HAMILTONIAN PATHS IN PROJECTIVE CHECKERBOARDS

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ABSTRACT. Place a checker in some square of an $n \times n$ checkerboard. The checker is allowed to step either to the east or to the north, and is allowed to step off the edge of the board in a manner suggested by the usual identification of the edges of the square to form a projective plane. We give an explicit description of all the routes that can be taken by the checker to visit each square exactly once.

1. Introduction

Place a checker in some square of an $n \times n$ checkerboard. The checker is allowed to step (a distance of one square) either to the east or to the north — in particular, it does not move diagonally. We ask whether it is possible to move the checker in this manner through a route that visits each square exactly once. We feel free to adopt graph-theoretic terminology whenever it is convenient, so we refer to such a route as a *hamiltonian path.*

It is traditional to allow the checker to step off the edge of the board (otherwise, it is clear that no hamiltonian path exists). Namely, when the checker steps off the east edge of the board, it moves to the westernmost square of the same row; when the checker steps off the north edge of the board, it moves to the southernmost square of the same column. This can be described succinctly by saying that the checkerboard has been made into a torus, by gluing each of its edges to the opposite edge. For these toroidal checkerboards, it is not difficult to see that every square is the starting point of some hamiltonian path, and several authors have studied more delicate properties of paths and cycles in toroidal checkerboards [1], [2], [4], [5], [6], [7].

In this paper, we do not study toroidal checkerboards. Instead, the checker is allowed to step off the edge of the board in a somewhat different manner, corresponding to the usual procedure for creating a projective plane by applying a twist when gluing each edge of a square to the opposite edge.

1.1. Definition. The squares of the $n \times n$ checkerboard can be naturally identified with the set B_n of ordered pairs (p,q) of integers with $0 \le p,q \le n-1$. Define $E\colon B_n \to B_n$ and $N\colon B_n \to B_n$ by

$$(p,q)E = \begin{cases} (p+1,q) & \text{if } p < n-1 \\ (0,n-1-q) & \text{if } p = n-1 \end{cases}$$

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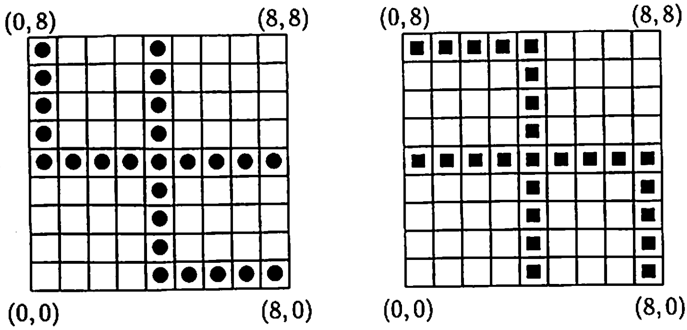

FIGURE 1. Locations of the initial (●) and terminal (■) squares of hamiltonian paths on the 9×9 projective checkerboard.

and

$$(p,q)N = \begin{cases} (p, q+1) & \text{if } q < n-1 \\ (n-1-p, 0) & \text{if } q = n-1 \end{cases}$$

The $n \times n$ *projective checkerboard* $\mathcal{B}_n$ is the digraph whose vertex set is B_n, with a directed edge from α to αE and from α to αN, for each $\alpha \in B_n$. We usually refer to the vertices of $\mathcal{B}_n$ as *squares*.

For these projective checkerboards, we show that only certain squares can be the initial square of a hamiltonian path, and only certain squares can be the terminal square. The form of the answer depends on the parity of n; illustrative examples appear in Figures 1 and 2.

1.2. Notation. For convenience, we let $\not\ell = \lfloor n/2 \rfloor$ and $\not\ell^- = \lfloor (n-1)/2 \rfloor$. Thus, if n is odd, we have $\not\ell = \not\ell^- = (n-1)/2$, whereas, if n is even, we have $\not\ell = n/2$ and $\not\ell^- = \not\ell - 1$.

1.3. Theorem. *A square (p,q) of $\mathcal{B}_n$ (with $0 \leq p, q \leq n-1$) is the initial square of a hamiltonian path if and only if either*

(1) $q = 0$ *and* $p \geq \not\ell^-$; *or*
(2) $q = \not\ell^-$ *and* $p \leq \not\ell$; *or*
(3) $q = \not\ell$ *and* $p \geq \not\ell^-$; *or*
(4) $p = 0$ *and* $q \geq \not\ell^-$; *or*
(5) $p = \not\ell^-$ *and* $q \leq \not\ell$; *or*
(6) $p = \not\ell$ *and* $q \geq \not\ell^-$.

It is the terminal square of a hamiltonian path if and only if either

(1) $q = n-1$ *and* $p \leq \not\ell$; *or*
(2) $q = \not\ell$ *and* $p \geq \not\ell^-$; *or*
(3) $q = \not\ell^-$ *and* $p \leq \not\ell$; *or*
(4) $p = n-1$ *and* $q \leq \not\ell$; *or*
(5) $p = \not\ell$ *and* $q \geq \not\ell^-$; *or*
(6) $p = \not\ell^-$ *and* $q \leq \not\ell$.

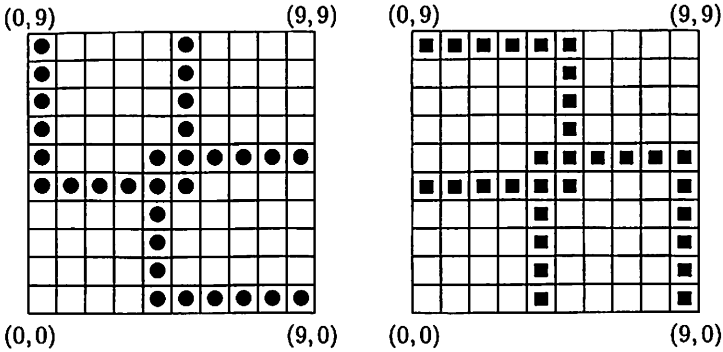

FIGURE 2. Locations of the initial (●) and terminal (■) squares of hamiltonian paths on the 10×10 projective checkerboard.

Furthermore, we give an explicit description of all the hamiltonian paths in $\mathcal{B}_n$. This more detailed result is stated in Section 2, after the necessary terminology has been introduced. The proof is in Section 3.

2. Direction-forcing diagonals

2.1. Definition. Define a symmetric, reflexive relation $\sim$ on the set of squares by $\alpha \sim \beta$ if

$$\{\alpha E, \alpha N\} \cap \{\beta E, \beta N\} \neq \emptyset.$$

The equivalence classes of the transitive closure of $\sim$ are *direction-forcing diagonals*. For short, we refer to them simply as *diagonals*. Thus, the diagonal containing α is

$$\{\alpha, \alpha N E^{-1}, \alpha (N E^{-1})^2, \ldots, \alpha E N^{-1}\}.$$

2.2. Notation. For $\alpha \in \mathcal{B}_n$, we let α_x and α_y be the components of α, that is, $\alpha = (\alpha_x, \alpha_y)$.

2.3. Notation. For $0 \leq i \leq 2n-2$, let $S_i = \{\alpha \in \mathcal{B}_n \mid \alpha_x + \alpha_y = i\}$.

2.4. Proposition. *For each i with $0 \leq i \leq 2n-3$, the set $D_i = S_i \cup S_{2n-3-i}$ is a diagonal. The only other diagonal D_{2n-2} consists of the single square $(n-1, n-1)$.* □

Notice that $D_i = D_{2n-3-i}$ if $0 \leq i \leq 2n-3$.

2.5. Notation. Let D be a diagonal, other than D_{2n-2}. Then, from Proposition 2.4, we may write $D = S_a \cup S_b$ with $a < b$. We let $D_- = S_a$ and $D_+ = S_b$.

To find the hamiltonian paths in $\mathcal{B}_n$, it is helpful to focus attention on the following class of subgraphs of $\mathcal{B}_n$, which contains all the hamiltonian paths.

2.6. Definition. A *spanning quasi-path* is a spanning subdigraph $\mathcal{H}$ of $\mathcal{B}_n$, such that exactly one component of $\mathcal{H}$ is a (directed) path, and each of the other components of $\mathcal{H}$ is a (directed) cycle. We usually use ι to denote the initial square of the path, and τ to denote the terminal square.

Thus, a spanning quasi-path has two distinguished vertices ι and τ, such that every square except ι has indegree 1, and every square except τ has outdegree 1. The square ι has indegree 0, and the square τ has outdegree 0.

2.7. Definition. If $\mathcal{H}$ is a spanning quasi-path, then the diagonal containing the terminal square τ is called the *terminal* diagonal of $\mathcal{H}$. All other diagonals are *non-terminal* diagonals.

The following fundamental propositions, which are used throughout the paper, justify the name *direction-forcing* diagonal, and reveal the special status of the terminal diagonal. The results are adapted from ideas of Housman [3].

2.8. Definition. Let $\mathcal{H}$ be a spanning quasi-path in $\mathcal{B}_n$. A square α *travels east* (in $\mathcal{H}$) if the edge from α to αE is in $\mathcal{H}$. Similarly, α *travels north* (in $\mathcal{H}$) if the edge from α to αN is in $\mathcal{H}$.

2.9. Proposition (cf. [1, Lemma 6.4c]). *If $\mathcal{H}$ is a spanning quasi-path in $\mathcal{B}_n$, then, for each non-terminal diagonal D, either every square in D travels north, or every square in D travels east. For short, we say that either D travels north or D travels east.* □

2.10. Proposition (cf. [1, Lemma 6.4b]). *Let T be the terminal diagonal of a spanning quasi-path $\mathcal{H}$ in $\mathcal{B}_n$, with initial vertex ι and terminal vertex τ, and let $\alpha \in T$.*

(1) *if $\tau N \neq \iota$, then $\tau N E^{-1}$ travels east;*
(2) *if $\tau E \neq \iota$, then $\tau E N^{-1}$ travels north;*
(3) *if α travels east and $\alpha N \neq \iota$, then $\alpha N E^{-1}$ travels east;*
(4) *if α travels east, then $\alpha E N^{-1}$ does not travel north;*
(5) *if α travels north and $\alpha E \neq \iota$, then $\alpha E N^{-1}$ travels north; and*
(6) *if α travels north, then $\alpha N E^{-1}$ does not travel east.* □

2.11. Corollary (cf. [1, Lemma 6.4a]). *If $\mathcal{H}$ is a spanning quasi-path in $\mathcal{B}_n$, then the diagonal that contains ιE^{-1} and ιN^{-1} is the terminal diagonal.* □

The following corollary follows from Proposition 2.10 by induction.

2.12. Corollary. *Let T be the terminal diagonal of a spanning quasi-path $\mathcal{H}$ in $\mathcal{B}_n$, with initial vertex ι and terminal vertex τ, and let $|T|$ denote the cardinality of T. For each $\alpha \in T$, there is a unique integer $u(\alpha) \in \{1, 2, \ldots, |T|\}$ with $\alpha = \tau(NE^{-1})^{u(\alpha)}$; the square α travels east if and only if $u(\alpha) < u(\iota E^{-1})$. (Similarly, there is a unique integer $v(\alpha) \in \{1, 2, \ldots, |T|\}$ with $\alpha = \iota E^{-1}(EN^{-1})^{v(\alpha)}$; the square α travels east if and only if $v(\alpha) < v(\tau)$.)* □

From Corollary 2.12, we see that the location of the initial square and terminal square determine which vertices in the terminal diagonal travel east, and which of them travel north, so we have the following corollary.

initial vertex ι	terminal vertex τ	path directions
$(n-k,0)$ $1 \le k \le \not{n}$	$(\not{n}-k, \not{n})$	$(E^{2n-1}N)^{\not{n}}E^{n-1}$
$(n-k,0)$ $1 \le k \le \not{n}+1$	$(\not{n}+k-1, \not{n})$	$(E^{2n-1}N)^{k-1}E^{2k-2}$ $(NE^{2i-1}NE^{2(n-k-i+1)}NE^{2k-2})_{i=1}^{\not{n}-k+1}$
$(0,n-1)$	$(\not{n}, \not{n})$	$(E^{2n-1}N)^{\not{n}}E^{n-1}$
$(0,n-1)$	$(\not{n}, \not{n}-1)$	$(N^2E^{2i}NE^{2(n-i)-1})_{i=1}^{\not{n}}$

FIGURE 3. Some hamiltonian paths in $\mathcal{B}_n$, when n is odd (where $\not{n} = (n-1)/2$). All non-terminal diagonals travel east.

2.13. Corollary. *A hamiltonian path (or, more generally, a spanning quasi-path) is uniquely determined by specifying*

(1) *its initial vertex;*
(2) *its terminal vertex; and*
(3) *which of its non-terminal diagonals travel east.* □

Our main results determine precisely which choices of (1), (2), and (3) lead to a hamiltonian path, rather than some other spanning quasi-path.

The following theorem shows that, to find the possible initial squares and terminal squares of hamiltonian paths, one need only consider hamiltonian paths of two special types: those in which all non-terminal diagonals travel east, and those in which all non-terminal diagonals travel north.

3.6′. Theorem. *If there is a hamiltonian path from ι to τ, then there is a hamiltonian path from ι to τ, such that either all non-terminal diagonals travel east, or all non-terminal diagonals travel north.*

Figures 3 and 4 list several hamiltonian paths. We now define some simple transformations that can be applied to create additional hamiltonian paths from these.

2.14. Definition. For $\alpha = (\alpha_x, \alpha_y) \in \mathcal{B}_n$, let $\tilde{\alpha} = (n-1-\alpha_x, n-1-\alpha_y)$ be the *inverse* of α, and let $\alpha^* = (\alpha_y, \alpha_x)$ be the *transpose* of α.

2.15. Proposition. *Let $[\alpha_0, \ldots, \alpha_m]$ be the sequence of squares visited by a hamiltonian path $\mathcal{H}$ in $\mathcal{B}_n$, from $\iota = \alpha_0$ to $\tau = \alpha_m$. Then the following are also hamiltonian paths in $\mathcal{B}_n$:*

the inverse of $\mathcal{H}$	*from $\tilde{\tau}$ to $\tilde{\iota}$:*	$[\tilde{\alpha}_m, \tilde{\alpha}_{m-1}, \ldots, \tilde{\alpha}_0]$;
the transpose of $\mathcal{H}$	*from ι^* to τ^*:*	$[\alpha_0^*, \ldots, \alpha_m^*]$;
the inverse-transpose of $\mathcal{H}$	*from $\tilde{\tau}^*$ to $\tilde{\iota}^*$:*	$[\tilde{\alpha}_m^*, \tilde{\alpha}_{m-1}^*, \ldots, \tilde{\alpha}_0^*]$. □

Our main theorem can be stated as follows.

initial vertex ι	terminal vertex τ	path directions
$(n-k,0)$ $1 \le k \le \not{n}$	$(\not{n}-k, \not{n}-1)$	$(E^{2n-1}N)^{\not{n}-1}E^{2n-1}$
$(n-k,0)$ $1 \le k \le \not{n}$	$(\not{n}+k-2, \not{n})$	$(E^{2n-1}N)^{k-1}E^{2k-2}$ $(NE^{2i-1}NE^{2(n-k-i+1)}NE^{2k-2})_{i=1}^{\not{n}-k}$ $NE^{n-2k+1}NE^{2k-2}$
$(\not{n}-1,0)$	$(n-1, \not{n})$	$(E^{2n-1}N)^{\not{n}-1}E^{2n-1}$
$(0,n-1)$	$(\not{n}-1, \not{n}-1)$	$(N^2E^{2i}NE^{2n-2i-1})_{i=1}^{\not{n}-1} N^2E^{n-1}$
$(0,n-1)$	$(\not{n}, \not{n}-1)$	$(E^{2n-1}N)^{\not{n}-1}E^{2n-1}$

FIGURE 4. Some hamiltonian paths in $\mathcal{B}_n$, when n is even (where $\not{n} = n/2$). All non-terminal diagonals travel east.

3.12′. Main Theorem. *If $\mathcal{H}$ is any hamiltonian path in $\mathcal{B}_n$, such that all non-terminal diagonals travel east, then either $\mathcal{H}$ is one of the hamiltonian paths listed in Figures 3 and 4, or $\mathcal{H}$ is the inverse of one of these hamiltonian paths.*

By considering the transposes of these paths, we obtain the following corollary.

2.16. Corollary. *If $\mathcal{H}$ is any hamiltonian path in $\mathcal{B}_n$, such that all non-terminal diagonals travel north, then $\mathcal{H}$ is either the transpose or the inverse-transpose of one of the hamiltonian paths listed in Figures 3 and 4.* □

The Main Theorem and its corollary, combined with Theorem 3.6′, suffice to determine all the initial and terminal squares of hamiltonian paths. These conclusions are detailed in Theorem 1.3.

By combining the Main Theorem with Proposition 3.3 and Theorem 3.5 below, we obtain the following precise description of the hamiltonian paths in $\mathcal{B}_n$, not just those in which all non-terminal diagonals travel the same direction. The point is that all hamiltonian paths can be derived by applying certain simple transformations to the basic paths listed in Figures 3 and 4.

2.17. Corollary. *The following construction always results in a hamiltonian path in $\mathcal{B}_n$, and every hamiltonian path can be constructed in this manner.*

(1) *Start with a basic hamiltonian path $\mathcal{H}_1$ from Figure 3 or 4.*
(2) *Write the terminal diagonal of $\mathcal{H}_1$ in the form $S_a \cup S_b$ (with $a < b$), and let $\mathcal{N}$ be any subset of $\{D_j \mid 0 \le j < a\}$.*
(3) *Let $\mathcal{H}_2$ be the spanning quasi-path of $\mathcal{B}_n$ in which the non-terminal diagonals in $\mathcal{N}$ travel north, all the other non-terminal diagonals travel east, and each square in the terminal diagonal travels exactly the same way as it does in $\mathcal{H}_1$.*
(4) *Let $\mathcal{H}$ be $\mathcal{H}_2$, or the inverse, or transpose, or inverse-transpose of $\mathcal{H}_2$.*

Then $\mathcal{H}$ is a hamiltonian path in $\mathcal{B}_n$. □

3. The Main Theorem

In this section, we prove Main Theorem 3.12.

3.1. Notation. We use $[\alpha](X_1X_2\dots X_m)$, where $X_i \in \{E, N\}$, to denote the path (or cycle) in $\mathcal{B}_n$ that visits the squares

$$\alpha,\ \alpha X_1,\ \alpha X_1X_2,\ \dots,\ \alpha X_1X_2\dots X_m.$$

Note that a 2×2 projective checkerboard has hamiltonian cycles, namely, the cycles $(NNNN)$ and $(EEEE)$. However, the following theorem shows that this is the only such board; for $n > 2$, there is no hamiltonian cycle in $\mathcal{B}_n$.

3.2. Theorem. *On an $n \times n$ projective checkerboard with $n \geq 3$, there are no hamiltonian paths with initial square $\iota = (0,0)$. Therefore, D_{2n-2} is not the terminal diagonal of any hamiltonian path in $\mathcal{B}_n$.*

Proof. Suppose $\mathcal{H}$ is a hamiltonian path in $\mathcal{B}_n$, with initial square $\iota = (0,0)$. From Corollary 2.11, we see that the terminal square τ must be $(n-1, n-1)$.

Let $[\alpha_0, \alpha_1, \dots, \alpha_m]$ be the sequence of squares visited by $\mathcal{H}$. By induction on i, we show that $\alpha_{m-i} = \tilde{\alpha}_i$ (the inverse of α_i) for $0 \leq i \leq m$. Assume inductively that $\alpha_{m-i} = \tilde{\alpha}_i$. Suppose, without loss of generality, that α_i travels east in $\mathcal{H}$. From Proposition 2.4, it is not difficult to see, for all $\alpha \in \mathcal{B}_n$, that the squares α and $\tilde{\alpha}E^{-1}$ are in the same diagonal. Therefore, $\tilde{\alpha}_iE^{-1}$ must travel east, as α_i does. Then, since $(\tilde{\alpha}_iE^{-1})E = \tilde{\alpha}_i = \alpha_{m-i}$, we conclude that $\tilde{\alpha}_iE^{-1} = \alpha_{m-i-1}$. Therefore,

$$\alpha_{m-i-1} = \tilde{\alpha}_iE^{-1} = \widetilde{\alpha_iE} = \tilde{\alpha}_{i+1},$$

as desired.

It is clear that $\mathcal{H}$ steps off the edge of the board somewhere, for, otherwise, it could not visit more than $2n-1$ squares. Let (p,q) be the square from which $\mathcal{H}$ steps off the edge of the board (the first time). We may assume, without loss of generality, that $p = n-1$, and that $\mathcal{H}$ steps east, from $\alpha_{n-1+q} = (n-1, q)$ to $\alpha_{n+q} = (0, n-1-q)$. Then, from the conclusion of the preceding paragraph, we have

$$\alpha_{m-(n-1+q)} = \tilde{\alpha}_{n-1+q} = (n-1,q)^{\sim} = (0, n-1-q) = \alpha_{n+q}.$$

Therefore, $m - (n-1+q) = n+q$. Since $(p,q) \neq \tau = (n-1, n-1)$ and $p = n-1$, we must have $q \leq n-2$, so this implies that $m = 2n-1+2q \leq 4n-5$. Because $n \geq 3$, we have $4n-5 < n^2-1$, so this contradicts the fact that the length of a hamiltonian path must be n^2-1. $\square$

Henceforth, we may (and do) always assume that the terminal diagonal T of a hamiltonian path $\mathcal{H}$ is not D_{2n-2}. Therefore, Proposition 2.4 implies that we may write T in the form $T = S_a \cup S_b$, with $a < b$.

3.3. Proposition. *Suppose the terminal diagonal of a hamiltonian path $\mathcal{H}$ is $S_a \cup S_b$, with $a < b$. Then all D_j, where $a < j < b$, travel in the same direction.*

Proof by induction. Given k with $a < k < n-2$, suppose that all D_j, where $k < j \leq n-2$, travel in the same direction, but D_k travels in the other direction.

Assume without loss of generality that D_k travels north and all D_j with $j > k$ travel east. Then, letting $\alpha = (k+1, 0) \in S_{k+1}$, we see that $\mathcal{H}$ contains the cycle $[\alpha](E^{2(n-k)-3}N)$. This contradicts the fact that $\mathcal{H}$ is a path. □

The preceding proposition shows that the "middle" diagonals (that is, those D_j with $a < j < b$) must all travel in the same direction. Theorem 3.5 shows that the other non-terminal diagonals have no effect on whether a spanning quasi-path is a hamiltonian path or not. First, we prove a lemma.

3.4. Lemma. *Let $\mathcal{H}$ be a spanning quasi-path, with terminal diagonal $S_a \cup S_b$ (where $a < b$). Let $(p, q) \in S_{b+1}$, and let P be the unique path in $\mathcal{H}$ that starts at (p, q) and ends in S_a, without passing through S_a. Then the terminal vertex of P is $(n-1-p, n-1-q)$, which is the inverse of (p, q).*

Proof. Let us begin by showing that P steps off the edge of the board exactly once. Since steps that do not go off the edge of the board move from S_k to S_{k+1}, and $a < b+1$, it is clear that P steps off the edge of the board at least once. Let α be the square from which P steps off the edge of the board (the first time). If $\alpha \in S_k$, then the square immediately immediately after α on P belongs to S_{2n-2-k}. Since $k \geq b+1 = 2n-2-a$, we have $2n-2-k \leq a$. Therefore, because $a < n-1$, we know that P reaches S_a before it reaches S_{n-1}, which is the first subdiagonal from which it is possible to step off the edge of the board.

Let $[\alpha_0, \alpha_1, \cdots, \alpha_m]$ be the sequence of squares visited by P. We now show that m is odd, that P steps off the edge of the board from $\alpha_{(m-1)/2}$, and that, for $0 \leq i \leq m-1$, the squares α_i and α_{m-1-i} lie on the same diagonal. If P steps off the edge of the board from α_k, then $\alpha_k \in S_{b+1+k}$, so $\alpha_{k+1} \in S_{2n-2-(b+1+k)}$. On the other hand, because $\alpha_m \in S_a$ (and P does not step off the edge of the board between α_{k+1} and α_m), we must have $\alpha_{k+1} \in S_{a-(m-k-1)}$. Therefore,

$$2n - 2 - (b + 1 + k) = a - (m - k - 1).$$

We know, from Proposition 2.4, that $a + b = 2n - 3$, so this implies $m = 2k + 1$, which means that m is odd, and that P steps off the edge of the board from $\alpha_{(m-1)/2}$. Furthermore, for $i < k$, we have $\alpha_i \in S_{b+1+i}$ and $\alpha_{m-1-i} \in S_{a-1-i}$. Since $a + b = 2n - 3$, we have $(b + 1 + i) + (a - 1 - i) = 2n - 3$, which means that α_i and α_{m-1-i} lie on the same diagonal, as desired.

We are now ready to prove that the terminal square of P is $(n-1-p, n-1-q)$. Assume, without loss of generality, that $\alpha_{(m-1)/2}$ travels east. From the preceding paragraph, we know that P steps off the edge of the board from $\alpha_{(m-1)/2}$, so $\alpha_{(m-1)/2}$ must be of the form $\alpha_{(m-1)/2} = (n-1, y)$. Thus, exactly $n-1-p$ of the squares $\alpha_0, \alpha_1, \ldots, \alpha_{(m-3)/2}$ travel east. Since α_i and α_{m-1-i} are on the same diagonal, then exactly $n-1-p$ of the squares $\alpha_{(m+1)/2}, \alpha_{(m+3)/2}, \ldots, \alpha_{m-1}$ travel east. Similarly, exactly $y - q$ of the squares $\alpha_{(m+1)/2}, \ldots, \alpha_{m-1}$ travel north. Because $\alpha_{(m-1)/2}$ travels east, off the edge of the board, we have $\alpha_{(m+1)/2} = (0, n-1-y)$. Then $n-1-p$ steps east and $y-q$ steps north result in the square $(n-1-p, n-1-q)$, as desired. □

3.5. Theorem. *Let $\mathcal{H}$ be a hamiltonian path in $\mathcal{B}_n$, with terminal diagonal $S_a \cup S_b$ (where $a < b$), and let $\mathcal{H}'$ be any spanning quasi-path in $\mathcal{B}_n$, with the same initial* *and terminal squares as $\mathcal{H}$ and such that each non-terminal diagonal D_j, with $a < j < b$, travels exactly the same way in $\mathcal{H}'$ as it does in $\mathcal{H}$. Then $\mathcal{H}'$ is also a hamiltonian path.*

Proof. From Proposition 2.10, we see that each square in the terminal diagonal travels exactly the same way in $\mathcal{H}'$ as it does in $\mathcal{H}$. To show that $\mathcal{H}'$ is a hamiltonian path, it suffices to show that $\mathcal{H}'$ has no cycles.

Suppose that there is a cycle C' in $\mathcal{H}'$. This cycle cannot be contained in $\mathcal{H}$, so it is not difficult to see that it must contain at least one square in S_{b+1}. We may write C' as the concatenation $C' = P_1 + Q_1' + \cdots + P_r + Q_r'$ of paths, such that each P_i is contained in $\mathcal{H} \cap \mathcal{H}'$, and each Q_j' starts in S_{b+1} and ends in S_a, without passing through S_a. Lemma 3.4 shows that, for each j, there is a path Q_j in $\mathcal{H}$ that starts in S_{b+1} and ends in S_a, without passing through S_a, and has the same endpoints as Q_j'. Then the concatenation $C = P_1 + Q_1 + \cdots + P_r + Q_r$ is a cycle in $\mathcal{H}$. This is a contradiction. □

3.6. Corollary. *If there is a hamiltonian path $\mathcal{H}$ from an initial square ι to a terminal square τ, then there is a hamiltonian path $\mathcal{H}'$ from ι to τ, such that all non-terminal diagonals travel in the same direction.*

Proof. We may write the terminal diagonal of $\mathcal{H}$ in the form $T = S_a \cup S_b$, with $a < b$. We know from Proposition 3.3 that all D_j where $a < j < b$ travel in the same direction. Assume without loss of generality that they travel east. Let $\mathcal{H}'$ be the spanning quasi-path of $\mathcal{B}_n$ in which every non-terminal diagonal travels east, and each square in the terminal diagonal travels exactly the same way as it does in $\mathcal{H}$. Theorem 3.5 shows that $\mathcal{H}'$ is a hamiltonian path from ι to τ. □

We now embark on the proof of Main Theorem 3.12. Thus, our goal is to describe precisely which squares can be joined by a hamiltonian path (in which all non-terminal diagonals travel east). To streamline the required case-by-case analysis, we start with a few lemmas. These preliminary results provide helpful information on the locations of initial and terminal squares of hamiltonian paths.

3.7. Lemma. *Let T be the terminal diagonal of a hamiltonian path from ι to τ, let α be the square on T_+ with $\alpha_y = n-1$, and let β be the square on T_- with $\beta_y = 0$.*

(1) *Suppose $\tau \in T_+$ and $\tau_y < n-1$. If some square $\lambda \in T_-$ travels east, then both α and β travel east.*
(2) *Suppose $\iota N^{-1} \in T_-$. If some square in T_+ travels east, then both α and β travel east.*
(3) *Suppose $\tau \in T_-$. If some square in T_+ travels north, then α travels north.*
(4) *Suppose $\iota E^{-1} \in T_+$. If some square in T_- travels north, then α travels north.*

Proof. We prove only (1), because the other cases are similar. In the notation of Corollary 2.12, it is not difficult to see that

$$u(\alpha) = n - 1 - \tau_y = u(\beta) - 1 < u(\beta) = u(\lambda) - \lambda_y \le u(\lambda).$$

Since λ travels east, we know from Corollary 2.12 that $u(\lambda) < u(\iota E^{-1})$, so this implies $u(\alpha), u(\beta) < u(\iota E^{-1})$, so both α and β travel east. □

3.8. Lemma. *Let $\mathcal{H}$ be a hamiltonian path in $\mathcal{B}_n$ from ι to τ, such that every non-terminal diagonal travels east. Assume n is odd. Let T be the terminal diagonal of $\mathcal{H}$, and let α be the square on T_+ with $\alpha_y = n - 1$. Then either:*

(1) *$\tau_y = \not{n}$; or*
(2) *$\iota_y = \not{n}$; or*
(3) *α travels north, and there is a square λ on T_- with $\lambda_y = \not{n}$, such that $\tau = \lambda E N^{-1}$; or*
(4) *α travels north, and there is a square ρ on T_+ with $\rho_y = \not{n}$, such that $\iota = \rho N$.*

Proof. Let ρ be the square on T_+ with $\rho_y = \not{n}$, if such a square exists; otherwise, let $\rho = (n-1, \not{n})$. Let λ be the square on T_- with $\lambda_y = \not{n}$, if such a square exists; otherwise, let $\lambda = (0, \not{n})$. We may assume $\tau_y \neq \not{n}$ and $\iota_y \neq \not{n}$; in particular, $\{\iota, \tau\} \cap \{\rho, \lambda, \rho E, \lambda E\} = \emptyset$.

Case 1. λ travels east. Because $\rho \neq \tau$, the square ρ must travel north, for, otherwise, $\mathcal{H}$ contains the cycle $[\lambda](E^n)$. (Note that, because all non-terminal diagonals travel east, this implies $\rho \in T_+$.) Since $\iota \neq \rho E$, we see from Proposition 2.10 that $\rho E N^{-1}$ must also travel north. Therefore $\rho \neq (n-1, \not{n})$, for, otherwise, $\mathcal{H}$ contains the cycle $[\rho](NENE^{n-1})$. Therefore $n - \rho_x - 2 \geq 0$, so $(n - \rho_x - 2, \not{n})$ is a square of $\mathcal{B}_n$. Then Proposition 2.4 implies $(n - \rho_x - 2, \not{n}) \in T_-$, so we have $\lambda \in T_-$. We must have $\lambda E N^{-1} = \tau$, for, otherwise, Proposition 2.10 implies that $\lambda E N^{-1}$ travels east, which would mean that $\mathcal{H}$ contains the cycle $[\rho](NE^{n+1}NE^{n-1})$. Hence Lemma 3.7(3) implies conclusion (3) holds.

Case 2. λ travels north. (Note that $\lambda \in T_-$, so Proposition 2.4 implies $(n - \lambda_x - 2, \not{n}) \in T_+$, which means $\rho \in T_+$.) Since $\iota \neq \lambda E$, we see from Proposition 2.10 that $\lambda E N^{-1}$ travels north, as λ does. Then, since $\rho \neq \tau$, we see that ρ must travel east, for, otherwise, $\mathcal{H}$ contains the cycle

$$[\lambda E](E^{\rho_x - \lambda_x - 1} N E^{n - \rho_x + \lambda_x + 1} N).$$

However, $\rho N E^{-1}$ does not travel east, for, otherwise, $\mathcal{H}$ contains the cycle

$$[\lambda](NE^{n+1}NE^{n-1}).$$

Therefore, Proposition 2.10 implies $\rho N = \iota$. Hence Lemma 3.7(4) implies conclusion (4) holds. □

3.9. Lemma. *Let $\mathcal{H}$ be a hamiltonian path in $\mathcal{B}_n$, such that every non-terminal diagonal travels east. Assume n is even. Let T be the terminal diagonal of $\mathcal{H}$, let α be the square on T_+ with $\alpha_y = n - 1$, and let β be the square on T_- with $\beta_y = 0$. Then either:*

(1) *$\iota_y = \not{n}$ and $\iota_x \geq \not{n} - 1$; or*
(2) *$\iota_y = \not{n} - 1$ and $\iota_x \leq \not{n}$; or*
(3) *$\tau_y = \not{n}$ and $\tau_x \geq \not{n} - 1$; or*
(4) *$\tau_y = \not{n} - 1$ and $\tau_x \leq \not{n}$; or*
(5) *$T \neq D_{n-1}$, and both α and β travel east.*

Proof. Let ρ be the square on T_+ with $\rho_y = \not n$, if such a square exists; otherwise, let $\rho = (n-1, \not n)$. Let λ be the square on T_- with $\lambda_y = \not n - 1$, if such a square exists; otherwise, let $\lambda = (0, \not n - 1)$. We may assume $\{\lambda, \rho\} \cap \{\iota, \tau\} = \emptyset$, for, otherwise, one of conclusions (1)-(4) holds.

Case 1. λ travels north. Note that we must have $\lambda \in T_-$, because all non-terminal diagonals travel east. Then we conclude from Proposition 2.4 that $(n - 2 - \lambda_x, \not n) \in T_+$, so $\rho \in T_+$ and $\rho_x < n - 1$.

The square ρ must travel north, for, otherwise, $\mathcal{H}$ contains the cycle $[\lambda](NE^n)$. We may assume that $\lambda E N^{-1}$ travels north, as λ does, for, otherwise, we see from Proposition 2.10 that $\iota = \lambda E$, which implies that conclusion (2) holds. Similarly, we may assume that $\rho E N^{-1}$ travels north, as ρ does, for, otherwise, we must have $\iota = \rho E$, which implies that conclusion (1) holds. Hence, $\mathcal{H}$ contains the cycle

$$[\lambda](NE^{\rho_x - \lambda_x} NE^{n-\rho_x+\lambda_x+1} NE^{\rho_x-\lambda_x} NE^{n-\rho_x+\lambda_x-1}).$$

Case 2. λ travels east. Let ρ' be the square on T_+ with $\rho'_y = \not n - 1$, if such a square exists; otherwise, let $\rho' = (n-1, \not n - 1)$.

Suppose $\rho' = \tau$. (Note that Proposition 2.4 then implies $(n-1-\rho'_x, \not n - 1) \in T_-$, so $\lambda \in T_-$.) We may assume $T \neq D_{n-1}$, for, otherwise, this implies that $\tau = (\not n, \not n - 1)$, so conclusion (4) holds. Furthermore, we may assume that $\rho' N E^{-1}$ travels east, for, otherwise, Proposition 2.10 implies $\iota = \rho' N$, in which case conclusion (1) holds. Therefore, Lemma 3.7(1) implies that α and β also travel east, so conclusion (5) holds.

We may now assume $\rho' \neq \tau$. Then ρ' must travel east, for, otherwise, $\mathcal{H}$ contains the cycle $[\rho'](NE^n)$.

Suppose $\lambda N E^{-1}$ does not travel east. Then $\lambda \in T_-$, and, because λ travels east, we must have $\iota = \lambda N$. Therefore, we may assume $T \neq D_{n-1}$, for, otherwise, we have $\iota = (\not n - 1, \not n)$, so conclusion (1) holds. Since $\lambda \in T_-$, Proposition 2.4 implies $(n - 1 - \lambda_x, \not n - 1) \in T_+$, so $\rho' \in T_+$. Therefore, Lemma 3.7(2) implies that conclusion (5) holds.

We may now assume $\lambda N E^{-1}$ travels east. Furthermore, we may assume that $\rho' N E^{-1}$ travels east, as ρ' does, for, otherwise, we see from Proposition 2.10 that $\rho' N = \iota$, which implies that conclusion (1) holds. Also, we may assume that $\rho' N$ travels north, for, otherwise, either conclusion (3) holds (if $\rho' N = \tau$) or $\mathcal{H}$ contains the cycle $[\lambda](E^{2n})$ (if $\rho' N$ travels east). Because all non-terminal diagonals travel east, this implies $\rho' N \in T_+$, so $\rho' N = \rho$. Then it is clear that $\rho' \notin T_+$, so we must have $\rho' = (n-1, \not n - 1)$, which means $\rho = (n-1, \not n)$. Therefore, $\rho E = \lambda \neq \iota$, so Proposition 2.10 implies that $\lambda N^{-1} = \rho E N^{-1}$ travels north, as ρ does. Therefore, $\mathcal{H}$ contains the cycle $[\lambda](E^{2n-1} NEN)$. □

3.10. Lemma. *Let $\mathcal{H}$ be a hamiltonian path in which all non-terminal diagonals travel east, and $\tau = (n-1, 0)$.*

- *If n is odd, then $\iota \in \{(\not n, \not n), (\not n, \not n + 1)\}$.*
- *If n is even, then $\iota \in \{(\not n - 1, \not n), (\not n, \not n)\}$.*

Proof. We consider two cases.

Case 1. n is odd. Let $\lambda = (\not n - 1, \not n)$ and $\rho = (\not n, \not n)$. We have $\tau_y = 0 \neq \not n$ and $\tau \neq \lambda E N^{-1}$, so we see from Lemma 3.8 that either $\iota_y = \not n$ or $\iota = \rho N =$

$(\not n, \not n + 1)$. Thus, we may assume $\iota_y = \not n$, and we wish to show $\iota = (\not n, \not n)$. Because $\tau \in D_{n-1}$, we see from Corollary 2.11 that $\iota N^{-1} \in D_{n-1} = S_{n-2} \cup S_{n-1}$. Thus, $\iota \in \{(\not n, \not n), (\not n + 1, \not n)\}$. If $\iota = (\not n + 1, \not n)$, then $\mathcal{H}$ contains the cycle $[\iota E^{-1}](NE^n N)$.

Case 2. n is even. If $n = 2$, the desired conclusion may be verified directly, so we may assume $n > 2$. Since $\tau_y = 0 \neq \not n$, and $\tau_x = n - 1 > \not n$, and $T = D_{n-1}$, we see that either (1) or (2) of Lemma 3.9 holds. Then, since we must have $\iota N^{-1} \in D_{n-1}$, we conclude that $\iota \in \{(\not n, \not n), (\not n - 1, \not n), (\not n, \not n - 1)\}$. However, if $\iota = (\not n, \not n - 1)$, then $\mathcal{H}$ contains the cycle $[\iota E^{-1}](NE^n)$. □

By considering the inverses of these hamiltonian paths, we obtain the following corollary.

3.11. Corollary. *Let $\mathcal{H}$ be a hamiltonian path in which all non-terminal diagonals travel east, and $\iota = (0, n-1)$.*

- *If n is odd, then $\tau \in \{(\not n, \not n), (\not n, \not n - 1)\}$.*
- *If n is even, then $\tau \in \{(\not n - 1, \not n - 1), (\not n, \not n - 1)\}$.* □

3.12. Main Theorem. *Let ι and τ be two squares on the $n \times n$ projective checkerboard $\mathcal{B}_n$, with $n \geq 3$. There is a hamiltonian path from ι to τ in which all non-terminal diagonals travel east if and only if τ is in the diagonal containing ιN^{-1} and either:*

(1) *n is odd and:*
  (a) *$\iota_y = 0$ and $\iota_x \geq \not n$ and $\tau_y = \not n$; or*
  (b) *$\iota = (0, n-1)$ and $\tau \in \{(\not n, \not n), (\not n, \not n - 1)\}$; or*
  (c) *$\tau_y = n - 1$ and $\tau_x \leq \not n$ and $\iota_y = \not n$; or*
  (d) *$\tau = (n-1, 0)$ and $\iota \in \{(\not n, \not n), (\not n, \not n + 1)\}$; or*

(2) *n is even and:*
  (a) *$\iota_y = 0$ and $\iota_x \geq \not n - 1$ and either:*
    (i) *$\tau_y = \not n - 1$ and $\tau_x \leq \not n - 1$; or*
    (ii) *$\tau_y = \not n$ and $\tau_x \geq \not n - 1$; or*
  (b) *$\iota = (0, n-1)$ and $\tau \in \{(\not n - 1, \not n - 1), (\not n, \not n - 1)\}$; or*
  (c) *$\tau_y = n - 1$ and $\tau_x \leq \not n$ and either:*
    (i) *$\iota_y = \not n$ and $\iota_x \geq \not n$; or*
    (ii) *$\iota_y = \not n - 1$ and $\iota_x \leq \not n$; or*
  (d) *$\tau = (n-1, 0)$ and $\iota \in \{(\not n - 1, \not n), (\not n, \not n)\}$.*

Furthermore, such a hamiltonian path is unique, if it exists.

Proof. If ι and τ are as specified, then the desired hamiltonian path from ι to τ either appears explicitly in Figure 3 or Figure 4 (in cases 1a, 1b, 2a, and 2b), or is the inverse of a hamiltonian path in Figure 3 or Figure 4 (in cases 1c, 1d, 2c, and 2d). The uniqueness follows from Corollary 2.13.

Now suppose there is a hamiltonian path $\mathcal{H}$ from ι to τ, such that all non-terminal diagonals travel east. Let T be the terminal diagonal of $\mathcal{H}$, let α be the square on T_+ with $\alpha_y = n - 1$, and let $\gamma = (n-1, 0)$. From Corollary 2.11, we know that $\iota N^{-1} \in T$. From Lemmas 3.8 and 3.9, we see that it is not possible to have both $\alpha = \tau$ and $\iota_y = 0$, so we may assume $\alpha \neq \tau$. (If $\alpha = \tau$, replace $\mathcal{H}$ with its inverse $\tilde{\mathcal{H}}$. Note that if $\tilde{\mathcal{H}}$ satisfies condition 1a or 2a, then $\mathcal{H}$ satisfies condition 1c or 2c, respectively.)

Case 1. $T \neq D_{n-1}$. The square α must travel east, for, otherwise, $\mathcal{H}$ contains the cycle $[\alpha](NE^{2\alpha_x+1})$. Let β be the square on T_- with $\beta_y = 0$. We see that β does not travel east, for, otherwise, $\mathcal{H}$ contains the cycle $[\alpha](E^{2n})$. Then, since $\beta EN^{-1} = \alpha$ travels east, but β does not, we conclude from Proposition 2.10 that $\iota = \beta E$. (In particular, $\iota_y = 0$.)

Suppose n is odd. Then, because α does not travel north, we see from Lemma 3.8 that $\tau_y = \not{n}$. Therefore, since $0 \leq \tau_x \leq n-1$, we have $\tau_x + \tau_y \geq \not{n}$ and $2n-3-(\tau_x + \tau_y) \geq \not{n} - 1$. Then, since $\beta \in T$, and $\beta_y = 0$, Proposition 2.4 implies $\beta_x \geq \not{n} - 1$, so $\iota_x \geq \not{n}$. Hence, conclusion 1a holds.

Now suppose n is even. Note that

$$\tau \notin \{(\not{n}, \not{n}-1), (\not{n}-1, \not{n}), (\not{n}-1, \not{n}-1)\},$$

because $T \neq D_{n-1}$. Then, because $\iota_y = 0 \notin \{\not{n}, \not{n}-1\}$ and β does not travel east, we see from Lemma 3.9 that either

(1) $\tau_y = \not{n}$ and $\tau_x \geq \not{n}$; or
(2) $\tau_y = \not{n} - 1$ and $\tau_x \leq \not{n} - 2$.

Therefore, $\tau_x + \tau_y \geq \not{n} - 1$ and $2n - 3 - (\tau_x + \tau_y) \geq \not{n} - 2$. Then, since $\iota E^{-1} \in T$ (and $\iota_y = 0$), Proposition 2.4 implies $\iota_x - 1 \geq \not{n} - 2$. Therefore, conclusion 2a holds.

Case 2. $T = D_{n-1}$ and α travels north. Note that $\alpha = (0, n-1)$. We know that γ does not travel east, for otherwise $\mathcal{H}$ contains the cycle $[\alpha](NE)$. If $\gamma = \tau$, then Lemma 3.10 implies that conclusion 1d or 2d holds.

We may now assume γ travels north. We see immediately that αN^{-1} does not travel north, for, otherwise, $\mathcal{H}$ contains the cycle $[\alpha](NNENEN)$. Thus, $\alpha E^{-1} = \gamma$ travels north, but αN^{-1} does not, so Proposition 2.10 implies $\iota = \alpha$. Hence, Corollary 3.11 implies that conclusion 1b or 2b holds.

Case 3. $T = D_{n-1}$ and α travels east. We may assume $\gamma \neq \tau$, for, otherwise, Lemma 3.10 implies that conclusion 1d or 2d holds. Therefore, γ travels either north or east.

Suppose $\gamma = \iota$. If n is odd, then Lemma 3.8 implies that conclusion 1a holds, whereas, if n is even, then Lemma 3.9 implies either that conclusion 2a holds, or that $\tau = (\not{n}, \not{n} - 1)$. (If n is even, note that $\tau \neq (\not{n}, \not{n})$, because $(\not{n}, \not{n}) \notin D_{n-1}$.) However, in the latter case, $\mathcal{H}$ would contain the cycle $[\tau E^{-1}](NE^n)$. Thus, we henceforth assume $\gamma \neq \iota$.

Suppose γ travels east. Let $\beta = \gamma E^{-1}$. Since the square $\beta EN^{-1} = \alpha$ travels east, and $\beta E = \gamma \neq \iota$, we see from Proposition 2.10 that that β also travels east. Thus, $\mathcal{H}$ contains the cycle $[\alpha](E^{2n})$.

We may now assume γ travels north. Furthermore, we may assume $\alpha \neq \iota$, for, otherwise, Corollary 3.11 implies that conclusion 1b or 2b holds. Then, since γ travels north, and $\gamma E = \alpha \neq \iota$, Proposition 2.10 implies that γEN^{-1} also travels north. Similarly, since α travels east, and $\alpha N = \gamma \neq \iota$, it must be the case that αNE^{-1} also travels east. Thus, $\mathcal{H}$ contains the cycle $[\alpha](E^{2n-1}NEN)$. □

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