	Cx	h8	M6	*4	J2	b0	Gy	L9	d7	i5	a3	f1	gz	mx	H8	l6	e4	k2	B0	jy	c9	D7	I5	A3	F1
F5	g2	L0	Dy	i9	A7	*5	K3	c1	Hz	Mx	e8	j6	b4	G2	h0	ay	I9	m7	f5	l3	C1	kz	dx	E8	J6	B4
G6	c5	h3	M1	Ez	jx	B8	*6	L4	d2	I0	Ay	f9	k7	C5	H3	i1	bz	Jx	a8	g6	m4	D2	l0	ey	F9	K7
H7	l8	d6	i4	A2	F0	ky	C9	*7	M5	e3	J1	Bz	gx	L8	D6	I4	j2	c0	Ky	b9	h7	a5	E3	m1	fz	Gx
I8	hy	m9	e7	j5	B3	G1	lz	Dx	*8	A6	f4	K2	C0	Hy	M9	E7	J5	k3	d1	Lz	cx	i8	b6	F4	a2	g0
J9	D1	iz	ax	f8	k6	C4	H2	m0	Ey	*9	B7	g5	L3	h1	Iz	Ax	F8	K6	l4	e2	M0	dy	j9	c7	G5	b3
Kx	M4	E2	j0	by	g9	l7	D5	I3	a1	Fz	*x	C8	h6	c4	i2	J0	By	G9	L7	m5	f3	A1	ez	kx	d8	H6
Ly	i7	A5	F3	k1	cz	hx	m8	E6	J4	b2	G0	*y	D9	I7	d5	j3	K1	Cz	Hx	M8	a6	g4	B2	f0	ly	e9
Mz	Ex	j8	B6	G4	l2	d0	iy	a9	F7	K5	c3	H1	*z	fx	J8	e6	k4	L2	D0	Iy	A9	b7	h5	C3	g1	mz

Section 3 exhibited an appropriate choice of values for the parameters a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 when $k = 9$, which enabled us to construct the 9×19 DYR given in Section 2. Now we consider $k > 9$, where as before the integer k is a prime power congruent to 1 (modulo 4), and we show for each k how to make a parameter selection which satisfies the conditions listed above. Suppose the λ distinct values of the exponent m for which m is even but $\overline{m}$ is odd are $r_1, \ldots, r_\lambda$, and likewise $s_1, \ldots, s_\lambda$ are the λ distinct values of m with m odd and $\overline{m}$ even. There are λ^2 sums $r_i + s_j$ (modulo $k-1$) and their values are all odd. But the series $1, 3, \ldots, 4\lambda - 1$ of odd integers has 2λ members. So when $\lambda > 2$ there exist indices $i_1 \neq i_2$, $j_1 \neq j_2$ that satisfy $r_{i_1} + s_{j_2} = r_{i_2} + s_{j_1}$. We may set $a_1 = b_1 = c_1 = r_{i_1}$, $d_1 = s_{j_1}$, $a_2 = b_2 = c_2 = r_{i_2}$, $d_2 = s_{j_2}$, as is readily verified. For example when $k = 13$, $\lambda = 3$, the finite field $\mathbf{F} = \mathbf{F}_{13}$ corresponds to arithmetic of the natural numbers modulo 13, a primitive element is $x = 2$, $r_1 = 4$, $r_2 = 6$, $r_3 = 8$, $s_1 = 1$, $s_2 = 7$, $s_3 = 9$, $r_1 + s_1 = 5$ and $r_3 + s_3 = 17 \equiv 5$ (modulo 12): Table 4 displays the 13×27 DYR arising from the corresponding parameter choice $a_1 = b_1 = c_1 = 4$, $d_1 = 9$ $a_2 = b_2 = c_2 = 8$, $d_2 = 1$ (where, for the display, we have employed $X = \{*, A, B, C, \ldots, M, a, b, c, \ldots, m\}$ and $Y = \{0, 1, 2, \ldots, 9, x, y, z\}$).

Acknowledgement

D.A. Preece was a source of much valuable advice during the preparation of this paper, and I thank my colleague for his sustained interest in my work.

References

[1] R.A. Bailey, Designs: mappings between structured sets, in J. Siemons, ed., *Surveys in Combinatorics*, 1989, 22–51, (Cambridge Univ. Press, Cambridge, 1989).

[2] V.N. Bhat and S.S. Shrikhande, Non–isomorphic solutions of some balanced incomplete block designs: I, *J.Combin.Theory* **9** (1970) 174–191.

[3] M. Hall Jr., *Combinatorial Theory*, (Wiley, New York, 1967).

[4] R.E.A.C. Paley, On orthogonal matrices, *J.Math.Phys.* **12** (1933) 311–320.

[5] D.A. Preece, Double Youden rectangles – an update with examples of size 5×11, *Discrete Mathematics* **125** (1994) 309–317.

[6] D.A. Preece, Youden squares, in C.J. Colbourn and J.H. Dirinitz, eds., *The CRC Handbook of Combinatorial Designs*, 511–514, (CRC Press, Boca Raton, 1996).

[7] D.A. Preece, B.J. Vowden and N.C.K. Phillips, Double Youden rectangles of sizes $p \times (2p+1)$ and $(p+1) \times (2p+1)$, *Ars Combinatoria* (to appear).

[8] B.J. Vowden, Infinite series of double Youden rectangles, *Discrete Mathematics* **125** (1994) 385-391.